Real Analysis.
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Assignment VI.

1. Problem 1.
   a) Let \( \mu \) be a measure on \( X \), \( f : X \to [0, \infty] \) be measurable, \( \int_X f(x) d\mu = c \), where \( 0 < c < \infty \), and \( \alpha > 0 \) be a constant. Find
   \[
   \lim_{n \to \infty} \int_X \log \left( 1 + \left( \frac{f(x)}{n} \right)^\alpha \right)^n d\mu.
   \]

   **Hint:** If \( \alpha \geq 1 \), the integrands are dominated by \( \alpha f(x) \). If \( 0 < \alpha < 1 \), Fatou’s lemma can be applied.

   b) Put \( f_n(x) := 1_A(x) \) if \( n \) is odd, and \( f_n(x) := 1 - 1_A(x) \) if \( n \) is even. Show that strict inequality can occur in Fatou’s lemma.

   c) It is easy to guess the limits of
   \[
   \int_0^n \left( 1 - \frac{x}{n} \right)^n e^{x/2} dx, \quad \int_0^n \left( 1 + \frac{x}{n} \right)^n e^{-2x} dx
   \]
as \( n \to \infty \). Prove that your guesses are correct.

   d) Does
   \[
   \lim_{n \to \infty} \frac{1}{n} \int_0^1 \log \left( 1 + e^{n f(x)} \right) dx
   \]
eexist for every real \( f \in L([0, 1], dx) \)? If it exists, what is it?

2. Problem 2.
   a) Let \( \mu(X) < \infty \), \( (f_n)_{n=1}^\infty \) be a sequence of bounded measurable functions on \( X \). Assume also that \( f_n \to f \) uniformly as \( n \to \infty \). Prove that
   \[
   \lim_{n \to \infty} \int_X f_n(x) d\mu = \int_X f(x) d\mu,
   \]
and show that the hypothesis “\( \mu(X) < \infty \)” can not be omitted.

   **Hint:** Take \( X = \mathbb{R} \), \( \mu \) - Lebesgue measure, \( f_n(x) = 1_{[-n^2,n^2]}(x)/n \).
Definition. Let $\mu$ be a measure on $X$. We say that a sequence of integrable functions $(f_n)_{n=1}^{\infty}$ converges to integrable $f$ in $L^1(X, \mu)$-sense, if for any $\epsilon > 0$ there exist a natural $N = N(\epsilon)$, such that $\forall n > N$ we have

$$\int_X |f_n(x) - f(x)| d\mu < \epsilon.$$ 

b) Construct a sequence of integrable on $[0, 1]$ functions $(f_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n \to \infty} f_n(x) = 0 \; \forall x \in [0, 1], \quad \int_{[0,1]} |f_n(x)| dx \leq C \; \forall n,$$

but such that the sequence does not converge in $L^1([0, 1], dx)$.

**Hint:** Consider $f_n(x) := n 1_{(0,1/n]}(x)$.

c) Construct a sequence $(f_k)_{k=1}^{\infty}$ of continuous on $[0, 1]$ functions, such that

$$0 \leq f_k(x) \leq 1, \quad \lim_{k \to \infty} \int_{[0,1]} f_k(x) dx = 0,$$

but such that the sequence converges for no $x \in [0, 1]$.

**Hint:** Consider $f_k(x)$ defined in Assignment IV, Problem 4, b).

3. **Problem 3.** Let $E_i$, $i = 1, \ldots, n$ be measurable subsets of $[0, 1]$ such that every point $x \in [0, 1]$ belongs to at least $q$ sets $E_i$, $q \leq n$. Prove that there exists $i$ such that $m(E_i) \geq q/n$.

4. **Problem 4.** Let $f(x)$ be bounded on $[0, 1]$. Prove that $\int_{0}^{c} f(x) dx = 0 \; \forall c \in [0, 1]$ implies $f(x) = 0$ almost everywhere.

**Hint:**

a) Observe that for any open interval $(a, b) \subset [0, 1]$ we have

$$\int_{0}^{b} f(x) dx - \int_{a}^{b} f(x) dx = 0,$$

and the same is true for any open set.

b) Prove that the same is true for any closed subset of $[0, 1]$.

c) Prove that the same is true for any subset $X := \cup_{i=1}^{\infty} F_i \subset [0, 1]$ of type $F_r$ (a set $X$ is called of type $F_r$ if it is a union of countably many closed sets). To do this, assume at first that $F_i \subset F_{i+1}$ (otherwise put $F_2 = F_1 \cup F_2$, $F_3 = F_1 \cup F_2 \cup F_3$, ..., $F_i = \cup_{k=1}^{i} F_k$). Then observe that

$$X = F_1 \cup (F_2 \setminus F_1) \cup (F_3 \setminus F_2) \cup \ldots \cup (F_{i+1} \setminus F_i) \cup \ldots$$