QUALIFYING EXAMINATION IN PROBABILITY

January 5, 2004

Directions

1. Answer questions completely: a fully-done problem is far more revealing of your understanding than two half-done attempts.
2. State your reasons for your claims. You may cite and use standard results. However, you have to demonstrate that *you* know why things are so; you cannot assume that we know that you know. That's the purpose of this examination! An exercise that states a standard result of the theory requires that you supply a proof.

1. (a) State and prove the Borel-Cantelli Lemma and the "zero-one" law.

   (b) Prove that if $X_1, X_2, \cdots$ is a sequence of independent and identically distributed random variables then

$$E(|X_1|) < \infty \text{ if and only if } P(|X_n| > n \; i.o.) = 0$$

2. (a) Define convergence in probability, almost everywhere, and in L_p and discuss the relationship between them with illustrations by examples.

   (b) Give an example of a sequence $X_1, X_2, \cdots$ so that $E(|X_1|) < \infty$ and $E(X_n) \to 0$ but no subsequences of $\{X_n\}$ converge to zero in probability.

3. (a) Define vague convergence of a sequence of measures and convergence of a sequence of random variables in distribution.

   (b) State and prove that each sequence of subprobability measures (s.p.m.) contains a subsequence converging vaguely to a s.p.m.

   (c) State the "continuity" theorem.

4. Define the classes C_K, C_O, C_B, C. In terms of them state conditions of vague convergence of measures.

5. (a) State a version of the Strong Law of Large Numbers (SLLN).

(b) Let $X_1, X_2, \cdots$ be independent identically distributed random variables so that $P(X_n = i^\alpha) = P(X_n = (-1)^\alpha) = \frac{1}{2}$.
Find all the values of α so that SLLN holds.

6. (a) Discuss relations between convergence in distribution, in probability, and almost surely.

(b) Prove that if $X_n \to X$ in probability then there exists a sequence $\{n_K\}$ such that $X_{n_K} \to X$ a.e.

7. Let $\{X_i\}$ be a sequence of independent identically distributed random variables with

$$P(X_1 \leq t) = \begin{cases} 1-e^{\frac{-t^2}{2}} \text{ if } t>0 \\ 0 \quad \text{if } t \leq 0 \end{cases}$$

Let the random variable $N_n(t)$ be the number of X_i which are $\leq t$. Find

$$P(N_n(t) = K), K = 0, 1, \cdots, n.$$

8. (a) State and prove the Chebyshev inequality.

(b) Suppose X has density $f(x) = \frac{1}{m!} x^m e^{-x}, x \geq 0, m \geq 0$, where m is an integer. Using formula $\int_0^\infty x^m e^{-x} dx = m!$, show that $P(0 < X < 2(m+1)) > \frac{m}{m+1}$.

9. (a) State Kronecker's Lemma.

(b) Let $\{X_i\}$ be independent, identically distributed r.v.p. with

$$P(X_1 = c^k) = \frac{1}{2^k}; K = 1, 2 \cdots$$

For which value of c does $\{X_i\}$ obey the SLLN?

10. (a) State the Kolomogorov-Feller theorem.

(b) State and prove the WLLN in Khintchin's formulation.