

# Real Analysis

Instructor: Dmitry Ryabogin

## Assignment VIII.

### 1. Problem 1.

**Definition 1.** A set  $M \subset \mathbf{R}$  is called closed if it coincides with its closure. In other words,  $M \subset \mathbf{R}$  is closed, if it contains all of its limit points.

**Definition 2.** A set  $A \subset \mathbf{R}$  is called open if there is a closed set  $M \subset \mathbf{R}$  such that  $A = \mathbf{R} \setminus M$ . In other words,  $A$  is open if all of its points are **inner** points, (a point  $x$  is called the inner point of a set  $A$ , if there exists  $\epsilon > 0$ , such that the  $\epsilon$ -neighbourhood  $U_\epsilon(x)$  of  $x$ , is contained in  $A$ ,  $U_\epsilon(x) \subset A$ ).

Prove that any open set on the real line is a finite or infinite union of disjoint open intervals. (The sets  $(-\infty, \infty)$ ,  $(\alpha, \infty)$ ,  $(-\infty, \beta)$  are intervals for us).

**Hint.** Let  $G$  be open on  $\mathbf{R}$ . Introduce in  $G$  the equivalence relationship, by saying that  $x \sim y$  if there exists an interval  $(\alpha, \beta)$ , such that  $x, y \in (\alpha, \beta) \subset G$ .

a) Prove at first that this is indeed the equivalence relationship, i. e., that  $x \sim x$ ,  $x \sim y$  implies  $y \sim x$ , and  $x \sim y, y \sim z$  imply  $x \sim z$ . Conclude that  $G$  can be written as a disjoint union  $G = \cup_{\tau \in \mathfrak{I}} I_\tau$  of "classes" of points  $I_\tau, I_\tau = \{x \in G : x \sim \tau\}$ ,  $\mathfrak{I}$  is the set of different indices, i.e.  $\tau \in \mathfrak{I}$  if  $I_\tau \cap I_x = \emptyset, \forall x \in G$ .

b) Prove that each  $I_\tau$  is an open interval.

c) Show that there are at most countably many  $I_\tau$  (the set  $\mathfrak{I}$  is countable).

### 2. Problem 2.

**Definition 3.** A set  $K \subset \mathbf{R}$  is called **nowhere dense** in  $\mathbf{R}$ , if for any open interval  $(\alpha, \beta)$  there exists an open interval  $(\gamma, \delta) \subset (\alpha, \beta)$ , such that  $K \cap (\gamma, \delta) = \emptyset$ .

a) Prove that a **closed** set  $K \subset \mathbf{R}$  is nowhere dense, if and only if for any interval  $(\alpha, \beta)$  there exists a point  $x \in (\alpha, \beta) \setminus K$ .

b) Prove that the Cantor set is nowhere dense in  $\mathbf{R}$ .

c) Let  $B$  be an **open** dense (the closure of  $B$  is  $\mathbf{R}$ ) subset of the real line. Is it true that  $\mathbf{R} \setminus B$  is nowhere dense?

d) Let  $D$  be a dense subset of the real line. Is it true that  $\mathbf{R} \setminus D$  is nowhere dense?

e) Prove that the closure of a nowhere dense set is nowhere dense.

3. **Problem 3.** a) Let  $(r_k)_{k=1}^\infty$  be a sequence of all rational numbers on the real line, and let  $v_k = (r_k - k^{-2}, r_k + k^{-2})$ . Prove that  $E := \mathbf{R} \setminus (\cup_{k=1}^\infty v_k)$  is closed and nowhere dense.

**Definition 3.** A set  $K \subset \mathbf{R}$  is called **of first category** if it is a countable union of nowhere dense sets.

b) Prove that no interval in  $\mathbf{R}$  is of the first category.

**Hint.** Our statement is an obvious consequence of the following

**Lemma.** The complement of any set  $A$  of the first category on the line is dense.

To prove the Lemma you could argue as follows. Let  $A = \cup_{i=1}^{\infty} A_i$ , where  $A_i$  are nowhere dense. Let  $I$  be any interval on the real line, and let  $I_1$  be a closed subinterval of  $I \setminus A_1$ ,  $I_2$  be a closed subinterval of  $I_1 \setminus A_2$ , and so on. Show that

$$\emptyset \neq \cap_{n=1}^{\infty} I_n \subset (I \setminus A).$$

c)\* Prove that  $\mathbf{R}$  can be written as a disjoint union of  $A \cup (\mathbf{R} \setminus A)$ , where  $|A| = 0$ , and  $\mathbf{R} \setminus A$  is of first category.

**Hint.** Let  $(r_k)_{k=1}^{\infty}$  be a sequence of all rational numbers on the real line, and let  $I_{kj} = (r_k - 2^{-k-j}, r_k + 2^{-k-j})$ . Define

$$G_j := \cup_{k=1}^{\infty} I_{kj}, \quad A := \cap_{j=1}^{\infty} G_j.$$

Show that  $|A| < \epsilon$ ,  $\forall \epsilon > 0$ . On the other hand,  $G_j$  is open (why?) and dense (why?), hence  $\mathbf{R} \setminus G_j$  is nowhere dense (Problem 2, c)).

#### 4. Problem 4.

**Definition 4.** A set  $M \subset \mathbf{R}$  is called **perfect**, if it is closed, bounded, and has no isolated points. Any closed interval is perfect. Cantor set is perfect (why?). A set containing two different points is not perfect.

a) Prove that  $M$  is perfect if and only if disjoint intervals from  $\mathbf{R} \setminus M$  do not have "common ends" (intervals  $(a, b)$ ,  $(b, c)$  have common ends, intervals  $(a, b)$ ,  $(c, d)$ ,  $b < c$ , do not have).

b) Construct a nonempty perfect nowhere dense subset of  $\mathbf{R} \setminus \mathbf{Q}$ .

**Hint.** Let  $(r_k)_{k=1}^{\infty}$  be a sequence of all rational numbers on the real line. Take  $r_1$ , and let  $\epsilon_1$  be any irrational number. Define  $I_1 := U_{\epsilon_1}(r_1)$ . Now take the first point from  $(r_k)_{k=2}^{\infty} \setminus I_1$ . Let it be  $r_{k_2}$ . Observe that  $r_{k_2} \neq r_1 \pm \epsilon_1$  (why?). Now take  $\epsilon_2 \in \mathbf{R} \setminus \mathbf{Q}$  such that  $U_{\epsilon_2}(r_{k_2}) \cap I_1 = \emptyset$ , and such that  $U_{\epsilon_2}(r_{k_2})$ ,  $I_1$  have no common ends. Define  $I_2 := U_{\epsilon_2}(r_{k_2})$ . On the next step find among  $r_{k_2+1}, r_{k_2+2}, \dots$ , the first point out of  $I_1 \cup I_2$ . Let it be  $r_{k_3}$ . Observe that  $r_{k_3} \neq r_1 \pm \epsilon_1$ ,  $r_{k_3} \neq r_{k_2} \pm \epsilon_2$ . Now find  $\epsilon_3 \in \mathbf{R} \setminus \mathbf{Q}$  such that  $U_{\epsilon_3}(r_{k_3}) \cap I_1, I_2 = \emptyset$ , and such that  $U_{\epsilon_3}(r_{k_3})$ ,  $I_1, I_2$  have no common ends. Define  $I_3 := U_{\epsilon_3}(r_{k_3})$ .

Continue this way to obtain a sequence  $(I_k)_{k=1}^{\infty}$ . Then  $E := \mathbf{R} \setminus (\cup_{k=1}^{\infty} I_k)$  is the set you are looking for. Prove that  $E$  is closed. Prove that  $E \neq \emptyset$ . Prove that  $E$  does not contain isolated points. Prove that  $E$  does not contain any rational point. Finally, prove that  $E$  is nowhere dense (use the closeness of  $E$  and Problem 2, a)).

5. **Problem 5.** a) Is it possible to construct a countable family  $(E_i)_{i=1}^{\infty}$  of perfect nowhere dense sets  $E_i \subset [a, b]$  such that  $|\cup_{i=1}^{\infty} E_i| = b - a$ ?

**Hint.** Take  $E_i \subset [a, b]$ , such that  $|E_i| = b - a - 1/i$ , and  $E_i$  is perfect and nowhere dense (how to construct these sets?). To show that  $\cup_{i=1}^{\infty} E_i$  satisfies the requirements of the problem define  $A_i := \cup_{k=1}^i E_k$ , and observe that  $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} A_i$ .

b)\* Is it possible to construct a countable family  $(E_i)_{i=1}^{\infty}$  of **pairwise disjoint** perfect nowhere dense sets  $E_i \subset [a, b]$  such that  $|\cup_{i=1}^{\infty} E_i| \equiv \sum_{i=1}^{\infty} |E_i| = b - a$ ?

**Hint.** Consider the case  $[a, b] = [0, 1]$ . Construct a perfect nowhere dense set  $E_1 \subset [0, 1]$ , such that  $|E_1| = 1/2$ . Take any  $(a_{1,i}, b_{1,i}) \subset ([0, 1] \setminus E_1)$  and put there a perfect nowhere dense set  $E_{1,i} \subset (a_{1,i}, b_{1,i})$ , such that  $|E_{1,i}| = (b_{1,i} - a_{1,i})/2$ . Then  $|\cup_{i=1}^{\infty} E_{1,i}| = 1/2^2$ . Observe that  $E_2 := E_1 \cup (\cup_{i=1}^{\infty} E_{1,i})$  is perfect and nowhere dense, moreover,  $|E_2| = 3/4$ .

Take any  $(a_{2,i}, b_{2,i}) \subset ([0, 1] \setminus E_2)$  and put there a perfect nowhere dense set  $E_{2,i} \subset (a_{2,i}, b_{2,i})$ , such that  $|E_{2,i}| = (b_{2,i} - a_{2,i})/2$ . Then  $|\cup_{i=1}^{\infty} E_{2,i}| = 1/2^3$ . Observe that  $E_3 := E_2 \cup (\cup_{i=1}^{\infty} E_{2,i})$  is perfect and nowhere dense, moreover,  $|E_3| = 1 - 1/2^3$ .

Continue this way, and get a sequence of perfect nowhere dense sets

$$E_1; \quad E_{1,1}, E_{1,2}, E_{1,3}, \dots, E_{1,i}, \dots; \quad E_{2,1}, E_{2,2}, E_{2,3}, \dots, E_{2,i}, \dots; \\ E_{k,1}, E_{k,2}, E_{k,3}, \dots, E_{k,i}, \dots; \dots$$

Observe that all of them are disjoint, and make the desired conclusion.