Qualifying Examination; Statistics; August, 1995

Do all problems, presenting clear, complete and well-written solutions. Two partially done problems do not equal one completely solved problem. Direct any questions concerning possible misprints to the proctor. Show all your work. If you need any percentile values of any distribution, ask the proctor.

1. Let $g(x)$ and $h(x)$ be two given densities over the real line $\Re$. Define the density

$$f(x;\theta) = k(\theta)\ (g(x))^{\theta} (h(x))^{1-\theta},$$

for $\theta \in [0,1]$. A random sample $X_1, X_2, \cdots, X_n$ of size n is taken from a population having density $f(x;\theta)$, $\theta \in \Omega = [0,1]$.

(a) Determine the uniformly most powerful test of size α, $(0 < \alpha < 1)$, for $H : \theta = 0$ versus $K : \theta > 0$.

(b) If $g(x) = e^{-x} I\{x > 0\}$, and $h(x) = (1+\epsilon)e^{-x(1+\epsilon)} I\{x > 0\}$, where ϵ is a given positive number, state which distribution you need to use to find the exact constants in the UMP test of part (a). Also, find the power function of the test.

2. Let $X_1, X_2, \cdots, X_n$ $(n \geq 2)$, be a random sample form $f(x;\theta) = \theta e^{-\theta x} I\{x > 0\}$. Find the uniformly minimum variance unbiased (UMVU) estimators of

(a) θ

(b) $P_\theta(X_1 > 13)$

(Hint: (a) try $c/\sum_i X_i$. (b) The conditional density of $X_1 = x$ given $\sum_i X_i = s$ is $(n-1)(s-x)^{n-2} I\{0 < x < s\}/s^{n-1}$.)

3. Consider the family of bivariate normal distributions with zero mean and unit variances, i.e.,

$$\begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right).$$

Let

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \ldots, \begin{pmatrix} X_n \\ Y_n \end{pmatrix}$$

be a random sample from this family.

(i) What is the minimal sufficient statistic?

(ii) Is the minimal sufficient statistic complete?

(iii) Prove that

$$r = \frac{1}{n} \sum_{i=1}^{n} X_i Y_i$$

is an unbiased estimator of ρ.

(iv) Is r a UMVU estimator of ρ?

(v) What is the variance of r?

4. Let N be a stopping random variable. Prove that the following two statements are equivalent:

(a) $P(N > n) \leq C\rho^n$ for some $C > 0$ and $0 < \rho < 1$

(b) $E(e^{tN}) < \infty$ for all $t \leq t_0$ for some $t_0 > 0$.

5. Define the following terms clearly:

(a) Type I and type II errors

(b) Level of Significance

(c) Size of a test

(d) Power function of a test

(e) Maximum Likelihood Estimator

(f) Minimax Estimator

(g) Bayes Estimator.

6. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $N(13, \sigma^2)$. To test the hypotheses $H_0 : \sigma^2 = 1$ versus $H_1 : \sigma^2 = 2$, find the most powerful test at level α.

(ii) For a sample of size $n = 15$ and $\alpha = 0.05$ specify all the constants of your test of part (i). (Show your work).

7. If you were asked to find an "optimal" estimator, write a brief discussion as to how you will decide which type of estimator you will choose among Bayes, Minimax, Maximum Likelihood, UMVU and Sequential estimators. Describe the merits and drawbacks of these estimators.