QUALIFYING EXAMINATION

AUGUST, 1995

REAL VARIABLES

1. Take as the definition of a Lebesgue measurable function the following: a function $f : [0,1] \longrightarrow (-\infty, \infty)$ is *measurable* if the pre-image $f^{\leftarrow}(I)$ of every interval I is Lebesgue measurable.

   Prove that if $f : [0,1] \longrightarrow (-\infty, \infty)$ is measurable, then $f^{\leftarrow}(B)$ is Lebesgue measurable for each Borel set $B \subseteq (-\infty, \infty)$.

2. Let $f : [0,1] \longrightarrow (-\infty, \infty)$ be a bounded measurable function. Show that f is the uniform limit of simple measurable functions.

3. Prove, using an $\varepsilon - \delta$ argument, that the function

$$f(t) = \frac{t^2 + 2t + 3}{t - 1}$$

   is continuous at $t = 0$.

4. A subset K of $L^1[0,1]$ is *uniformly inferable* if for each $\varepsilon > 0$ there is an $M_\varepsilon > 0$ so that for all $f \in K$

$$\int_{[|f(t)| \geq M_\varepsilon]} |f(t)| dt \leq \varepsilon.$$

   Suppose $\Phi : [0, \infty) \longrightarrow [0, \infty)$ is an increasing function such that $\lim\limits_{x \to \infty} \frac{\Phi(x)}{x} = +\infty$. Prove that the set K of all $f \in L^1[0,1]$ such that $\int \Phi(|f(t)|)dt \leq 1$ is a uniformly integrable set.

5. Let f be a bounded real-valued function defined on the unit square $S = [0,1]^2$. Assume $f(x,t)$ is a measurable function of x for each $t \in [0,1]$. Suppose $\frac{df}{dt}$ exists and is bounded on S. Show that

$$\frac{d}{dt}\int_0^1 f(x,t)dx = \int_0^1 \frac{df(x,t)}{dt}dx.$$

6. Let $f : (-\infty,\infty) \longrightarrow [0,\infty)$ be a measurable function and let $F(t)$ be given by

$$F(t) = \lambda(\{x : f(x) > t\})$$

where λ is Lebesgue measure, be f's distribution function.

Show that $F : [0,\infty) \longrightarrow [0,\infty)$ is non-increasing (hence measurable) and

$$\int_{-\infty}^{\infty} f(t)dt = \int_0^{\infty} F(t)dt.$$

7. Prove that there is a constant $C > 0$ so that for each $f \in L^2[0,2\pi]$,

$$\|f\|_1 \le C\|f\|_2.$$

By means of example show that $L^1[0,2\pi] \not\subset L^2[0,2\pi]$.

8. Let $f : [-\pi, \pi] \longrightarrow I\!R$ be given by

$$f(x) = \begin{cases} +1 & \text{if } x \leq 0 \\ -1 & \text{if } x > 0 \end{cases}$$

(a) Find the Fourier series of f with respect to the system

$$\{1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots\}$$

(b) Does the above Fourier series converge to f in $L^2[-\pi, \pi]$? Explain.

(c) Does it converge to f uniformly? Explain.

9. Suppose $1 < p < \infty$ and $f : [0,1] \longrightarrow I\!R$ is absolutely continuous and $f' \in L^p[0,1]$. Show that $f \in Lip(\frac{1}{q})$ where $\frac{1}{p} + \frac{1}{q} = 1$.

[$Lip(\alpha)$ consists of those $g : [0,1] \longrightarrow I\!R$ for which there is $K > 0$ so

$$|g(x) - g(y)| \leq K|x - y|^\alpha$$

for all $x, y \in [0,1]$].