Real Analysis

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Assignment X.

1. Problem 1. a) Let \( E := \bigcup_{k=1}^{\infty} E_k \subset \mathbb{R} \), \( E_k := [2^{-2k-1}, 2^{-2k}] \cup [-2^{-2k}, -2^{-2k-1}] \). Show that at 0, \( E \) has upper density 2/3 and lower density 1/3. In other words, show that
\[
\lim_{\delta \to 0} \frac{m(E \cap (-\delta, \delta))}{2\delta} = \frac{2}{3}, \quad \lim_{\delta \to 0} \frac{m(E \cap (-\delta, \delta))}{2\delta} = \frac{1}{3}.
\]
b) Construct a set \( E \subset \mathbb{R} \) which has density 1/2 at the origin.

c*) Suppose that \( E \) is a measurable set of real numbers with arbitrary small periods, i.e., there exists a sequence \( (p_i)_{i=1}^{\infty} \), \( p_i > 0 \), \( p_i \to 0 \) as \( i \to \infty \), so that \( E + p_i = E \) \( \forall i \). Prove that either \( m(E) = 0 \) or \( m(\mathbb{R} \setminus E) = 0 \).

Hint: Pick \( \alpha \in \mathbb{R} \), and put \( f(x) := m(E \cap [\alpha, x]) \) for \( x > \alpha \). Show that
\[
f(x + p_i) - f(x - p_i) = f(y + p_i) - f(y - p_i), \quad \alpha + p_i < x < y.
\]
What does this imply about \( f'(x) \) if \( m(E) > 0 \)?

d*) Suppose that \( f \) is Lebesgue measurable on \( \mathbb{R} \) with periods \( s, t \), \( s/t \in \mathbb{Q} \) (\( \forall x \in \mathbb{R}, f(x + t) = f(x), f(x + s) = f(x) \)). Prove that \( f(x) = \text{const} \) almost everywhere, but \( f \) need not to be a constant.

Hint: Apply c) to the set \( \{ x \in \mathbb{R} : f(x) > \lambda \} \), where \( \lambda > 0 \).

2. Problem 2. In this exercise \( f \) is a continuous function on \([a, b], F, E \subset [a, b] \) are measurable.

a) Let \( F \) be closed. Prove that \( f(F) \) is closed.

b) Let \( F \) be \( F_\sigma \), then \( f(F) \) is \( F_\sigma \).

Definition (Lusin). We say that \( f \) has N-property, if \( m(f(E)) = 0 \ \forall E : m(E) = 0 \).

c) Prove that absolutely continuous function has N-property.

d) Show that \( f(E) \) is measurable \( \forall E \) measurable iff \( f \) has N-property.

Hint. To prove part if, observe that \( f \) does not have N-property, implies the existence of \( E, \) such that \( m(E) = 0 \), and \( \mu^*(f(E)) > 0 \). Then there exists \( B \subset f(E) \) which is not measurable (why?). Consider \( A := f^{-1}(B) \subset E \). Show that \( A \) is measurable, and get a contradiction.

e) Prove that an absolutely continuous function maps measurable sets onto measurable sets.

f) Construct a continuous one-to-one mapping of \([0, 1]\) onto \([0, 2]\) which does not have N-property.

Hint: Consider \( f(x) = c(x) + x \), where \( c(x) \) is the Cantor ladder, defined in Problem 3, Assignment 7. What is the measure of the image of the Cantor set?
3. Problem 3. This exercise helps to understand “the change of variables” in the Lebesgue integral.

Let \((a, b) \subset \mathbb{R}\) be bounded.

a) Choose intervals \(W_n \subset (a, b)\) in such a way that \(\bigcup_n W_n\) is dense in \((a, b)\), and the set \(K := (a, b) \setminus \bigcup_n W_n\) has positive measure.

b) Choose continuous functions \(\phi_n\) on \(W_n\) so that \(\phi_n(x) = 0\) outside \(W_n\), and \(0 < \phi_n(x) < 2^{-n}\) in \(W_n\).

Put
\[
\phi(x) := \sum_{n=1}^{\infty} \phi_n(x), \quad T(x) := \int_a^x \phi(t)dt, \quad x \in (a, b).
\]

c) Prove that \(T\) is one-to-one, and differentiable at every point.

d) Prove that \(T'\) is continuous, \(T'(x) = 0, \forall x \in K, m(T(K)) = 0\).

e) Prove that if \(E\) is a nonmeasurable subset of \(K\) (why it exists?), then \(1_{T(E)}(x)\) is measurable, but \(1_{T(E)}(T_x)\) is not.

Observe that \(\phi_n\) can be chosen such that \(T\) is \textit{infinitely differentiable homeomorphism} of \((a, b)\) onto some segment in \(\mathbb{R}\).

4. Problem 4. This exercise is a part of the proof of “the change of variables” in the Lebesgue integral.

Let \(\phi\) be increasing and absolutely continuous function on \([p, q]\), \(x = \phi(t), a = \phi(p), b = \phi(q)\).

a) Let \(E_t\) me a measurable subset of \([p, q]\), \(E_x := \phi(E_t)\). Prove that
\[
\frac{m_x(E_x) - \varepsilon}{m(E_x)} \leq m_x(E_x) = \int_{E_t} \phi'(t)dt \leq \int_{E_t} \phi'(t)dt = m_x(E_x) \leq \frac{m(E_x)}{\varepsilon - \delta_x}. \quad \varepsilon \leq \varepsilon
\]

\textbf{Hint.} Prove this at first for \(E_t = [\alpha, \beta]\). Then for \((\alpha, \beta)\), open and closed \(E_t\). In the general case, find closed \(F_x\), and open \(G_x\) such that \(\forall \varepsilon > 0, \quad F_x \subset E_x \subset G_x \subset (a, b), \quad m(F_x) > m(E_x) - \varepsilon, \quad m(G_x) < m(E_x) + \varepsilon\).

Consider \(F_t := \phi^{-1}(F_x), G_t := \phi^{-1}(G_x)\), and observe that
\[
m(F_x) \leq \int_{E_t} \phi'(t)dt \leq m(G_x).
\]

b) Let all notation be as in a), and let \(e_x \subset [a, b]\), \(m(e_x) = 0\), \(e_t = \phi^{-1}(e_x)\). Prove that
\[
m(e_t^c) = 0, \quad e_t^c := \{t \in e_t: \phi'(t) = 0 \text{ is false}\}.
\]
Hint: Observe that
\[ e_x \subset G_x^0 \subset G_x^{m-1} \subset \ldots \subset G_x^2 \subset G_x^1 \subset (a, b), \quad m(G_x^m) \to 0. \]

Define
\[ H_x := \cap_{n=1}^{\infty} G_x^n, \quad H_t := \phi^{-1}(H_x), \quad G_t^m := \phi^{-1}(G_x^m), \]
and prove that \( H_t \) is measurable, and \( m(H_x) = 0 \). Show that \[ \int_{H_t} \phi'(t) dt = 0. \]