Real Analysis.

Instructor: Dmitry Ryabogin

Assignment III.

1. Problem 1.

**Definition 1.** Let $E := [0, 1] \times [0, 1]$, and let $A \subseteq E$. The inner measure $\mu_*(A)$ (of $A$) is a number defined as

$$\mu_*(A) := 1 - \mu^*(E \setminus A).$$

Prove that $\mu_*(A) \leq \mu^*(A)$.

**Hint.** $\mu^*(E \setminus A) + \mu^*(A) \geq \mu^*(E)$.

2. Problem 2. Prove that $A \subseteq E$ is Lebesgue measurable if and only if $\mu_*(A) = \mu^*(A)$.

**Hint.** Let $A \subseteq E$ be Lebesgue measurable. Then $\forall \epsilon > 0$ there exists an elementary set $B$ such that $\mu^*(A \Delta B) < \epsilon$. Prove that

$$\mu^*(B) - \epsilon \leq \mu_*(A) \leq \mu^*(A) \leq \mu^*(B) + \epsilon.$$

Conversely, assume that $\mu_*(A) = \mu^*(A)$. Then prove the following chain of statements leading to the result.

a) $\forall \epsilon > 0$ there exist elementary sets $P_n, Q_n, n = 1, 2, ..., $ such that

$$A \subseteq \bigcup_{n=1}^{\infty} P_n, \quad \mu^*(A) \geq \sum_{n=1}^{\infty} m'(P_n) - \epsilon,$$

and

$$(E \setminus A) \subseteq \bigcup_{n=1}^{\infty} Q_n, \quad \mu^*(E \setminus A) \geq \sum_{n=1}^{\infty} m'(Q_n) - \epsilon.$$

Conclude that

$$\sum_{n=1}^{\infty} (m'(P_n) + m'(Q_n)) \leq 1 + 2\epsilon,$$

and that there exists $N$ such that

$$\sum_{n=N+1}^{\infty} (m'(P_n) + m'(Q_n)) < \epsilon.$$

b) Denote

$$\bigcup_{n=1}^{\infty} P_n = P, \quad \bigcup_{n=1}^{\infty} Q_n = Q, \quad \bigcup_{n=1}^{N} P_n = P_N, \quad \bigcup_{n=1}^{N} Q_n = Q_N.$$

Observe that

$$\mu^*(P_N \Delta A) \leq \mu^*(P_N \setminus A) + \mu^*(A \setminus P_N).$$
c) Prove that
\[ \mu^*(A \setminus P_N) \leq \sum_{n=N+1}^{\infty} m'(P_n). \]

d) Observe that
\[ P_N \setminus A \subset (P_N \cap Q_N) \cup (P_N \cap (Q \setminus Q_n)) \subset (P_N \cap Q_N) \cup (Q \setminus Q_n), \]
and conclude
\[ \mu^*(P_N \setminus A) \leq \mu^*(P_N \cap Q_N) + \sum_{n=N+1}^{\infty} m'(Q_n). \]

e) Observe that \( E \subseteq (P \cup Q) \), and show that
\[ 1 = \mu^*(E) \leq \mu^*(P_N \cup Q_N) + \mu^*(P \setminus P_N) + \mu^*(Q \setminus Q_N). \]

Use the fact that for elementary sets \( C, D \),
\[ m'(C \cup D) = m'(C) + m'(D) - m'(C \cap D), \]
and a) to conclude that
\[ 1 \leq \sum_{n=1}^{\infty} (m'(P_n) + m'(Q_n)) - \mu^*(P_N \cap Q_N) \leq 1 + 3\epsilon - \mu^*(P_N \cap Q_N), \]
and \( \mu^*(P_N \cap Q_N) \leq 3\epsilon. \)

f) Use d) to show that
\[ \mu^*(P_N \setminus A) \leq 2\epsilon + \sum_{n=N+1}^{\infty} m'(Q_n). \]

g) "Glue" pieces b), c), f) to obtain the desired result.

3. **Problem 3.**

**Definition 1.** Let \( U \) be a collection of all open subsets of the real line. Then \( R(U) \) is called **Borel sets** (the minimal ring containing \( U \)).

Prove that any Lebesgue measurable set on the real line is a union of a Borel set and a set of measure zero.

**Hint.** Let \( A \subseteq \mathbb{R} \) be measurable. According to Problem 2, \( \forall \epsilon > 0 \), there exists a closed set \( B_\epsilon \subseteq A \) such that \( \mu^*(A \setminus B_\epsilon) < \epsilon \). The set you are looking for is \( \bigcup_{n=1}^{\infty} B_1/n \).

4. **Problem 4.**

**Definition 2.** We say that a measure \( \mu \) (defined on a corresponding subring \( S \)) is invariant under the transformation \( T : S \to S \) if
\[ \forall A \in S, \quad \mu(T^{-1}(A)) \equiv \mu(A). \]
a) It is known (take it as granted) that a real number \( x \in [0, 1] \) can be written as a \textbf{continuous fraction}

\[
x = \frac{1}{n_1 + \frac{1}{n_2 + \ldots}}, \quad n_k \in \mathbb{N},
\]

where a rational number can be written as a finite fraction, and an irrational number as an infinite one. Define the transformation \( T \) on \([0, 1]\) as \( T := \{1/x\} \), where \( \{\cdot\} \) stands for the fractional part of a number. Prove that (in terms of sequences \( (n_k)_{k=1}^{\infty} \)), \( T \) has the form \( T((n_k)_{k=1}^{\infty}) = (n_{k+1})_{k=1}^{\infty} \).

b) Let \( \mu \) be a measure on \([0, 1]\), defined as

\[
\mu([\alpha, \beta]) := \log_2 \frac{1 + \beta}{1 + \alpha}.
\]

Prove that \( \mu \) is invariant under \( T \) defined in a).

5. **Problem 5.** Let \( E = [0, 1] \times [0, 1] \) be a unit square in the plane, and let

\[
A := \{(x, y) \in E : |\sin x| < \frac{1}{2}, \cos(x + y) \in \mathbb{R} \setminus \mathbb{Q}\}.
\]

Find the Lebesgue measure of \( A \).

**Hint.** What is the complement of \( A \)?