Qualifying Examination

Probability

May 16, 1994

1. a) State the Borel-Cantelli Lemmas (both parts), and prove one part.

   b) Suppose X_n are mean zero *r.v.s* such that for any $\varepsilon > 0$,

$$\sum P\left[|X_n| > \varepsilon\right] < \infty.$$

Prove that $\sum X_n$ converges, or give a counter example.

2. a) Define

   i) convergence in probability;

   ii) almost sure convergence;

   iii) weak convergence of distribution functions.

   b) Discuss the relationships between i), ii), and iii) in part a).

   c) Prove that if $X_n \to X$ in probability, then there exist a sequence $\{n_k\}$ such that

$$X_{n_k} \longrightarrow X \quad a.e. \text{ as } k \longrightarrow \infty.$$

3. a) State Kolmogorov's 3 Series Theorem.

   b) Let X_n be i.i.d r.v.'s with

$$P\left[X_n = 0\right] = 1/2 = P\left[X_n = 1\right].$$

Show that $\sum a_n X_n$ converges *a. s.* iff $\sum a_n^2 < \infty$.

4. a) Define the conditional expectation of X with respect to the σ-field $\mathcal{A}$.

   b) Let $\Omega = [0, 1]$, and let P be Lebesgue measure on Ω. Let $\mathcal{A}$ be the σ-field generated by $\{[0,\ 1/4]\,,\ [1/4,\ 3/4]\,,\ [3/4,\ 1]\}$. Compute $E\left(X \mid \mathcal{A}\right)$ if $X(\omega) = \omega^2$.

5. a) State a version of the Central Limit Theorem. Give Liapunov's condition, and the Lindeberg-Feller condition. Give a brief outline of a proof.

   b) Let $P\left[X_n = n\right] = P\left[X_n = -n\right] = \frac{1}{n^2}$ and $P\left[X_n = 0\right] = 1 - \frac{2}{n^2}$. Does the CLT hold for $\{X_n\}$? Justify your answer.

6. a) Define: X_n are uniformly integrable.

b) Discuss the significance of uniform integrability.

c) Suppose $\{X_k\}$ is a sequence of *i.i.d. r.v.s* $\ni P[X_k = k] = 1 - P[X_k - 0] = \frac{1}{k}$ for $k = 1, 2, \cdots$. Prove or disprove: $\{X_k\}$ are uniformly integrable.

7. Let f be a non-negative, measurable function and let X be a random variable.

a) Prove that for any $c > 0$,

$$P[f(X) \geq c] \leq \frac{Ef(X)}{c}.$$

b) Derive Chebychev's inequality from a).

8. a) Define the characteristic function of a random variable X.

b) State the Continuity Theorem for characteristic functions and describe the connection between convergence of distribution functions and convergence of characteristic functions.

c) Prove that an infinitely divisible characteristic function never vanishes.