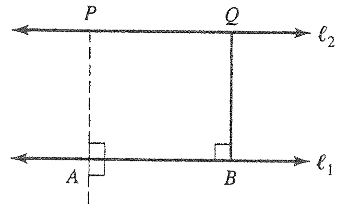


Figure 2.9: Parallel lines are everywhere equidistant

PROBLEMS

2.1. Let the lines ℓ_1 and ℓ_2 be cut by the transversal $\overleftrightarrow{AB}$ forming the eight angles a, b, c, d, e, f, g, h, as shown in Fig. 2.10. Prove that the following are equivalent:

(a) $\ell_1 \| \ell_2$

(b) $\angle a \cong \angle e$

(c) $\angle c \cong \angle g$

(d) $\angle b \cong 180° - \angle e$

(e) $\angle d \cong \angle h$

Note: The pairs a and e, b and f, c and g, d and h are called corresponding angles.

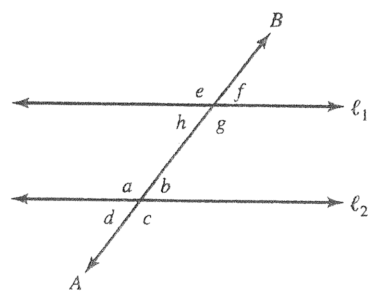


Figure 2.10: Exercise 1

2.2. Prove that the sum of the angles in a convex n-sided figure is $(n-2)180°$.

***2.3.** Given quadrilateral $ABCD$, prove that the following are equivalent:

(a) $ABCD$ is a parallelogram

(b) $\overline{AB} \| \overline{CD}$ and $\overline{AB} \cong \overline{CD}$

(c) $\angle A \cong \angle C$ and $\angle B \cong \angle D$

***2.4.** Prove that a parallelogram $ABCD$ is a rectangle if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ are congruent.

2.5. Prove that a parallelogram $ABCD$ is a rhombus if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular to each other.

2.6. Given triangles $\triangle ABC$ and $\triangle DEF$ such that $\overline{AB}$ and $\overline{DE}$ are parallel and congruent and $\overline{BC}$ and $\overline{EF}$ are parallel and congruent, prove that $\overline{AC}$ and $\overline{DF}$ are parallel and congruent.

2.7. Given a quadrilateral $ABCD$ such that $\overline{AB} \parallel \overline{CD}$

(a) Prove that $\angle C \cong \angle D$ if and only if $\overline{AD} \cong \overline{BC}$.

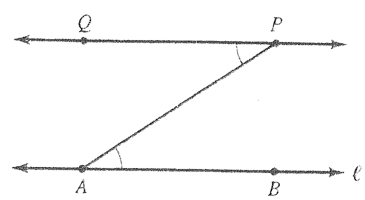
(b) Prove that $\angle C < \angle D$ if $AD > BC$.

2.8. Our proof that parallel lines are at a constant distance apart used the parallel postulate. (Can you see where?) Prove the following, without using the parallel postulate: Assume that lines ℓ_1 and ℓ_2 are at a constant distance apart. Then, if ℓ_1 and ℓ_2 are cut by a transversal, alternate interior angles must be congruent.

2.9. Euclid's postulate P.5, the famous "parallel postulate," states that if a straight line falling on twop straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles. Prove that Euclid's P.5 and Playfair's parallel postulate are equivalent (i. e. each implies the other).

CHAPTER SUMMARY

- The line through P such that $\angle APQ \cong \angle PAB$ will be parallel to ℓ. Moreover, if $\overleftrightarrow{QP}$ and $\overleftrightarrow{AB}$ are parallel and $\overleftrightarrow{PA}$ is a transversal, alternate interior angles, $\angle QPA$ and $\angle PAB$ are congruent.



- The shortest distance between a point P and a line $\overleftrightarrow{AB}$ is measured along the line through P perpendicular to $\overleftrightarrow{AB}$.

- The sum of the measures of the angles of a triangle is 180°.
  The sum of the measures of the angles of a quadrilateral is 360°.

- Given quadrilateral ABCD, the following are equivalent:

  (1) Opposite sides are parallel.

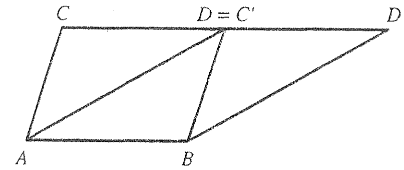


Figure 3.20: Shifting a parallelogram

.., T_n into a rectangle $R_1, \ldots, R_n$, such that each of these rectangles has base 1. Then these rectangles can be stacked to form a bigger rectangle R, also of base 1. So, any polygon can be decomposed to a rectangle of base 1. As we remarked earlier, the transitive property of equidecomposibility now implies that any two polygons with equal area are equidecomposible. □

PROBLEMS

3.1. Let $ABCD$ be a parallelogram. Define the base to be $\overline{AB}$ and the height to be the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Prove that ABCD has area = base × height.

3.2. Assume that in quadrilateral $ABCD$ that $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$. (Such a figure is called a *trapezoid.*) Let $AB = b_1$, $CD = b_2$ and let $h =$ the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Prove that $ABCD$ has area $\frac{1}{2}h(b_1 + b_2)$.

3.3. In Fig. 3.21, $ABCD$ is a rectangle and E lies on $\overline{AC}$. Prove that rectangle I and rectangle II have the same area. Use this fact to solve the construction problem: Given a rectangle R and a line segment $\overline{AB}$, construct a rectangle $ABCD$ with area equal to the area of R.

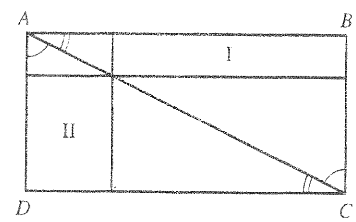


Figure 3.21: Exercise 3

3.4. Use exercise 3 to solve this construction problem: Given two rectangles R_1 and R_2 find a rectangle R such that area $R =$ area $R_1 +$ area R_2. Can you do it for more than two rectangles?

***3.5.** Prove that in a right triangle, if the hypotenuse is the base of length c, then the height is $h = \frac{ab}{c}$.

***3.6.** Given a convex quadrilateral $ABCD$ with $AC \perp BD$, prove that $AB^2 + CD^2 = BC^2 + AD^2$.

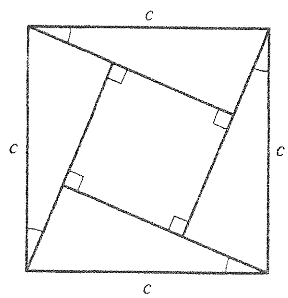


Figure 3.22: Exercise 7. Proof of Pythagorean theorem

3.7. Give another proof of the Pythagorean theorem based on Fig. 3.22.

3.8. Fig. 3.23 shows a sketch of Euclid's proof of the Pythagorean theorem. Fill in the details.

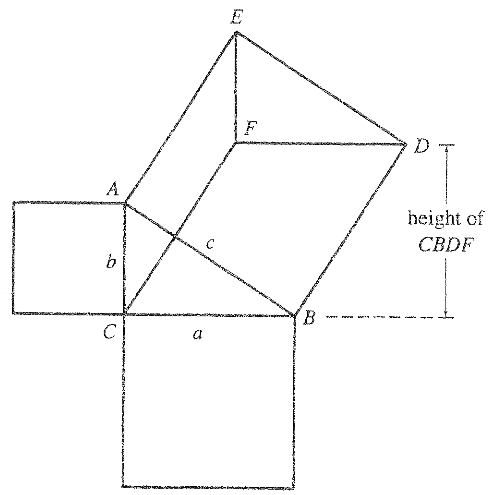


Figure 3.23: Exercise 8. Proof of Pythagorean Theorem

Construct F such that $\overleftrightarrow{EF} \parallel \overleftrightarrow{AC}$ and $\overleftrightarrow{DF} \parallel \overleftrightarrow{BC}$. Then $c^2 =$ area of $ACBDFE =$ area of $CBDF+$ area $AEFC$. Note that the area of $CBDF = CB^2$, since it is a parallelogram with base $\overline{BC}$ and height BC. Likewise, $AEFC$ has area AC^2.

3.9. Let $\triangle ABC$ be such that $\angle C = 90°$, $\angle A = 60°$, and $\angle B = 30°$. Prove that $AB = 2AC$ and $BC = \sqrt{3}AC$.

3.10. (a) Given $\overline{AB}$ and $\overline{CD}$ construct $\overline{EF}$ such that $EF^2 = AB^2 + CD^2$.

(b) Given $\overline{AB}$ and $\overline{CD}$ such that $AB \geq CD$, construct $\overline{EF}$ such that $EF^2 = AB^2 - CD^2$.

(c) Given $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ construct $\overline{GH}$ such that $GH^2 = AB^2 + CD^2 + EF^2$.

3.11. Prove the analogue of the Pythagorean theorem that uses equilateral triangles instead of squares.

***3.12.** Let $\triangle ABC$ and $\triangle DEF$ be such that $\angle A$ and $\angle D$ are supplementary (i.e., $\angle A + \angle D = 180°$). Prove that

$$\frac{\text{area }(ABC)}{\text{area }(DEF)} = \frac{AB \cdot AC}{DE \cdot DF}$$

3.13. (a) Assume that in $\triangle ABC$, $a = 4$, $b = 9$, and $c = 11$. Calculate the area.
(b) Calculate the length of each of the three altitudes in the triangle of part (a).
(c) If we let $a = 2$, $b = 3$, and $c = 7$ in Heron's formula, we get a problem. What is this problem and why does it happen?

3.14. In our proof of the first theorem in this chapter we omitted the last case of α any positive real number. Complete the proof using the following property of real numbers: Let $x < y$ be any two real numbers. Then there exists a rational number $\frac{n}{m}$ such that $x < \frac{n}{m} < y$.
[*Hint*: Use a proof by contradiction. First assume that area $(ABCD) < \alpha \cdot$ area $(EFGH)$. By the property of real numbers we can find $\frac{n}{m}$ such that

$$\alpha > \frac{n}{m} > \frac{\text{area } ABCD}{\text{area }(EFGH)}.$$

Construct a rectangle R with base $\overline{AB}$ and height $\frac{n}{m} \cdot FG$. Compare the area of R with that of $ABCD$ and $EFGH$. What is the contradiction? Next, do the case of area $(ABCD) > \alpha \cdot$ area $EFGH$.] *Remark*: This property of the real numbers can be expressed by saying that the rational numbers are dense in the reals. The application of this fact to geometry was known by the ancient Greeks. It is the basis of the method of exhaustion, and it plays a role in geometry similar to the role of limits in calculus.

3.15. Show how each of these pairs of polygons can be decomposed into congruent polygons:
(a) A 1×1 square and a $\frac{2}{3} \times \frac{3}{2}$ rectangle
(b) An isosceles right triangle and a square
(c) A 1×1 square and a rectangle with one side equal to $\sqrt{2}$
(d) An equilateral triangle with each side of length 1 and an isosceles triangle with base of length $\frac{1}{2}$