**Lemma** Assume \( A \) is measurable on \( X \). Then \( \exists B \supseteq A: \mu(A) = \mu(B) \), where \( B = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} B_{n_k} \) and \( B_n \subseteq B_{n+1} \subseteq B_{n_k} \) for \( n \geq k \).

\[
B_{n_k} \subseteq R(x)
\]

**Proof:** Let \( \phi_A(x) = \mu_2(A_x) \) \( \forall x \in A \), then \( \phi_A(x) = \begin{cases} \mu_2(A_x), & x \in A, \\ 0, & \text{otherwise} \end{cases} \)

First, assume \( A \) is of the form \( B_{n_k} \). For \( A \) fixed and \( A_{y_0} \) constant for \( y_0 \),

\[
\mu_1 \otimes \mu_2 (B_{n_k}) = \int \phi_{B_{n_k}}(x) \, d\mu_1
\]

\[
\sum_{\text{finite } k} \mu_1 \otimes \mu_2 (U_k) = \sum_{\text{finite } k} \int \phi_{UB_{n_k}}(x) \, d\mu
\]

By continuity of the measure, \( \phi_B(x) = \lim_{n \to \infty} \phi_{B_n}(x) \)

\[
\phi_{B_1} \geq \phi_{B_2} \geq \phi_{B_3} \geq \ldots
\]

\[
\phi_{B_1} \leq \phi_{B_2} \leq \phi_{B_3} < \ldots \Rightarrow \phi_B(x) = \lim_{k \to \infty} \phi_{B_{n_k}}(x)
\]

So now: \( \mu_1 \otimes \mu_2 (B) = \int \phi_B(x) \, d\mu \)
Assume that $\mu_1 \otimes \mu_2 (C) = \int_X \varphi_c (x) \, d\mu_1$.

$$A = B \setminus C$$

$$\mu (C) = 0$$

$$\int_X \varphi_B (x) \, d\mu_1 - \int_X \varphi_C (x) \, d\mu_1 =$$

$$\int_X \left( \varphi_B (x) - \varphi_C (x) \right) \, d\mu_1 \leq \int_X \varphi_{B \setminus C} (x) \, d\mu_1$$

To prove $\mu (C) = 0$, change the "then" at the beginning to $\exists \, B \supseteq C : \mu (C) = \mu (B)$.

Then $\mu_1 \otimes \mu_2 (C) = 0 \Rightarrow \mu_1 \otimes \mu_2 (B) = 0 = \int_X \mu_2 (B_x) \, d\mu_1$.

Using Chebyshev

$$\Rightarrow \mu_2 (B_x) = 0 \Rightarrow \forall x \mu_2 (C_x) = 0$$

$$C \subseteq B \Rightarrow C_x \subseteq B_x \Rightarrow \int_X \mu_2 (C_x) \, d\mu_1 = 0$$

The rest of the proof of the Fubini Thm. will be twenty minutes next semester. We use this

$$\int_X f(x) \, d\mu = A_x$$

3 times with a clever trick.