1. Let $\triangle ABC$ be a right triangle and $D$ be its centroid. If $AB = 3$ and $BC = 4$, find $BD$.

![Diagram of a right triangle with centroid D]

$$D_{\text{centroid}} = \frac{3}{A_1 A_c}$$

$$A_1 B_1 = B_1 C_1 = \frac{1}{3}$$

$$A D = 2 \cdot D A_1$$

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2. $\triangle AA_1 B$ (Pythagoras) \(\Rightarrow\) $A_1 A^2 = 9 + 16 = 25$

$$A_1 A = 5$$

If you forgot that $B_1 B = \frac{AC}{2}$ you can remember by constructing $AA' || BC, |AA'| = |BC|$

$$\Rightarrow \triangle AA'CB$$ is rectangle \(\Rightarrow |BA'| = |AC|, |BA'\cap AC| = \frac{1}{3}$$
2. In a right triangle with hypotenuse 6, the inscribed circle splits the hypotenuse in two segments, one being twice as long as the other. Find the perimeter of the triangle.

Because $MI \perp AB$
$\angle B = 90^\circ$
$\therefore MI \parallel AB$

But $|MI| = |IP| \Rightarrow MI$ is a square.

Perimeter $= x + x + 2x + 2x + r + r = 6x + 2r$

Also:

$\begin{align*}
(2x + x)^2 &= (x + r)^2 + (2x + r)^2 \\
9x^2 &= x^2 + 2xr + r^2 + 4x^2 + 4xr + 4r^2 \\
4x^2 - 4xr + 2r^2 &= 0 \\
2x^2 - 3xr + r^2 &= 0 \\
(2x - r)(x - r) &= 0 \\
x &= \frac{r}{2} \quad \text{or} \quad x = r
\end{align*}$
3. What is the orthocenter of $\triangle BCH$ where $H$ is the orthocenter of the $\triangle ABC$? Give complete explanations.

Let $AD, BE, CF$ heights in $\triangle ABC$.

His orthocenter.

Then in $\triangle BHC$:
- $HD \perp BC \Rightarrow HD$ height
- $CH \perp AB$ because $CH = CF$
- $AB \perp CH \Rightarrow BA$ height.
- $CA \perp BE \Rightarrow CA \perp BH \Rightarrow CA$ height.

Therefore, the orthocenter of $\triangle BHC$ is $CA \cap BA \cap HD = \{A\}$. 

\[ 3 \]
4. Let \( \triangle A''B''C'' \) be the anti complimentary triangle of \( \triangle ABC \). Identify the
a) centroid
b) circumcenter
d) center of the nine-point circle of \( \triangle A''B''C'' \).
Give complete explanations.

Because \( A''A, C''B, B''C \) are medians

In \( \triangle A''B''C'' \), the centroid of \( A''B''C'' \)
is \( A'' \cap B'' \cap C'' \cap B = G'' \).

But \( B''C'' \cap AC = B' \) with \( |AB| = |BC'| \) (by similar triangles, for example) \( \Rightarrow B'B \) is median in \( \triangle AAB \).

Similarly \( C'B' \cap AA'' \cap A'B' = G'' \).

b) let \( O'' \) be the circumcenter of \( \triangle A''B''C'' \). Then \( O''C \perp A''C'' \)

\( \Rightarrow O''C \perp AB \). Similarly \( O''B \perp AC \), etc.

\( O''C \) is height in \( \triangle A''B''C'' \)

So \( O''C, O''B, O''A \) are heights in \( \triangle A''B''C'' \Rightarrow (O'' = H) \).
5. Let $\triangle ABC$ have a right angle $C$. Prove that the median $CC'$ is the Euler line. Is there any other type of triangle in which $CC'$ will be the Euler line?

\[ \text{The Euler line is the line containing } O, H, E, C. \]

$CC'$ is median, therefore $\triangle G \sim \triangle GEC'$. Now, because $\triangle ABC$ is a right triangle, $AB$ is diameter in its circumcircle $\Rightarrow \quad C'$ is its center. So $C' = O, O \in CC' \Rightarrow CC'$ is Euler line.

In an isosceles $\triangle$, with $|AC| = BC$, this is also true because the median is also the height; therefore $H \in CC'$, $E \in CC'$ and, of course, equilateral $\triangle$. 