Real Analysis

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Assignment IX.

1. Problem 1.

a) Is it possible to construct $f(x)$, $x \in [0, 1]$, such that $f'(x) = D(x)$? Here $D(x) = 1$ for $x \in [0, 1] \cap \mathbb{Q}$, and $D(x) = 0$ for $x \in [0, 1] \cap \mathbb{R} \setminus \mathbb{Q}$. 

Hint: Prove the Darboux Theorem: If $f(x)$ is differentiable on $[0, 1]$, then $\forall C \in [f'(0), f'(1)]$ there exists $x \in [0, 1]$ such that $f'(x) = C$.

b) Construct a function $f(x)$ on $[0, 1]$ such that $f'(x)$ exists at every $x \in [0, 1]$, (and bounded), but $f'(x)$ is not continuous for every $x \in F$, where $F \subset [0, 1]$ is closed, nowhere dense, $\inf F = 0$, $\sup F = 1$, and $m(F) > 0$.

Hint: Define

$$f(x) = (x - a_n)^2(x - b_n)^2 \sin \frac{1}{(b_n - a_n)(x - a_n)(x - b_n)}, \quad x \in (a_n, b_n),$$

where $[0, 1] \setminus F = \bigcup_{i=1}^\infty (a_i, b_i)$, and $f(x) = 0$ otherwise.

c) Construct a continuous function $f$ on $\mathbb{R}$ which is not differentiable at any point.

Hint: Put $\phi(x) = x$, $x \in [0, 1]$, and $\phi(x) = 2 - x$, $x \in [1, 2]$. Define $\phi_0(x) := \phi(x)$, $x \in [0, 2]$, and $\phi_0(x + 2) = \phi_0(x)$. Then, define $f(x) := \sum_{n=0}^{\infty} (\frac{3}{4})^n \phi_0(4^n x)$.

2. Problem 2. a) Let $f(0) = 0$, $f(1) = 5$, $f(x) = 1 - x$, for $x \in (0, 1)$. Use definition to find the total variation of $f(x)$ on $[0, 1]$.

b) Write out $f(x) = \cos^2 x$ on $[0, \pi]$ as a difference of two increasing functions.

c) Let $f(x) = x^2$, $x \in [0, 1]$, $f(x) = x + 3$, $x \in (1, 2]$, $f(1) = 5$. Check that $|V_0^1(f)| = V_0^1(f) + V^1_2(f)$. Write out $f(x)$ as a difference of two increasing functions.

3. Problem 3. a) Let $f : V_0^1(|f|) < \infty$. Is it true that $V_0^1(f) < \infty$?

b) Let $f$ be continuous on $[0, 1]$, and such that $V_0^1(|f|) < \infty$. Prove that $V_0^1(f) < \infty$.

Hint: Use the mean-value theorem.
4. **Problem 4**. a) Construct a continuous \( f(x) \) on \([a, b]\) such that \( V^b_a(f) < \infty \), but \( f(x) \) “is not Holder” for any \( \alpha > 0 \), \( f(x) \) is said to satisfy the Holder condition for some \( \alpha > 0 \) on \([a, b]\), if there exists a constant \( K > 0 \) such that

\[
\forall x, y \in [a, b], \quad |f(x) - f(y)| \leq K|x-y|^\alpha.
\]

**Hint:** Take \([a, b] = [0, 1/2]\), and \( f(x) = -1/ \log x, \ x \in (0, 1/2], f(0) = 0 \).

b) Construct an example of a continuous \( f(x) \) on \([a, b]\), such that \( V^b_a(f) = \infty \), but \( f \) “is Holder” of the given \( 0 < \alpha < 1 \).

**Hint:** Let \((a_i)_{i=1}^\infty \) be such that \( a_i > a_{i+1} > 0 \) and \( \sum_{i=1}^\infty a_i = A \). Put \( f(x) = 0 \forall x = a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots; f(x) = 1/n \) at the point \( a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n/2, n = 1, 2, 3, \ldots \); \( f(1) = 0 \), and make \( f \) to be linear on any segment of the type \([\sum_{i=1}^{n-1} a_i, \sum_{i=1}^{n} a_i + a_n/2]\), \([\sum_{i=1}^{n-1} a_i + a_n/2, \sum_{i=1}^{n} a_i] \), and on the segments \([0, a_1/2], [a_1/2, a_1]\).

To show that \( f \) “is Holder” of the given \( 0 < \alpha < 1 \), take \( a_n := n^{-1/\alpha} \), and consider two cases, 1) points \( M_1(x_1, y_1), M_2(x_2, y_2) \) belong to “the same” part of the graph of \( f(x) \), 2) points \( M_1(x_1, y_1), M_2(x_2, y_2) \) do not belong to “the same” part of the graph of \( f(x) \).

5. **Problem 5**. a) Let \((f_n(x))_{n=1}^\infty \) be a sequence of functions having bounded variation on \([a, b]\). Assume also that \( \sum_{n=1}^\infty V^b_a(f_n) < \infty \), and \( f_n(a) = 0 \), \( \forall n \in \mathbb{N} \). Prove that the series \( \sum_{n=1}^\infty f_n(x) \) is convergent \( \forall x \in [a, b] \), and \( V^b_a(\sum_{n=1}^\infty f_n) \leq \sum_{n=1}^\infty V^b_a(f_n) \).

b) Let \((f_n(x))_{n=1}^\infty \) be a sequence of continuous functions having bounded variation on \([a, b]\). Assume also that the series \( \sum_{n=1}^\infty f_n(x) \) converges uniformly on \([a, b]\). Is it true that \( V^b_a(\sum_{n=1}^\infty f_n) < \infty \)?

**Hint:** Consider \((f_n(x))_{n=1}^\infty \) on \([0, 1]\), \( f_n(x) := \sin(n\pi(x(n+1)-1))/n \) on \([1/(n+1), 1/n]\), \( f_n(x) := 0 \) on \([0, 1] \setminus [1/(n+1), 1/n]\). You may also use an example from Problem 1, c).

c* ) Construct a function \( f \), which is of bounded variation on any finite segment (and hence is a difference of two monotonic functions), but, nevertheless, is not monotonic on any segment.

**Hint:** Let \( \phi_0(x) = |x| \) for \( x \in [-1/2, 1/2] \), \( \phi_0(x + 1) = \phi_0(x) \), and let \( \phi_n(x) := \min(\phi_0(x), 8^{-n}) \). Consider \( f(x) := \sum_{n=0}^\infty 2^{-n}\phi_n(8^n x) \).