

PROBABILITY Ph.D. QUALIFYING EXAMINATION
August 23, 1996.

Do as many problems as you can. Show your work. All problems carry equal weight.

Problem 1. (1) State the definitions of convergence in probability and almost sure convergence.

(2) State the following theorems precisely.

(i) Kolmogorov's three series theorem.

(ii) Any version of the weak law of large numbers.

Problem 2. (1) State a version of the central limit theorem.

(2) Let $X_i, i = 1, 2, \dots$ be an independent sequence of random variables, so that $P(X_i = i^\alpha) = P(X_i = -i^\alpha) = 1/(2i^{2\alpha})$ and $P(X_i = 0) = 1 - (1/i^{2\alpha})$, where $0 < \alpha$ is a fixed constant. For which values of α , if any, does the central limit theorem hold? That is, find all values of α so that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \xrightarrow{dist} N(0, 1).$$

Problem 3. (1) State the definition of convergence in distribution and the continuity theorem.

(2) Discuss the relationship between convergence in distribution, almost sure convergence and convergence in probability (with examples, if possible).

Problem 4. (1) State a version of the strong law of large numbers (SLLN).

(2) Let X_1, X_2, \dots be independent random variables so that the density of X_i is $P(X_i = i^\alpha) = P(X_i = -i^\alpha) = \frac{1}{2}$. Find all the values of α so that the SLLN holds.

Problem 5. (1) Define uniform integrability of a collection of random variables and discuss the importance of this concept.

(2) Let X, X_1, X_2, \dots be integrable random variables defined over the same probability space. Prove that the following two statements are equivalent.

(i) $\lim_{n \rightarrow \infty} E|X_n - X| = 0$,

(ii) $X_n \xrightarrow{p} X$ and $\{X_1, X_2, \dots\}$ is uniformly integrable.

Problem 6. (1) Define the characteristic function of a random variable.
(2) Find the characteristic function of a random variable whose density is $f(x) = \lambda e^{-\lambda|x|}/2$, for $-\infty < x < \infty$.
(3) Using an inversion formula (or directly) find the characteristic function of the Cauchy density, i.e., $g(x) = 1/(\pi(1+x^2))$, for $-\infty < x < \infty$.

Problem 7. (1) State and prove any one of the two Borel-Cantelli lemmas.
(2) Prove that $E|X_1| < \infty$ if and only if $P\{|X_n| > n \text{ i.o.}\} = 0$ where X_1, X_2, \dots is a sequence of independent and identically distributed random variables.

Problem 8. (1) State the Radon-Nikodym theorem.
(2) Define conditional expectation, $E(X|\mathcal{F})$, with respect to a sigma field \mathcal{F} . Then prove that $E(X) = E(E(X|\mathcal{F}))$ for any sigma field \mathcal{F} .

Problem 9. (1) Define a martingale and a submartingale sequence.
(2) Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $E(X_n) = 0$ and $Var(X_n) = \sigma^2$. Prove that the sequence $\{Y_n, n \geq 1\}$ defined as follows is a martingale sequence.

$$Y_n = (X_1 + X_2 + \dots + X_n)^2 - n\sigma^2.$$

Problem 10. (1) State the martingale convergence theorem.
(2) State and prove Doob's (maximal) martingale inequality for submartingales.