

Real Analysis.

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Assignment VI.

1. Problem 1.

a) Let μ be a measure on X , $f : X \rightarrow [0, \infty]$ be measurable, $\int_X f(x) d\mu = c$, where $0 < c < \infty$, and $\alpha > 0$ be a constant. Find

$$\lim_{n \rightarrow \infty} \int_X \log \left(1 + \left(\frac{f(x)}{n} \right)^\alpha \right)^n d\mu.$$

Hint: If $\alpha \geq 1$, the integrands are dominated by $\alpha f(x)$. If $0 < \alpha < 1$, Fatou's lemma can be applied.

b) Put $f_n(x) := 1_A(x)$ if n is odd, and $f_n(x) := 1 - 1_A(x)$ if n is even. Show that **strict** inequality can occur in Fatou's lemma.

c) It is easy to guess the limits of

$$\int_0^n \left(1 - \frac{x}{n} \right)^n e^{x/2} dx, \quad \int_0^n \left(1 + \frac{x}{n} \right)^n e^{-2x} dx$$

as $n \rightarrow \infty$. Prove that your guesses are correct.

d) Does

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \log \left(1 + e^{nf(x)} \right) dx$$

exist for every real $f \in L([0, 1], dx)$? If it exists, what is it?

2. Problem 2.

a) Let $\mu(X) < \infty$, $(f_n)_{n=1}^\infty$ be a sequence of bounded measurable functions on X . Assume also that $f_n \rightarrow f$ uniformly as $n \rightarrow \infty$. Prove that

$$\lim_{n \rightarrow \infty} \int_X f_n(x) d\mu = \int_X f(x) d\mu,$$

and show that the hypothesis " $\mu(X) < \infty$ " can not be omitted.

Hint: Take $X = \mathbf{R}$, μ - Lebesgue measure, $f_n(x) = 1_{[-n^2, n^2]}(x)/n$.

Definition. Let μ be a measure on X . We say that a sequence of integrable functions $(f_n)_{n=1}^\infty$ converges to integrable f in $L^1(X, \mu)$ -sense, if for any $\epsilon > 0$ there exist a natural $N = N(\epsilon)$, such that $\forall n > N$ we have

$$\int_X |f_n(x) - f(x)| d\mu < \epsilon.$$

what is this? (single integral?)

b) Construct a sequence of integrable on $[0, 1]$ functions $(f_n)_{n=1}^\infty$ satisfying

$$\lim_{n \rightarrow \infty} f_n(x) = 0 \quad \forall x \in [0, 1], \quad \int_{[0,1]} |f_n(x)| dx \leq C \quad \forall n,$$

but such that the sequence does not converge in $L^1([0, 1], dx)$.

Hint: Consider $f_n(x) := n1_{(0, 1/n]}(x)$.

c) Construct a sequence $(f_k)_{k=1}^\infty$ of continuous on $[0, 1]$ functions, such that

$$0 \leq f_k(x) \leq 1, \quad \lim_{k \rightarrow \infty} \int_{[0,1]} f_k(x) dx = 0,$$

but such that the sequence converges for no $x \in [0, 1]$.

Hint: Consider $f_k(x)$ defined in Assignment IV, Problem 4, b).

3. **Problem 3.** Let $E_i, i = 1, \dots, n$ be measurable subsets of $[0, 1]$ such that every point $x \in [0, 1]$ belongs to at least q sets $E_i, q \leq n$. Prove that there exists i such that $m(E_i) \geq q/n$.

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4. **Problem 4.** Let $f(x)$ be bounded on $[0, 1]$. Prove that $\int_0^c f(x) dx = 0 \quad \forall c \in [0, 1]$ implies $f(x) = 0$ almost everywhere.

must be measurable (assumed)

Thm. of Lebesgue
Just play with it.

Hint:

a) Observe that for any open interval $(a, b) \subset [0, 1]$ we have

$$\int_0^b f(x) dx - \int_0^a f(x) dx = 0,$$

and the same is true for any open set.

b) Prove that the same is true for any closed subset of $[0, 1]$.

c) Prove that the same is true for any subset $X := \cup_{i=1}^\infty F_i \subset [0, 1]$ of type F_σ (a set X is called of type F_σ if it is a union of countably many closed sets). To do this, assume at first that $F_i \subset F_{i+1}$ (otherwise put $F_2 = F_1 \cup F_2, F_3 = F_1 \cup F_2 \cup F_3, \dots, F_i = \cup_{k=1}^i F_k$). Then observe that

$$X = F_1 \cup (F_2 \setminus F_1) \cup (F_3 \setminus F_2) \cup \dots \cup (F_{i+1} \setminus F_i) \cup \dots$$