Lecture 18

Math 50051, Topics in Probability Theory and Stochastic Processes

Property:
Doob's maximal L^2 inequality

$$E(max_{s\leq t}W^2(s)) \leq 4E(W(t)^2) = 4t$$

Definition
Brownian motion in several dimensions.
Let $X_t^1, X_t^2, ..., X_t^d$ be independent B.m.. The vector-valued stochastic process $X_t = (X_t^1, ..., X_t^d)$ is called d-dimensional B.m.

Properties:
The d-dimensional B.m. has the same properties from the definition of B.m. except that now the increment $X_t - X_s$ (a d-dimensional r.v.) has a joint normal distribution with mean 0 and covariance $(t-s)I$, i.e. has density

$$f(x_1, ..., x_d) = \frac{1}{\sqrt{2\pi r}}e^{-\frac{(x_1)^2}{2r}} \cdot ... \cdot \frac{1}{\sqrt{2\pi r}}e^{-\frac{(x_d)^2}{2r}} = \frac{1}{(2\pi r)^{d/2}}e^{-\frac{|x|^2}{2r}} \quad with \;\; r = t-s.$$

Remark
As in 1-dimensional case, the transition probability density of X_t assuming $X_0 = x$, is given by

$$p_t(x,y) = \frac{1}{(2\pi t)^{d/2}}e^{-\frac{|y-x|^2}{2t}}$$

which satisfies the Chapman-Kolmogorov equation

$$p_{s+t}(x,y) = \int_{\mathbb{R}^d} p_s(x,z)p_t(z,y)dz_1, ..., dz_d$$

Diffussion property of B.m.

We know that symmetric random walk is the movement of a particle among the integers. At each time unit a fair coin is tossed and according to the result the particle is moving to the left or to the right, one unit of distance. Now let's accelerate the random walk. The displacements are made every Δt units of time and the distance travled by the particle is equal to ϵ unit of distance to the left or to the right, where Δt and ϵ are positive numbers that we can choose as small as we want.As the Bm is a continuous time and continuous state process, we will take the limit as Δt and ϵ decrease to 0 so that the particle is moving continuously but will travel an infinitesimal distance each displacement. This is in accordance with our modeling of normal changes in stock prices.

However, in order to obtain a meaningfull process we can not allow the changes Δt and ϵ to become independently small.

Indeed for such a random walk,

$$E(S_{n\Delta t}) = 0$$

and

$$Var(S_{n\Delta t}) = n\epsilon^2$$

because $Var(X_{\Delta t}) = 1/2\epsilon^2 + 1/2\epsilon^2 = \epsilon^2$. Since $T = n\Delta t$ we have

$$n\epsilon^2 = \frac{T}{\Delta t}\epsilon^2$$

and if we want our process to have finite, non-zero variance, we need ϵ^2 and Δt to go to 0 at the same rate, ie, we need

$$\epsilon^2 = C\Delta t$$

Remark: The notation for C is σ^2. When $\sigma^2 = 1$ we say that the Bm is standard, or that we deal with **standard Brownian motion**. This is the one we have defined last time. If we do not mention anything about σ we assume we are talking about standard Bm.

Remark: From the above we see that the short-term properties of Bm are important. We need to specify the properties of $\Delta X = X(t+\Delta t) - X(t)$, the increment in X, over the small time-interval $(t, t+\Delta t)$.

We observe that:

1)

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}P[|X(t+\Delta t) - X(t) > \epsilon|X(t) = x] = 0$$

for all $\epsilon > 0$ 2)

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}E[X(t+\Delta t) - X(t)|X(t) = x] = m(t,x)$$

and 3)

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}E[(X(t+\Delta t) - X(t))^2|X(t) = x] = v(t,x)$$

with $m(x,t)$ and $v(x,t)$ continuous functions of x and t. Why?

A continuous time, continuous state Markovian process with properties 1),2) 3) is called a **difussion** process. The functions $m(x,t)$ and $v(x,t)$ are called infinitesimal mean and variance respectively. The most important case for the applications is the one when the difussion process is time-homogeneous., so $m(x,t) = m(x)$ and $v(x,t) = v(x)$. Observe, this is the case with Bm.

Suppose a large number of particles are distributed in $\mathbb{R}^d$ according to a density $f(y)$. Let $f(t,y)$ be the density of the particles at time t (so that $f(0,y) = f(y)$). Assume that the particles perform standard B.m. independent density.

If a particle starts at x then the probability density for its position at time t is $\underline{p_t(x,y)}$. Hence

$$f(t,y) = \int_{\mathbb{R}^d} f(x)p_t(x,y)dx_1, ..., dx_d$$

<u>Remark 1:</u>

Because $p_t(x,y) = p_t(y,x)$, we see that $f(t,y) = E(f(X_t))$ assuming $X_0 = y$, or,

$$f(t,y) = E^y(f(X_t))$$
$$f(t,x) = E^x(f(X_t))$$

Remark 2:(Property)

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \quad in \;\; one - dimension$$
$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta \text{ in d dimensions}$$

where Δ denotes the Laplacian

$$\Delta f(t, x_1, ...x_d) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}$$

This equation is often called the heat equation.

Brownian motion with drift

Consider a d-dimensional B.m. X_t starting at $x \in \mathbb{R}^d$. Let $\mu \in \mathbb{R}^d$ and

$$Y_t = X_t + t\mu$$

Then Y_t is called d-dimensional B.m. with drift μ starting at x.

Remark: The motion of Y_t consists of a "straight line" motion in the direction μ with random fluctuations.

Properties:

1) If has all the properties from the definition of B.m. except that the density of the increment is normal with mean $\mu(t-s)$ and covariance matrix $(t-s)I$.

2) $E(Y_t) = t\mu$ (see property 1)

3) $p_t(x,y) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y-x-t\mu|^2}{2t}}$ and the Chapman-Kolmogorov equation is

$$p_{s+t}(x,y) = \int_{\mathbb{R}^d} p_s(x,z) p_t(z,y) dz_1, ..., dz_d$$

4) Support we start with a density on $\mathbb{R}^d$, $f(x)$. Consider the function $f(t,x) = E^x[f(Y_t)]$. Then

$$\frac{\partial f}{\partial t} = \mu \frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \quad in \;\; one - dimension$$
$$\frac{\partial f}{\partial t} = \sum_{i=1}^{d} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\Delta f \quad in \;\; d - dimensions$$