Permutations and Determinants

The determinant of a square matrix "codes" much information about the matrix into a single number.
Permutations

A *permutation* of the set $S = \{1, 2, \ldots, n\}$ is a rearrangement of its elements.

A permutation $s$ of the set $S$ can be seen as a function $s : S \to S$, or as a sequence of numbers without repetitions:

$$s = 7463521$$

or

$$s(1) = 7, \ s(2) = 4, \ s(3) = 6, \ s(4) = 3, \ s(5) = 5, \ s(6) = 2, \ s(7) = 1.$$  

An *inversion* happens every time a larger number precedes a smaller one. The permutation $s$ above has 18 inversions.
A permutation is \textit{even} if its number of inversions is even, and \textit{odd} otherwise.

\textit{Example 1.} The permutation \(s\) from before is even.

Given a positive integer \(n\), the set \(S_n\) stands for the set of all permutations of \(\{1, 2, \ldots, n\}\). The total number of permutations in \(S_n\) is:

\[ n! = n(n - 1)(n - 2) \cdots 3 \cdot 2. \]

\textit{Example 2.} Find \(S_2\), \(S_3\), and \(S_4\).
Sneak Preview: Permutations

- Matrices can code permutations.
- Matrix product corresponds to composition of permutations.
- These permutations can also be represented by graphs.
- The matrices that represent both objects are related.
- Permutation matrices satisfy $A^{-1} = A^T$.
- How many permutation matrices of size $n \times n$ exist?
- Permutations can be odd or even: Count the number of inversions, or the number of transpositions.
The Determinant

Definition 1. Given an $n \times n$ matrix $A = [a_{ij}]$, we define the determinant of $A$ by:

$$\text{det}(A) = |A| = \sum_{s \in S_n} (\pm) a_{1s(1)} a_{2s(2)} \cdots a_{ns(n)},$$

where the sign is taken as $+$ or $-$ according to whether $s$ is even or odd, respectively.
Example 3.

\[
\det \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]
Example 4.

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
\]

\[
= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.
\]

There are mnemonic rules for $3 \times 3$ determinants.

Example 5. Do some.
Homework: 5.1