QUALIFYING EXAMINATION IN REAL VARIABLES

January 7, 2004

Directions

1. Five (5) problems done completely will be an *outstanding* performance.
2. Proper attention to detail is reflected by a corresponding appreciation on the part of graders.
3. Two half-done solutions do *not* add up to one full-done; go for complete solutions!

1. (a) State Fatou's Lemma.

(b) Either prove Fatou's lemma or use it to show that if (f_n) is a sequence in $L^1[0,1]$ with $||f_n||_1 \leq K$ for each n and if f_n converges to f_0 almost everywhere then $f_0 \in L^1[0,1]$ and $||f_0||_1 \leq K$.

2. Let K be a closed linear subspace of the Hilbert space H and suppose f is a bounded linear functional defined on K. Show that there is a unique bounded linear functional F on H such that $F(x) = f(x)$ for each $x \in K$ and $||F|| = ||f||$.

3. Suppose $f \in L^2(\mathbb{R})$. Evaluate

$$\overline{\lim}_{|x|\to\infty} \int_{\mathbb{R}} |f(x+t) - f(t)|^2 dt.$$

Suggestion: you might find it reasonably easy to start with f continuous and having compact support.

4. Let $f \in L^1(0,1)$ and define $F(x) = \int_0^x f(t)dt$ for $x \in [0,1]$. Show that $f \in L^\infty(0,1)$ if and only if there is $K > 0$ so that for any $x, y \in [0,1]$, $|F(x) - F(y)| \leq K|x - y|$.

5. Prove that if A and B are subsets of $[0, 1]$ and if there are disjoint open subintervals I and J of $[0, 1]$ such that $A \subseteq I$ and $B \subseteq J$, then

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

YOU MAY ASSUME THAT $m^*(A \cup B) \leq m^*(A) + m^*(B)$.

6. Suppose that $F : [0,1] \to \mathbb{R}$ is continuous and satisfies

$$F\left(\frac{x+y}{2}\right) \leq \frac{F(x)+F(y)}{2}$$

for any $x, y \in [0,1]$. Show that F is convex, that is, that for any $0 \leq t \leq 1$ and any $x, y \in [0,1]$, $F(tx + (1-t)y) \leq tF(x) + (1-t)F(y)$.

7. Let E be the set of points in $[0, 1]$ whose binary expansion has zeroes in all the *even* places. Show that E has measure zero.

8. (a) What does it mean to say a set $E \subseteq [0,1]$ is measurable?

(b) Show that if $m^*(E) = 0$ then E is measurable.

9. Suppose that g is a strictly increasing absolutely continuous function on $[0, 1]$ and that $|g(x)| \leq M$ for each $0 \leq x \leq 1$. Show that if f is absolutely continuous on $[-M, M]$, then $f \circ g$ is absolutely continuous on $[0, 1]$.

10. Suppose $F_n(x) = n^2 x(1-x)^n$ for $0 \leq x \leq 1$.

(a) Show that $\lim_{n\to\infty} F_n(x) = 0$ for each $x \in [0, 1]$.

(b) Show that $\int_0^1 F_n(x)dx \to 1$ as $n \to \infty$.

(c) Does there exist $g \in L^1[0, 1]$ such that $f_n \leq g$ for all n? If so: then what g works? If not, why not?