1. **Problem 1.** a) Compute the Hardy-Littlewood maximal function $M1_{[0,1]}(x)$, of the characteristic function of segment $[0,1]$.

b) Let $f(x) = x^{-1}(\log x)^{-2}$ if $x \in (0,1/2)$, and zero on the rest of $\mathbb{R}$. Prove that $f$ is integrable. On the other hand, show that $Mf(x) \geq |2x \log(2x)|^{-1}$, if $x \in (0,1/4)$, so that $\int_0^1 Mf(x)dx = \infty$.

c) Assume that both $f$ and $Mf$ are integrable on $\mathbb{R}^n$. Prove that $f(x) = 0$ almost everywhere.

**Hint.** Show that there exists a constant $c = c(f)$ such that $Mf(x) \geq c(f)|x|^{-n}$ whenever $x$ is sufficiently large.

2. **Problem 2.** Let $f(x) = 1$ if $x \in [0,1]$, and zero otherwise. Define $h_c(x) := \sup_{n \in \mathbb{N}} n^{c}f(nx)$. Prove that

a) $h_c$ is integrable on $\mathbb{R}$ if $c \in (0,1)$.

b) $h_c$ is in weak-type $L^{\infty}$ but not integrable on $\mathbb{R}$.

c) $h_c$ is not in weak-type $L^{r}$ if $c > 1$.

**Hint.** For a.e. $x \in \mathbb{R}, h_c(x) = \sum_{n=1}^{\infty} 1_{(1/(n+1),1/n)}(x)n^c$.

3. **Problem 3.** For any set $E \subset \mathbb{R}^2$, the boundary $\partial E$ of $E$ is, by definition, the closure of $E$ minus the interior of $E$.

a) Show that $E$ is Lebesgue measurable whenever $m(\partial E) = 0$.

b) Suppose that $E$ is the union of a non-density uncountable collection of closed discs in $\mathbb{R}^2$, whose radii are at least 1 and at most 2. Use a) to show that $E$ is Lebesgue measurable.

c) Show that the conclusion of b) is true even when the radii are unrestricted.

d) Show that some unions of closed discs of radius 1 are not Borel sets.

**Hint.** What if all the discs touch the straight line?

4. **Problem 4.** Let $f$ be periodic with the period 1 and Lebesgue integrable with respect to the Lebesgue measure on $[0,1]$. Define the **Riemann sum**

$$f_n(x) := \frac{1}{n} \sum_{k=1}^{n} f(x + \frac{k}{n}), \quad n \in \mathbb{N}.$$
Note that a sequence \((f_n(x))_{n=1}^{\infty}\) converges almost everywhere to \(A = \int \limits_0^1 f(x)dx\) by completing the following steps.

a) Assume that \(f \geq 0\). Show that \(s_N f(x) := \sup_{0 \leq n \leq N} f_n(x)\) is in weak type. More precisely, prove that
\[
\forall \lambda > 0 \quad \lambda m(E_{\lambda}) \leq \int \limits_{E_{\lambda}} f(x)dx, \quad E_{\lambda} := \{x \in [0,1] : s_N f(x) > \lambda\}.
\]

**Hint.** Define \(E_n := \{x \in [0,1] : f_n(x) > \lambda\}, 0 \leq n \leq N, N \in \mathbb{N}\), and observe that
\[
E_{\lambda} = \bigcup_{n=1}^{N} E_n = \bigcup_{n=1}^{N} A_n, \quad A_N := E_N, \quad A_n := E_n \setminus (\bigcup_{i=n+1}^{N} E_i), \quad n = 0, \ldots, N - 1,
\]
where \(A_n\) are disjoint. Prove that all \(A_n\) are periodic with period \(2^{-n}\), and conclude that
\[
\int_{A_n} f(x)dx = \int_{A_n} f_n(x)dx \geq \lambda m(A_n).
\]
This implies \(\mathcal{N}\).

b) Letting \(N \to \infty\) prove this.

\[
\lambda m(\{x \in [0,1] : \sup_{n \in \{0,1,\ldots,N\}} s_n f(x) > \lambda\}) \leq \int \limits_0^1 f(x)dx.
\]

c) Define \(\Phi(x) := \lim_{n \to \infty} f_n(x)\). Prove that \(\Phi\) has arbitrary small periods, hence \(\Phi(x) = \text{const}\) almost everywhere.

**Hint.** Use Problem 1, d) of Assignment XI.

d) Prove that \(\Phi \leq A\) by taking any \(\lambda < \Phi\). Conclude that \(\Phi = A\).

**Hint.** Replace \(f(x)\) by \(-f(x)\).

Ξ) Show that in fact any sequence \((f_n(x))_{n=1}^{\infty}\) converges to \(A\) almost everywhere, provided \(m_{n-1}\) divides \(m_n\).

Ξ) Find an integrable periodic function \(f\) such that a sequence of Dedekind sums \((f_n(x))_{n=1}^{\infty}\) diverges almost everywhere.

**Hint.** Assume (do not prove) that for every irrational \(x\) there are infinitely many integers \(n\) such that \(|x - m/n| < 1/n^2\) for some integer \(m\).

Fix \(0 < \alpha < 1/2\). Take \(f(x) = |x|^{-1-\alpha}\) for \(|x| \leq 1/2\), and define \(f(x)\) for all other \(x\) by periodicity. Then \(f(x - m/n) > f(1/n^2) = n^{2-2\alpha}\), so \(f_n(x) > n^{1-2\alpha}\).