Quiz

Math 50051

REMARK: The problems denoted (**) are the trickiest.

1. (50 pts) The joint density function of X and Y is

$$f(x,y) = \begin{cases} xy & 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the density function of X;
b) Find the density function of Y;
c) Are X and Y independent?
d) Find $E(X)$.
e) Find $Var(Y)$.

a) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 xy\,dy = x\frac{y^2}{2}\Big|_0^2 = 2x$

b) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 xy\,dy = y\frac{x^2}{2}\Big|_0^1 = \frac{y}{2}$

c) X and Y are independent iff $f(x,y) = f_X(x) f_Y(y)$

Indeed since $\begin{cases} xy & 0<x<1,\ 0<y<2 \\ 0 & \text{otherwise} \end{cases} = \left(\begin{cases} 2x, & 0<x<1 \\ 0 & \text{otherwise} \end{cases}\right)\left(\begin{cases} \frac{y}{2}, & 0<y<2 \\ 0 & \text{otherwise} \end{cases}\right)$

we conclude X & Y are independent

d) $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 2x^2\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$

e) $Var(Y) = E(Y^2) - (E(Y))^2$

$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^2 y \cdot \frac{y}{2}\,dy = \int_0^2 \frac{y^2}{2}\,dy = \frac{y^3}{6}\Big|_0^2 = \frac{8}{6} = \frac{4}{3}$

$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_0^2 y^2 \cdot \frac{y}{2}\,dy = \int_0^2 \frac{y^3}{2}\,dy = \frac{y^4}{8}\Big|_0^2 = \frac{16}{8} = 2$

$Var(Y) = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$

2. (25 pts) The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-\frac{x}{y}} e^{-y}}{y}, \quad 0 < x < \infty, \;\; 0 < y < \infty.$$

Compute $E[X^2|Y = y]$.

b) $$E(X^2|Y=y) = \int_{-\infty}^{\infty} x^2 \, f_{X|Y=y}(x,y)\, dx$$

$$= \int_0^\infty x^2 \, f_{X|y=y}(x,y)\, dx$$

$$= \int_0^\infty x^2 \, \frac{f(x,y)}{f_y(y)}\, dx$$

$$\therefore f_y(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty \frac{e^{-\frac{x}{y}} e^{-y}}{y}\, dx = \frac{e^{-y}}{y} \int_0^\infty e^{-\frac{x}{y}}\, dx$$

$$= \frac{e^{-y}}{y} \cdot -e^{-\frac{x}{y}} \cdot y \Big|_0^\infty$$

$$= e^{-y}$$

$$E(X|Y=y) = \int_0^\infty x^2 \cdot \frac{e^{-\frac{x}{y}} \cancel{e^{-y}}}{y\, \cancel{e^{-y}}}\, dx = \frac{1}{y} \int_0^\infty \underbrace{x^2}_{u} \underbrace{e^{-\frac{x}{y}}}_{v'}\, dx$$

$$= \frac{1}{y} \Big[x^2 e^{-\frac{x}{y}} (-y) \Big|_0^\infty + \int_0^\infty 2x\, y\, e^{-\frac{x}{y}}\, dx$$

$$\cancel{= 2 \int_0^\infty \underbrace{x}_{u} \underbrace{e^{-\frac{x}{y}}}_{v} dx = 2\Big[x e^{-\frac{x}{y}}(-y)\Big|_0^\infty + \int_0^\infty y e^{-\frac{x}{y}} dx}$$

$$\cancel{= 2y \int_0^\infty e^{-\frac{x}{y}} dx = -2y^2 e^{-\frac{x}{y}} \Big|_0^\infty}$$

$$= 2y^2$$

$$-xe^{-\frac{x}{y}}\Big|_0^\infty + \int_0^\infty e^{-\frac{x}{y}}\, dx$$

$$= \frac{-x}{e^{x/y}}\Big|_0^\infty - y e^{-\frac{x}{y}}\Big|_0^\infty = 0 + y = y$$

3. (20pts) For each of the following, answer **TRUE** or **FALSE**. Give a brief justification for your answer.

(a) TRUE FALSE Given events $A \subset B$, we always have $P(B) \geq P(A)$.

$B = A \cup B \setminus A$

(b) TRUE FALSE $E(X/Y) = E(X)/E(Y)$, for any independent random variables X and Y.

In general we don't know if $E(1/Y) = \frac{1}{E(Y)}$

so $E(X/Y) = E(X)E(1/Y) \neq E(X)/E(Y)$

(c) TRUE FALSE If A and B are disjoint events then $P(A \cap B) = P(A)P(B)$.

$P(A \cap B) = \emptyset$

(d) TRUE FALSE Tossing a dice is a binomial experiment.

We need 2 outcomes and counting Successes.