Lecture 3

Math 50051, Topics in Probability Theory and Stochastic Processes

In real life, we are often not so much interested in the actual outcome ω in the sample space Ω, as in the numerical results that depend on ω. For example, in any bet, one is usually more interested in the resulting gain or loss than in the outcome of the randomizer that decides the issue. Also, if we look at the previous example about the USDA report, "favorable report" might contain several judgmental statements, but people are interested in the numbers accompanying these judgments.

A random variable X on $(\Omega, \mathcal{F}, P)$ is a real valued function on Ω (the sample space):

$$X : \Omega \to V_X \subset R$$

where V_X is the set of all possible values for the random variable X, such that the set $\{X \in B\} \in \mathcal{F}$ for every set B in the smallest $\sigma-$field generated by the open intervals (this $\sigma-$field is called the Borel $\sigma-$field). For our purposes, think of B as containing unions and intersections of any countable collection of intervals (although this is not true!).

An element $a \in V_X$ is called a state of the random variable X.
Note: $\{X \in B\} = \{\omega \in \Omega : X(\omega) \in B\} = X^{-1}(B)$
Examples:

1. Roll a die and let
$$X = \begin{cases} 0, & \text{if we get an odd number} \\ 1, & \text{if we get an even number} \end{cases} \tag{1}$$
Then $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $V_X = \{0, 1\}$

2. Toss 3 coins and let X =number of heads obtained. Then $\Omega = \{(H, H, H), (H, H, T), \cdots, (T, T, T)\}$ and $V_X = \{0, 1, 2, 3\}$. Here $X((H, H, H)) = 3$, $X((H, T, T)) = 1$, etc.

3. Ask somebody to pick a number at random from $(1, 2)$ and let X be 10 times the picked number. Then $\Omega = (1, 2)$, $V_X = (10, 20)$.

In example 1) and 2) V_X is a discrete set, so we call this type of random variables (rv) **discrete random variables**. Very commonly V_X is a subset of integers.

In example 3) V_X is a continuum of points, so we call the associated rv a **continuous rv**. By definition, a continuous rv is a rv that is not discrete.

Remark 1: A discrete set is not necessarily finite. The set $\{1, 2, 3, \cdots\}$ is discrete, but not finite.
Remark 2: Random variable are denoted, ussually, by capital letters X, Y, etc, while their values by small letters x, y, etc.

Distribution function

A mathematical model for the probabilities associated with a random variable X is given by the distribution function $F_X(x)$, defined by

$$F_X(x) = P(X \leq x) = P(\{\omega : X(\omega) \leq x\}).$$

Note 1: Since $\{\omega : X(\omega) \leq x\}$ or, in other words, the event $\{X \leq x\}$ is nothing else than $X \in (x, \infty)^c$ and X is a random variable, we have $\{X \leq x\} \in \mathcal{F}$, and hence we can assign a probability to it.
Note 2: F_X is a function in x.

Properties: 1) $P(X > x) = 1 - F_X(x)$
2) $P(a < X \leq b) = F_X(b) - F_X(a)$.

Remark 3: In the discrete random variable case, because the rv takes only countable many values, we are dealing with a probability distribution function, or probability mass function $f_X(x)$, where

$$f_X(x) = P(X = x).$$

Remark 4: If there is a $f_X : R \to R$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du,$$

then X is a rv with an absolutely continuous distribution, and f_X is called the **density** of X. We like to work with the density function because it has a few nice properties:

1. $P(a < X \leq b) = \int_a^b f_X(u)du$
2. $f_X(x) = \frac{dF}{dx}$, except, perhaps, at those places where $\frac{dF}{dx}$ does not exist. At these points, in our course, we will set $f_X(x) = 0$.
3. We can write the previous properties in a more informal way $P(a < X \leq a + h) \approx f_X(x)h$, for small h.
4. $$P(X \in A) = \int_A f_X(u)du,$$

   for any A Borel set.

It can be shown that under some technical conditions there always exists a distribution function. However, this function might not be written as a convenient formula. We will review a few basic models that are frequently used in pricing derivative products.

Examples:

1. **Binomial distribution** If you perform n independent trials with probability p of success at each trial, and probability $1-p$ of failure, then the number X of successes has the distribution given by
   $$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \;\; 0 \leq x \leq n.$$
   This is the Binomial distribution and it is denoted by $B(n, p)$. What type of distribution is it?

2. **Poisson distribution** If we consider the case of the Binomial distribution with n large, and p small, such that $np = \lambda$ for a fix λ, we can take $n \to \infty$, and obtain the Poisson distribution:

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \;\; x = 0, 1, 2, \cdots$$

and it is denoted by Poisson(λ). What type of distribution is it?

3. **Normal distribution and density** For any constants μ and $\sigma^2 > 0$ we have

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(x-\mu)^2), \;\; -\infty < x < \infty$$

and it is denoted by N(μ, σ^2). If $\mu = 0$ and $\sigma^2 = 1$ then the density is called the standard Normal density. Since this density can not be integrated simply (except on $(-\infty, \infty)$, and even then one needs a nifty trick), we have special notations for it and its distribution:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)$$

and

$$\Phi(x) = P(X \leq x) = \int_{-\infty}^{x} \phi(u)du.$$

What type of rv is a normally distributed rv?

Joint distribution

The joint distribution of several random variables $X_1, X_2, ..., X_n$ is a probability measure $P_{X_1,X_2,...,X_n}$ such that

$$P_{X_1,X_2,...,X_n}(B) = P((X_1, X_2, ..., X_n) \in B)$$

for any Borel set B in $\mathbb{R}^n$. If I can write

$$P_{X_1,X_2,...,X_n}(B) = \int_B f_{X_1,X_2,...,X_n}(t_1, ..., t_n)dt_1...dt_n$$

for all B in $\mathcal{B}(\mathbb{R}^n)$, then $f_{X_1,X_2,...,X_n}$ is called the joint density of $X_1, X_2, ..., X_n$.

In practical terms: For two rvs X, Y we say they are jointly continuous with **joint density** $f_{X,Y}(x, y)$ if

$$F_{X,Y}(x, y) = P(\{X \leq x\} \cap \{Y \leq y\}) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)dvdu.$$

and as in the case of a single rv

$$f_{X,Y}(x, y) = \begin{cases} \frac{\delta^2 f}{\delta x \delta y}, & \text{where the derivative exists,} \\ 0, & \text{elsewhere.} \end{cases} \tag{2}$$

In the discrete case we are talking about a **joint mass function**, or simply the distribution of (X, Y). Here we have

$$f_{X,Y}(x, y) = P(X = x, Y = y)$$

In both cases we can recover the separate distributions of X and Y, by:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy, \;\; f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx,$$

in the continuous case, and

$$f_X(x) = \sum_y f_{X,Y}(x, y), \;\; f_Y(y) = \sum_x f_{X,Y}(x, y),$$

in the discrete case.

Independence of rv Two rv X and Y are independent if, for all x and y we have

$$F_{X,Y}(x, y) = F_X(x)F_Y(y).$$

In particular, if X and Y are jointly continuous, they are independent, if for all x and y we have

$$f_{X,Y}(x, y) = f_X(x)f_Y(y).$$

Please, read Example 17 on page 15 of the attached Chapter 1.

Expectation and moments

The expectation of a random variable X, denoted by $E(X)$ is defined by

$$E(X) = \int_\Omega X dP$$

A random variable is said to be integrable, if

$$\int_\Omega |X| dP < \infty.$$

Then the expectation exists.