Instructions: Your solutions must be written clearly enough so that a grader will have no question about what you mean. Make sure that the essential points in each solution are clearly explained; obvious details may be omitted. You may use the standard results in the subject (except the result you are being asked to prove, or one that is trivially equivalent to it), but clearly identify any theorem you are using, verify that its hypotheses are satisfied and explain how it is being used.

You will be graded primarily on those problems which are essentially completely solved. Do not submit work that you do not want to be graded. Complete as many questions as you can in the time allotted to you.
1. (a). Let \((X,d)\) and \((Y,\rho)\) be two metric spaces and let \(f : X \to Y\) be a function. Give three possible definitions of the statement \(f\) is continuous on \(X\) and prove that your definitions are equivalent.

(b). Let \(f : \mathbb{R} \to \mathbb{R}\) be a continuous, one-to-one, onto function. Prove or disprove: \(f\) is a homeomorphism. (Here \(\mathbb{R}\) has the usual metric topology.)

2. Prove or disprove each of the following assertions.
   (a). \(\text{Int } (A \cap B) = \text{Int } A \cap \text{Int } B\).
   (b). \(\overline{A} = \overline{\text{Int } A}\). (Here and in part (a), \(A\) and \(B\) are arbitrary subsets of a topological space \((X, \tau)\).)
   (c). There is a subset \(F \subset \mathbb{R}\) such that \(F\) is closed, uncountable, and disjoint from \(\mathbb{Q}\).

3. (a). For each \(n \in \mathbb{N}\), let \(X_n\) be a separable topological space. Prove that the product

\[ \Pi_{n \in \mathbb{N}} X_n \]

is also separable.

(b). Let \(I\) be an index set. For each \(i \in I\), let \(X_i = \{0,1\}\) with the discrete topology. Prove that if the product

\[ \Pi_{i \in I} X_i \]

is separable, then card \(I \leq c\) (\(= 2^\mathbb{R}\)). (Hint: Fix \(i_0\). Let \(U_{i_0} = \{x = (x_i) : x_{i_0} = 0\}\) and let \(V_{i_0} = \{x = (x_i) : x_{i_0} = 1\}\). If \(D \subset X\) is a countable dense set, show that there is a point \(d_{i_0} \in D \cap U_{i_0} \cap V_j\), where \(j \in I, j \neq i_0\). Let \(D_{i_0}\) be all such points \(d_{i_0}\), and show that the mapping \(i_0 \sim D_{i_0} \in \mathcal{P}(\mathbb{N})\) is one-to-one.)

4. A topological space \(X\) is said to be pseudocompact provided every continuous function \(f : X \to \mathbb{R}\) is bounded.

   (a). Prove that if a metric space \(X\) is pseudocompact, then \(X\) is compact.

   (b). Prove or disprove: If an arbitrary topological space \(X\) is pseudocompact, then \(X\) is compact.
5. Let $C \subset [0,1]$ be the Cantor set. Prove or disprove each of the following assertions:

(a). There is a continuous onto function $f : C \to [0,1]$.
(b). There is a continuous onto function $f : [0,1] \to C$.
(c). There is a continuous one-to-one function $f : C \to [0,1]$.
(d). There is a continuous one-to-one function $f : [0,1] \to C$.

6. (a). Prove that every completely regular Hausdorff space can be embedded into a product of real lines, with the product topology.
(b). Let $X$ be a closed subset of $\mathbb{R}$, a product of real lines. It is known that if $\varphi : C(\mathbb{R}) \to \mathbb{R}$ is a homomorphism, then there is $c \in \mathbb{R}$ such that $\varphi(f) = f(c)$ for every $f \in C(\mathbb{R})$. Let $\psi : C(X) \to \mathbb{R}$ be a homomorphism. Prove that there is $c \in X$ such that $\psi(f) = f(c)$ for every $f \in C(X)$. (Here $C(Y)$ denotes the continuous real-valued functions on the topological space $Y$, and $\theta : C(Y) \to \mathbb{R}$ is a homomorphism means that $\theta$ is a linear mapping with the additional property that $\theta(f \cdot g) = \theta(f) \cdot \theta(g)$.)

7. Consider the following subspace $X \subset \mathbb{R}^2$.

$$X = \bigcup_{n=1}^{\infty} (0,1] \times \left( \frac{1}{n} \right) \cup \{(0,0), (1,0)\}.$$  

Let $V \subset X$ be a closed and open subset (for the relative topology from $\mathbb{R}^2$). Suppose that $(0,0) \in V$. Prove that $(1,0) \in V$.

8. Recall that the cofinite topology $\tau$ on a set $X$ is defined as follows: $V \in \tau$ if and only if $V = \emptyset$ or the complement of $V$ is finite.

(a). Prove that $\tau$ is indeed a topology.
(b). Characterize all $X$ such that $(X, \tau)$ is Hausdorff.
(c). Prove or disprove: $(\mathbb{N}, \tau)$ is connected.
(d). Characterize the sequences $(x_j) \subset (\mathbb{N}, \tau)$ that converge to the number 1.

9. Let $(X,d)$ be a metric space. In each case, either prove the assertion or explain why the assertion is false.

(a). $B_r(a) = \{ x \in X : d(x,a) \leq r \}$. (Here, $B_r(a)$ is the open ball, center $a$ and radius $r$.)
(b). $B_r(a) = B_r(c)$ if and only if $a = c$.
(c). The function $f : X \to \mathbb{R}$, $f(x) \equiv d(x,a)$, is uniformly continuous (where $a \in X$ is fixed).
10. (a). Show that the set of all sequences of the form $x = .x_1x_2\ldots x_k0101010101\ldots$
(where $x_i \in \{0, 1, \ldots, 9\}$ and $k \in \mathbb{N}$ are arbitrary) is countable.
(b). Prove or disprove: There is a one-to-one correspondence between the power set (set of all subsets) of \(\mathbb{N}\), \(\mathcal{P}(\mathbb{N})\), and the set of all functions \(f : \mathbb{N} \to \{0, 1, 2\}\).