Qualifying Examination

Real Variables

August 24, 1998

Directions: Answer as many questions as you can. Questions answered completely and correctly carry **much more** weight with the examiners than two questions that are half-answered. Of course, assertions should be accompanied with reasons; conjectures are out of place.

1. Let M be the set of Lebesgue measurable functions on $[0,1]$. Define $d(f,g)$ by

$$d(f,g) = \int_0^1 \frac{|f(t)-g(t)|}{1+|f(t)-g(t)|}\,dt.$$

Prove that for $f_n, f \in M$,

$$\lim_n d(f_n, f) = 0$$

if and only if

$$(f_n)\,\text{converges to}\,f\,\text{in measure.}$$

2. A set K of Lebesgue integrable functions defined on $[0,1]$ is said to be *uniformly integrable* if given $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $\lambda(E) \leq \delta$ (here λ denotes the Lebesgue measure) then $\int_E |f| \leq \varepsilon$ for all $f \varepsilon K$.

(a) Show that the closed unit ball B_{L^2}

$$B_{L^2} = \left\{ f : \int_0^1 |f(t)|^2 dt \leq 1 \right\}$$

of $L^2(0,1)$ **is** uniformly integrable.

(b) Show that the closed unit ball of $L^1[0,1]$ is **NOT** uniformly integrable.

3. Let $f_n = nX_{[1/n^2, 2/n^2]}$

(a) Prove that

$$\lim_{n\to\infty} f_n(x) = 0$$

for all $x \in [0,1]$ but that the convergence is **NOT** uniform.

(b) Is it true that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0?$$

Explain.

4. (a) Show that if a linear subspace E of a normed linear space X has a non-empty interior, then $E = X$.

(b) Show that if E is a linear subspace of a normed linear space X, then the closure $\overline{E}$ of E is also a linear subspace of X.

(c) Suppose X and Y are normed linear spaces and $T : X \to Y$ is a linear map that's continuous at some point $x_0 \in X$. Show that there's a $K > 0$ so that $\|Tx\| \leq K\|x\|$ for all $x \in X$.

5. (a) Let $f : [0,1] \to \mathbb{R}$ be a *bounded Lebesgue measurable* function. Show that there is a sequence (g_n) of **simple** measurable functions so that f is the *uniform* limit of (g_n).

(b) Keeping the Dominated Convergence theorem in mind and modifying the construction of (a), what can you say about approximating a *Lebesgue integrable* function $f : [0,1] \to \mathbb{R}$ by simple functions?

6. Let λ denote Lebesgue measure on $[0,1]$ and f, g be Lebesgue measurable functions on $[0,1]$. We say f, g are *independent* if

$$\lambda\left(f^{-1}(A)\right)\lambda\left(g^{-1}(B)\right) = \lambda\left(f^{-1}(A) \cap g^{-1}(B)\right)$$

for any Borel sets $A, B \subseteq \mathbb{R}$.

(a) Prove that if f, g are simple, non-negative and independent Lebesgue measurable functions then

$$\int_0^1 fg d\lambda = \int_0^1 f d\lambda \int_0^1 g d\lambda.$$

(b) Prove that if f, g are non-negative independent Lebesgue integrable functions then the same identity holds.

7. (a) State and prove the Baire Category Theorem.

(b) Using (a), show the "principle of uniform boundedness", namely, show that if F is a family of real-valued continuous functions on a **complete** metric space X such that for each $x \in X$, $\{f(x) : f \in F\}$ is bounded then there is a non-empty open set $O \subseteq X$ and a constant M so that $|f(x)| \leq M$ for all $x \in O$ and $f \in F$.

8. (a) Show that every $f \in L^2[0,1]$ is also in $L^1[0,1]$.

(b) Prove that the closed unit ball B_{L^2}

$$B_{L^2} = \left\{ f : \int_0^1 |f(t)|^2 dt \leq 1 \right\}$$

is also closed in $L^1[0,1]$.

9. Let f be absolutely continuous on $[0,1]$ with $f' \in L_2[0,1]$. Suppose $f(0) = 0$. Prove that $x^{-1/2} f(x)$ is in $L_\infty[0,1]$.

10. (a) State the Radon-Nikodym Theorem.

(b) Define the Borel measures μ, v on $[0,1]$ by

$$\mu(E) = \int_E x^{-1/2} dx\,,\ v(E) = \int_E x^{-1/4} dx.$$

Which Radon-Nikodym derivative can be defined: $\frac{d\mu}{dv}$ or $\frac{dv}{d\mu}$? Whatever the answer, compute it.

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