

QUALIFYING EXAMINATION IN REAL VARIABLES

August 22, 2003

Directions

1. Answer questions completely: a fully-done problem is far more revealing of your understanding than two half-done attempts.
2. State your reasons for your claims. You may cite and use standard results. However, you have to demonstrate that *you* know why things are so; you cannot assume that we know that you know. That's the purpose of this examination! An exercise that states a standard result of the theory requires that you supply a proof.
3. Write your answers on separate blank paper (not the exam). Do not write all the way to the edge, as it will be lost when copied. Thank You!

1. A real valued function f defined on an interval $[a, b]$ satisfies a Lipschitz condition with constant M if $|f(s) - f(t)| \leq M|s - t|$ for all $s, t \in [a, b]$. Prove that f satisfies a Lipschitz condition with constant M if and only if

(i) f is absolutely continuous on $[a, b]$

and

(ii) $|f'(x)| \leq M$ a.e. (relative to Lebesgue measure)

2. Let μ be a measure. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers such that

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in $L^1(\mu)$ with bounded L^1 norm; i.e., there exists $M > 0$ such that for all n

$$\int |f_n| \leq M$$

Prove that $g(x) = \sum_{n=1}^{\infty} a_n f_n(x)$ converges a.e. (relative to μ), $g \in L^1(\mu)$,

and $\int g d\mu = \sum_{n=1}^{\infty} a_n \int f_n d\mu$.

3. A subset K of $L^1(0, 1)$ is called uniformly integrable if given $\epsilon > 0$ There is a $\delta > 0$ such that if $m(E) < \delta$ (m is the Lebesgue measure) then $\int_E |f| dm < \epsilon$ for all $f \in K$.

(1) State Holder's inequality and then using it show that if $1 < p < \infty$. Then the closed unit ball of $L^p(0, 1)$ is uniformly integrable.

(2) Show that the closed unit ball of $L^1(0, 1)$ is NOT uniformly integrable, by constructing a suitably 'misbehaved' sequence in the ball.

4. In each of the following, state whether the given statement is true or false and indicate briefly the reason for your answer:

(a) If $F : [0, 1] \rightarrow [0, 1]$ is a homeomorphism of $[0, 1]$ onto $[0, 1]$, then F takes Borel sets to Borel sets.

(b) If $F : [0, 1] \rightarrow [0, 1]$ is a homeomorphism of $[0, 1]$ onto $[0, 1]$, then F takes Lebesgue measurable sets onto Lebesgue measurable sets.

(c) If $F : (0, 1) \rightarrow \mathbb{R}$ is everywhere differentiable, then $F'(x)$, the derivative of F at $x \in (0, 1)$, has some points of continuity.

(d) If $F : (0, 1) \rightarrow \mathbb{R}$ is differentiable at each point of $(0, 1)$ and $F'(x_0) = 0$ and $F'(x_1) = 1$ where x_0 and x_1 are points in $(0, 1)$. Then there is a point $x \in (0, 1)$ so that $F'(x) = \frac{\sqrt{2}}{2}$.

5. Suppose $1 \leq p < \infty$, $f \in L^p(-\infty, \infty)$ and $g(x) = \int_x^{x+1} f(t)dt$. Prove that g is continuous and show that $\lim_{x \rightarrow \pm\infty} g(x) = 0$.

6. If f and g are non negative integrable functions on $[0, 1]$ with

$$\int_0^1 f(x)dx = \int_0^1 g(x)dx = 1,$$

then the set of points $x \in [0, 1]$ for which both $f(x)$ and $g(x)$ are ≤ 3 has Lebesgue measure $\geq \frac{1}{3}$.

7. Using Fatou's lemma, show that if (f_n) is a bounded sequence in L^2 that converges almost everywhere to the measurable function f then f is in L^2 , too.

8. Let m^* denote Lebesgue outer measure on R . Suppose A and B are two measurable subsets with the property that there are two disjoint open intervals I and J with $A \subset I$ and $B \subset J$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$. You may assume subadditivity of m^* .

9. Suppose that f is a non negative Lebesgue integrable function on $[0, \infty)$ and that $\int_0^\infty f(t)dt = 1$. Evaluate

$$\int_0^\infty \left(\frac{1}{x} \int_x^{ex} f(t)dt \right) dx.$$

10. Suppose X is a set, \mathcal{M} a sigma-algebra of subsets of X , and μ, ν , and η are (positive, sigma-finite) measures on \mathcal{M} with $\mu \ll \nu \ll \eta$. Show that $\mu \ll \eta$ and

$$\frac{d\mu}{d\eta} = \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\eta}.$$