Real Analysis Qualifying Exam January 2005

Answer as many questions as possible. A complete answer is generally worth more than two partial answers. Good Luck!

1. For any interval I in $(0,1)$, let $l(I)$ denote the length of I. Define the Lebesgue outer regular measure of a subset E of $(0,1)$ by
$$m(E) = \inf\{\sum_j l(I_j) \,|\, E \subset \cup_j I_j\}.$$
Let E be a measurable subset of $(0,1)$. Prove for every $\epsilon > 0$ there exists a closed set F and an open set U with $F \subset E \subset U$ and
$$m(E) - \epsilon \leq m(F) \leq m(U) \leq m(E) + \epsilon.$$

2. Let $(X, \mathcal{M}, \mu)$ denote a measure space and assume $f : X \to \mathbb{R}$ is integrable. Prove that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every measurable subset E of X with $\mu(E) < \delta$,
$$\int_E f(x)\, d\mu(x) < \epsilon.$$

3. Let f be a bounded function over the unit square $S = [0,1] \times [0,1]$. Let $f(x,t)$ be a measurable function of x for each fixed $t \in [0,1]$. Prove that if $\frac{\partial f}{\partial t}$ exists and is bounded on S, then
$$\frac{d}{dt}\int_0 1 f(x,t)dx = \int_0 1 \frac{\partial f(x,t)}{\partial t} dx$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and assume $\mu(X) < \infty$. Let $L^p(X)$ denotes the collections of measurable functions f on X such that $||f||_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}$ is finite. Let $1 \leq r \leq s < \infty$. Prove that $L^s(X) \subset L^r(X)$.

5. (a) State the Hahn-Banach Theorem.

   (b) Let X be a normed linear space and let x_0 be a fixed non-zero vector in X. Prove there exists a bounded linear functional $f : X \to \mathbb{R}$ such that $f(x_0) = ||x_0||$ and $||f|| = 1$.

6. A function $f : [0,1] \to \mathbb{R}$ is called *Lipschitz continuous* with *Lipschitz constant* α if $|f(x) - f(y)| \leq \alpha|x - y|$ and $\alpha > 0$. Let $M(f) = \inf \alpha$ be the infinmum over all Lipschitz constants of f. For any Lipschitz continuous function f, let
$$||f|| = |f(0)| + M(f). - - - (*)$$
Show that the collection of Lipschitz continuous functions is a Banach space with respect to this norm. Specifically, prove $(*)$ defines a norm, and the Lipschitz continuous functions on $[0,1]$ a real vector space which is complete in this norm.

7. Define $f : [0,1] \to [0,1]$ as follows: let $f = \frac{1}{2}$ on $[\frac{1}{3}, \frac{2}{3}]$; then let $f = \frac{1}{4}$ on $[\frac{1}{9}, \frac{2}{9}]$ and $f = \frac{3}{4}$ on $[\frac{7}{8}, \frac{8}{9}]$.In the next step take the middle third out of each of the four remaining intervals in $[0,1]$ and define f to be $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, on these middle thirds. Continuing in this way, we get the the Cantor function f. Prove that f is uniformly continuous and $f' = 0$ almost everywhere on $[0,1]$. Is f absolutely continuous on $[0.1]$?

8. State the Monotone Convergence Theorem, then use it to prove Fatou's Lemma: If $f_1, f_2, \cdots,$ is a sequence of Lebesgue integrable non-negative functions on $[0,1]$ which converges pointwise almost everywhere to a function f, then $\int_0 1 f(x)dx \leq \liminf_n \int_0 1 f_n(x)dx$.

9. Let f and g be measurable, real-valued functions on $[0,1]$. Prove or disprove the following:

   (i) If $\int_0 1|f(x)|^2dx = \int_0 1|g(x)|^2dx = \frac{1}{2}\int_0 1|f(x)+g(x)|^2dx < \infty$ then $f = g$ almost everywhere on $[0,1]$.

   (ii) If $\int_0 1|f(x)|dx = \int_0 1|g(x)|dx = \frac{1}{2}\int_0 1|f(x)+g(x)|dx < \infty$ then $f = g$ almost everywhere on $[0,1]$.

10. Let $f : [0,1] \to \mathbb{R}$. Decide the truth or falsehood of each of the statements below. Justify your answer for each statement with a proof or a counterexample.

    (a) If f is absolutely continuous then f is of bounded variation.

    (b) If f is uniformly continuous then f is absolutely continuous.

    (c) If f is Lipschitz then f is absolutely continuous.

    (d) If f is absolutely continuous then f is Lipschitz.