

Real Analysis

Instructor: Dmitry Ryabogin

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Assignment IX.

About fcn. of bounded variation.

1. Problem 1.

a) Is it possible to construct $f(x)$, $x \in [0, 1]$, such that $f'(x) = D(x)$? Here $D(x) = 1$ for $x \in [0, 1] \cap \mathbb{Q}$, and $D(x) = 0$ for $x \in [0, 1] \cap \mathbb{R} \setminus \mathbb{Q}$. *impossible*

$$f'(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

possible

Hint: Prove the **Darboux Theorem:** If $f(x)$ is differentiable on $[0, 1]$, then $\forall C \in [f'(0), f'(1)]$ there exists $x \in [0, 1]$ such that $f'(x) = C$.

Converse of
interm. value
thm. on
 $f'(x)$

b) Construct a function $f(x)$ on $[0, 1]$ such that $f'(x)$ exists at every $x \in [0, 1]$, (and bounded), but $f'(x)$ is not continuous for every $x \in F$, where $F \subset [0, 1]$ is closed, nowhere dense, $\inf F = 0$, $\sup F = 1$, and $m(F) > 0$. *Check properties for*

Hint: Define

$$f(x) = (x - a_n)^2(x - b_n)^2 \sin \frac{1}{(b_n - a_n)(x - a_n)(x - b_n)}, \quad x \in (a_n, b_n),$$

where $[0, 1] \setminus F = \cup_{i=1}^{\infty} (a_i, b_i)$, and $f(x) = 0$ otherwise.

c) Construct a **continuous** function f on \mathbb{R} which is not differentiable at any point. *(use the hint and go through)*

Hint: Put $\phi(x) = x$, $x \in [0, 1]$, and $\phi(x) = 2 - x$, $x \in [1, 2]$. Define $\phi_0(x) := \phi(x)$, $x \in [0, 2]$, and $\phi_0(x + 2) = \phi_0(x)$. Then, define $f(x) := \sum_{n=0}^{\infty} (3/4)^n \phi_0(4^n x)$.

a) Fix any $x \in \mathbb{R}$ and any $m \in \mathbb{N}$. Observe that there exists $k \in \mathbb{Z}$ such that $k \leq 4^m x \leq k + 1$, and put $\alpha_m := 4^{-m}k$, $\beta_m := 4^{-m}(k + 1)$. Prove that $|\phi_0(4^n \beta_m) - \phi_0(4^n \alpha_m)| = 0$, for $n > m$, and $|\phi_0(4^n \beta_m) - \phi_0(4^n \alpha_m)| = 4^{n-m}$ for $n \leq m$.

b) Conclude that $|f(\beta_m) - f(\alpha_m)| \geq 1/2(3/4)^m$, and show that f is not differentiable at x .

2. **Problem 2.** a) Let $f(0) = 0$, $f(1) = 5$, $f(x) = 1 - x$, for $x \in (0, 1)$. Use definition to find the total variation of $f(x)$ on $[0, 1]$.

b) Write out $f(x) = \cos^2 x$ on $[0, \pi]$ as a difference of two increasing functions. *fcn. of bd. variation*

c) Let $f(x) = x^2$, $x \in [0, 1]$, $f(x) = x + 3$, $x \in (1, 2]$, $f(1) = 5$. Check that $V_0^2(f) = V_0^1(f) + V_1^2(f)$. Write out $f(x)$ as a difference of two increasing functions.

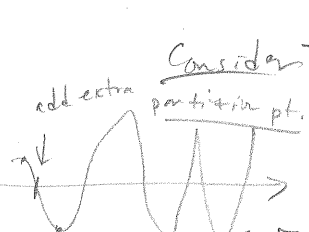
3. **Problem 3.** a) Let $f : V_0^1(|f|) < \infty$. Is it true that $V_0^1(f) < \infty$?

b) Let f be **continuous** on $[0, 1]$, and such that $V_0^1(|f|) < \infty$. Prove that $V_0^1(f) < \infty$.

Hint: Use the mean-value theorem.

1.5
 $\exists f'(x) \neq \frac{f(x+h)-f(x)}{h}$
 $\forall [\alpha_n, \beta_n] \ni x$

$\beta_n - \alpha_n \rightarrow 0$
 $\left| \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} \right| \rightarrow \infty$



add extra partition pt. Consider $V_a^b(f) = \sup_{\text{partitions}} \sum_{i=1}^n |f(t_i) - f(t_{i-1})| \leq C < \infty$

$a = t_0 < t_1 < \dots < t_n = b$

$V_0^1(|f|) < \infty \Rightarrow V_0^1(f) < \infty$

consider $D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$

$|D| = 1$

nice exercise needed to

f. bd deriv. ⇒ f bd variation on a segment

4. **Problem 4***. a) Construct a continuous $f(x)$ on $[a, b]$ such that $V_a^b(f) < \infty$, but $f(x)$ "is not Holder" for any $\alpha > 0$, (f is said to satisfy the Holder condition for some $\alpha > 0$ on $[a, b]$, if there exists a constant $K > 0$ such that

$$\forall x, y \in [a, b], \quad |f(x) - f(y)| \leq K|x - y|^\alpha.$$

Lipschitz^α Holder^α can we relax
 $\exists C > 0: \forall x, y \in [a, b] |f(x) - f(y)| \leq C|x - y|$
slope is always bd.

Hint: Take $[a, b] = [0, 1/2]$, and $f(x) = -1/\log x, x \in (0, 1/2], f(0) = 0$.

b) Construct an example of a continuous $f(x)$ on $[a, b]$, such that $V_a^b(f) = \infty$, but f "is Holder" of the given $0 < \alpha < 1$.

Hint: Let $(a_i)_{i=1}^\infty$ be such that $a_i > a_{i+1} > 0$ and $\sum_{i=1}^\infty a_i = A$. Put $f(x) = 0 \forall x = a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots$; $f(x) = 1/n$ at the point $a_1 + a_2 + \dots + a_{n-1} + a_n/2, n = 1, 2, 3, \dots$; $f(1) = 0$, and make f to be linear on any segment of the type $[\sum_{i=1}^{n-1} a_i, \sum_{i=1}^n a_i + a_n/2]$, and on the segments $[0, a_1/2], [a_1/2, a_1]$.

So trade for linear fun. Monotonic B.V. ↓ If of 2 monotonic ↓

To show that f "is Holder" of the given $0 < \alpha < 1$, take $a_n := n^{-1/\alpha}$, and consider two cases, 1) points $M_1(x_1, y_1), M_2(x_2, y_2)$ belong to "the same" part of the graph of $f(x)$, 2) points $M_1(x_1, y_1), M_2(x_2, y_2)$ do not belong to "the same" part of the graph of $f(x)$.

5. **Problem 5.** a) Let $(f_n(x))_{n=1}^\infty$ be a sequence of functions having bounded variation on $[a, b]$. Assume also that $\sum_{n=1}^\infty V_a^b(f_n) < \infty$, and $f_n(a) = 0, \forall n \in \mathbb{N}$. Prove that the series $\sum_{n=1}^\infty f_n(x)$ is convergent $\forall x \in [a, b]$, and $V_a^b(\sum_{n=1}^\infty f_n) \leq \sum_{n=1}^\infty V_a^b(f_n)$.

b) Let $(f_n(x))_{n=1}^\infty$ be a sequence of continuous functions having bounded variation on $[a, b]$. Assume also that the series $\sum_{n=1}^\infty f_n(x)$ converges uniformly on $[a, b]$. Is it true that $V_a^b(\sum_{n=1}^\infty f_n) < \infty$?

Hint: Consider $(f_n(x))_{n=1}^\infty$ on $[0, 1], f_n(x) := \sin(n\pi(x(n+1)-1))/n$ on $[1/(n+1), 1/n], f_n(x) := 0$ on $[0, 1] \setminus [1/(n+1), 1/n]$. You may also use an example from Problem 1, c).

c*) Construct a function f , which is of bounded variation on any finite segment (and hence is a difference of two monotonic functions), but, nevertheless, is not monotonic on any segment.

Hint: Let $\phi_0(x) = |x|$ for $x \in [-1/2, 1/2], \phi_0(x+1) = \phi_0(x)$, and let $\phi_n(x) := \min(\phi_0(x), 8^{-n})$. Consider $f(x) := \sum_{n=0}^\infty 2^{-n} \phi_n(8^n x)$.

$f \in B.V. \Rightarrow f = f_1 - f_2$
 $\uparrow \quad \uparrow$
 \geq monotonic

this is the precise construction to get