

QUALIFYING EXAMINATION

Probability

August 27, 1993

1. a. State Kolmogorov's Three Series Theorem.

b. Using Kolmogorov's Three Series Theorem, show that $\sum_n t_n r_n(\cdot)$ is almost surely convergent if and only if $\sum_n |t_n|^2 < \infty$, where (r_n) is the sequence of Rademacher functions and (t_n) is a sequence of real numbers.

2. a. Define a martingale sequence carefully.

b. State Doob's martingale convergence theorem carefully.

3. a. What does it mean to say " K is a uniformly integrable subset of $L^1(P)$ "?

b. Show that if (X_n, Σ_n) is an L^1 -bounded martingale sequence then (X_n) is L^1 -convergent if and only if $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable.

4. Suppose a number x is chosen at random from the unit interval. Find the probability that the second digit of \sqrt{x} is 4.

Hint: First, show that x belongs to the event of interest precisely when there is an $m = 0, 1, \dots, 9$ such that

$$m + \frac{4}{10} \leq 10\sqrt{x} < m + \frac{5}{10}.$$

5. a. Define convergence in probability.

b. Let (X_n) be a sequence of independent, identically distributed random variables. Show that if $S_n = \sum_{k=1}^n X_k$ then (S_n) is almost surely convergent whenever (S_n) is convergent in probability.

6. a. Define the characteristic function of a random variable.

b. Find a closed expression for the characteristic function of the exponential distribution.

7. a. Show that the characteristic function of any integrable random variable is uniformly continuous.

b. Let X and Y be independent normally distributed random variables. Show that the sum of $X + Y$ is also normally distributed.

8. a. Carefully define conditional probability and conditional expectation of X relative to the σ -field \mathcal{A} .

b. Consider the closed unit interval with Lebesgue measure. Let \mathcal{A} be the σ -field generated by $\{[0, 1/4], (\frac{1}{4}, \frac{3}{4}], (\frac{3}{4}, 1]\}$.

Compute $E(X | \mathcal{A})$ if $X(\omega) = \omega^2$.

9. a. State both parts of the Borel-Cantelli Lemma carefully, and prove one part.

b. Let (Ω, \mathcal{A}, p) be a probability space, and let $\{A_n\}$ be a sequence of members of \mathcal{A} . Prove that if $\sum_{n=1}^{\infty} P(A \cup A_n) < \infty$ for every $A \in \mathcal{A}$ with $P A > 0$, then $P[A_n \text{ i.o.}] = 0$.

10. State precisely some form of the Central Limit Theorem with an outline of its proof.