Qualifying Examination; Statistics; August, 2004

Please do all problems, presenting clear, complete and well-written solutions. Two partially done problems do not equal one completely solved problem. Direct any questions concerning possible misprints to the proctor. Show all your work. If you need any percentile values of any distribution, ask the proctor.

1. Let $X_1, X_2, \cdots, X_n$ be independent random variables so that $X_i \sim N(\alpha + \beta t_i, \sigma^2)$, where α, β, σ^2 are unknown parameters, $\alpha, \beta, \in \Re$, $\sigma > 0$, and $t_1, t_2, \cdots, t_n$ are known real numbers that are not all equal. Find the UMVU estimators of α, β. (Justify your steps.)

2. Let $X_1, X_2, \cdots, X_n \overset{iid}{\sim} Poisson(\lambda)$, where $\lambda \sim Gamma(\alpha, \gamma)$. That is, the (prior) density of λ is
$$f_\lambda(u) = \begin{cases} \frac{\alpha^\gamma}{\Gamma(\gamma)} u^{\gamma-1} e^{-\alpha u} & , \quad \text{if } u > 0, \\ 0 & , \quad \text{otherwise.} \end{cases}$$
(i) Find the Bayes estimator of λ for the squared error loss function. (ii) Determine the behavior of your estimator in part (i) as n gets large.

3. Forty eight measurements were recorded to several decimal places. However, their sum was calculated after rounding off to the nearest integers. Assume that the round off errors have uniform distribution over $(\frac{-1}{2}, \frac{1}{2})$ and are probabilistically independent random variables. Find an approximation of the probability that the error in the sum is within 2 (units) of the true sum. Explain your method of approximation. (OPTIONAL: Can you give a closed form expression for the exact probability?)

4. State the Neymann-Pearson (NP) lemma precisely and explain (by examples if necessary) why sometimes the NP-lemma can be used to obtain optimal (uniformly most powerful) tests even when the null and the alternative hypotheses are composite.

5. Let $X_1, X_2, \cdots, X_n \overset{iid}{\sim} N(\theta, 1)$, where $\theta \in \Omega := (0, \infty)$. Find the maximum likelihood estimator (MLE) of θ whenever it exists. Indicate clearly when the MLE does not exist.

6. (i) Let $X_1, X_2, \cdots, X_8 \overset{iid}{\sim} f_\theta(x)$, where

$$f_\theta(x) := \begin{cases} \theta x^{\theta-1} & , \quad \text{if } 0 < x < 1, \\ 0 & , \quad \text{otherwise,} \end{cases}$$

and $\theta \in \Omega := (0, \infty)$. For testing

$$H_0 : \theta \leq 1, \qquad \text{versus} \qquad H_1 : \theta > 1,$$

show that the UMP, level $\alpha = 0.05$, test rejects H_0 if $\sum_{i=1}^{8} \log X_i \geq C$, for some constant C.

(ii) Find the constant C of part (i).

7. (i) Define consistency of estimators.

(ii) Let $X_1, X_2, \cdots \overset{iid}{\sim} F$, for some distribution F which has finite variance σ^2. Show that

$$S_n := \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$$

is consistent for σ^2, where $\overline{X}_n := (X_1 + X_2 + \cdots + X_n)/n$.

8. (i) Define one-parameter exponential family.

(ii) Define sufficiency.

(iii) State the Central Limit Theorem.