Real Analysis

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Assignment XI.

1. **Problem 1.**
   a) Let $\mu$ be a complex measure on a $\sigma$-algebra $\Sigma$, and let $E \in \Sigma$. Define $\lambda(E) := \sup \sum |\mu(E_i)|$, the supremum being taken over all finite partitions \{E_i\} of $E$. Does it follow that $\lambda = |\mu|$?

   b) Define a measure $\lambda$ on Lebesgue measurable subsets of $\mathbb{R}^2$ as follows
   \[
   \lambda(A) := \int_{A \cap \mathbb{R}} f(x, 0) \, dx, \quad A \subset \mathbb{R}^2, \tag{closed and bounded}
   \]
   Here $f(x, 0)$ is a continuous function having a compact support on $\mathbb{R} = \{(x, y) : y = 0\}$. Prove that $\lambda$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, but is singular with respect to the Lebesgue measure on $\mathbb{R}^2$.

2. **Problem 2.**
   **Definition.** We say that two measures $\lambda_1, \lambda_2$, defined on a $\sigma$-algebra $\Sigma$ are mutually singular, $(\lambda_1 \perp \lambda_2)$, if there exists a pair of disjoint sets $A, B$ such that $\lambda_1$ is concentrated on $A$, and $\lambda_2$ is concentrated on $B$.
   \[ B = \text{supp}(\lambda_2) \]
   Suppose that $\mu, \lambda, \lambda_1, \lambda_2$ are measures on a $\sigma$-algebra $\Sigma$, and $\mu$ is positive. Prove the following chain of statements.
   a) If $\lambda$ is concentrated on $A$, so is $|\lambda|$.
   b) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.
   c) If $\lambda_1 \perp \mu$, and $\lambda_2 \perp \mu$, then $(\lambda_1 + \lambda_2) \perp \mu$.
   d) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $(\lambda_1 + \lambda_2) \ll \mu$.
   e) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.
   f) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.
   g) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.

3. **Problem 3.** Suppose $\mu$ and $\lambda$ are measures on a $\sigma$-algebra $\Sigma$, $\mu$ is positive, $\lambda$ is complex. Prove that the following two conditions are equivalent:
   \( \nabla \) $\lambda \ll \mu$.

   a) For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\lambda(E)| < \epsilon$ for all $E \in \Sigma$ with $\mu(E) < \delta$.
   **Hint.** To show that $\nabla$ implies $\nabla$), assume that $\nabla$ is false. Then there exists $\epsilon > 0$ and $E_n \in \Sigma$ such that $\mu(E_n) < 2^{-n}$, but $|\lambda(E_n)| \geq \epsilon$. Put $A_n := \cup_{i=n}^{\infty} E_i$, $A := \cup_{n=1}^{\infty} A_n$. Prove that $\mu(A) = 0$, but $|\lambda|(A) > 0$. Use Problem 2, e), to get a contradiction.
4. **Problem 4.** Suppose \( \mu \) is a positive measure on \( \Sigma \), \( g \in L(X, \mu) \), and \( \lambda(E) = \int_E f d\mu \), \( E \in \Sigma \). Prove that \( |\lambda|(E) = \int_E |f|d\mu \).

**Hint.** Observe that \( |\lambda|(E) \leq \int_E |f|d\mu \). To show the opposite inequality, construct a sequence \((g_n(x))_{n=1}^{\infty}\) of measurable simple functions such that \( |g_n(x)| = 1 \), and \( \lim_{n \to \infty} g_n(x) f(x) = |f(x)| \). Check that

\[
|\int_A g_n f d\mu| = \left| \sum_j a_{n,j} \int_{A \cap A_{n,j}} f d\mu \right| = \left| \sum_j a_{n,j} \lambda(A \cap A_{n,j}) \right| \leq \sum_j |\lambda(A \cap A_{n,j})| \leq |\lambda(A)|,
\]

where \( a_{n,j} \) are the values of \( g_n \), attained on the sets \( A_{n,j} \).

5. **Problem 5.** Let \((r_n)_{n=1}^{\infty}\) be an enumeration of the rational numbers, and for each positive integer \( n \), let \( f_n : \mathbb{R} \to \mathbb{R} \) be defined as \( f_n(x) = 2^{n+1}, x \in [r_n - 2^{-n}, r_n + 2^{-n}] \), and zero otherwise. Define a measure \( \lambda \) on Borel subsets of \( \mathbb{R} \) by

\[
\lambda(A) := \int_A f(x) dx, \quad f(x) := \sum_{n=1}^{\infty} f_n(x).
\]

a) Show that \( f(x) \) is finite almost everywhere with respect to \( m \), the Lebesgue measure on \( \mathbb{R} \).

**Hint.** Define \( A_k := \{ x \in \mathbb{R} : f(x) \geq 2^k \} \), \( k \) is a nonnegative integer. To show that \( m(\cap_{k=1}^{\infty} A_k) = 0 \) observe that \( \sum_{k=1}^{\infty} m(A_k) < \infty \).

b) Show that \( \lambda \) is \( \sigma \)-finite. In other words find a partition of \( \mathbb{R} \) into disjoint sets \( B_k \), such that \( \mathbb{R} = \cup_{k=1}^{\infty} B_k \), and \( \lambda(B_k) < \infty \).

**Hint.** One can take

\[
B_1 := A_1^c, \quad B_k := A_k^c \setminus (\cup_{l=1}^{k-1} A_l^c).
\]

c) Show that \( \lambda \ll m \).

d) Show that each non-empty open subset of \( \mathbb{R} \) has infinite measure under \( \lambda \).

**Hint.** Observe that \( f(x) \geq \sum_{k=1}^{\infty} f_{n_k}(x) \), where a subsequence \( n_k \) is chosen such that segments \([r_{n_k} - 2^{-n_k}, r_{n_k} + 2^{-n_k}]\), \([r_{n_l} - 2^{-n_l}, r_{n_l} + 2^{-n_l}]\) are disjoint (for \( k \neq l \)) and belong to the given open interval. Use \( \int f_n(x) dx = 1 \).