Now, as shown earlier, we can write

$$S_{t_k} = -(1 - 2p)(k + 1) + Z_{t_k}, \tag{94}$$

where Z_{t_k} *is* a martingale. Hence, we *decomposed* a submartingale into two components. The first term on the right-hand side is an increasing deterministic variable. The second term is a martingale that has a value of $S_{t_0} + (1 - 2p)$ at time t_0. The expression in (94) is a simple case of Doob–Meyer decomposition.[17]

8.2.1 The General Case

The decomposition of an upward-trending submartingale into a deterministic trend and a martingale component was done for a process observed at a finite number of points during a continuous interval. Can a similar decomposition be accomplished when we work with *continuously* observed processes?

The Doob–Meyer theorem provides the answer to this question. We state the theorem without proof.

Let $\{I_t\}$ be the family of information sets discussed above.

THEOREM: If $X_t, 0 \leq t \leq \infty$ is a right-continuous *sub*martingale with respect to the family $\{I_t\}$, and if $E[X_t] < \infty$ for all t, then X_t admits the decomposition

$$X_t = M_t + A_t, \tag{95}$$

where M_t is a right-continuous martingale with respect to probability P, and A_t is an increasing process measurable with respect to I_t.

This theorem shows that even if continuously observed asset prices contain occasional jumps and trend upwards at the same time, then we can convert them into martingales by subtracting a process observed as of time t.

If the original continuous-time process does not display any jumps, but is continuous, then the resulting martingale will also be continuous.

8.2.2 The Use of Doob Decomposition

The fact that we can take a process that is not a martingale and convert it into one may be quite useful in pricing financial assets. In this section we consider a simple example.

We assume again that time $t \in [0, T]$ is continuous. The value of a call option C_t written on the underlying asset S_t will be given by the function

$$C_T = \max[S_T - K, 0] \tag{96}$$

at expiration date T.

[17]This term is often used for martingales in continuous time. Here we are working with a discrete partition of a continuous-time interval.

According to this, if the underlying asset price is above the strike price K, the option will be worth as much as this spread. If the underlying asset price is below K, the option has zero value.

At an earlier time t, $t < T$, the exact value of C_T is unknown. But we can calculate a forecast of it using the information I_t available at time t,

$$E^P[C_T|I_t] = E^P[\max[S_T - K, 0] \mid I_t], \tag{97}$$

where the expectation is taken with respect to the distribution function that governs the price movements.

Given this forecast, one may be tempted to ask if the fair market value C_t will equal a properly discounted value of $E^P[\max[S_T - K, 0]|I_t]$.

For example, suppose we use the (constant) risk-free interest rate r to discount $E^P[\max[S_T - K, 0]|I_t]$, to write

$$C_t = e^{-r(T-t)}E^P[\max[S_T - K, 0] \mid I_t]. \tag{98}$$

Would this equation give the fair market value C_t of the call option?

The answer depends on whether or not $e^{-rt}C_t$ is a martingale with respect to the pair I_t, P. If it is, we have

$$E^P[e^{-rT}C_T|C_t] = e^{-rt}C_t, \qquad t < T, \tag{99}$$

or, after multiplying both sides of the equation by e^{-rt},

$$E^P\left[e^{-r(T-t)}C_T|C_t\right] = C_t, \qquad t < T. \tag{100}$$

Then $e^{-rt}C_t$ will be a martingale.

But can we expect $e^{-rt}S_t$ to be a martingale under the true probability P?

As discussed in Chapter 2, under the assumption that investors are risk-averse, for a typical risky security we have

$$E^P\left[e^{-r(T-t)}S_T|S_t\right] > S_t. \tag{101}$$

That is,

$$e^{-rt}S_t \tag{102}$$

will be a submartingale.

But, according to Doob–Meyer decomposition, we can decompose the

$$e^{-rt}S_t \tag{103}$$

to obtain

$$e^{-rt}S_t = A_t + Z_t, \tag{104}$$

where A_t is an increasing I_t measurable random variable, and Z_t is a martingale with respect to the information I_t.

If the function A_t can be obtained explicitly, we can use the decomposition in (104) along with (101) to obtain the fair market value of a call option at time t.

However, this method of asset pricing is rarely pursued in practice. It is more convenient and significantly easier to convert asset prices into martingales, not by subtracting their drift, but instead by changing the underlying probability distribution P.

9 The First Stochastic Integral

We can use the results thus far to define a new martingale M_{t_i}.[18] Let Z_t be any martingale with respect to I_t and to some probability measure P. Then the process defined by

Let $H_{t_{i-1}}$ be any random variable adapted to $I_{t_{i-1}}$.

$$M_{t_k} = M_{t_0} + \sum_{i=1}^{k} H_{t_{i-1}}[Z_{t_i} - Z_{t_{i-1}}] \tag{105}$$

will also be a martingale with respect to I_t.

The idea behind this representation is not difficult to describe. Z_t is a martingale and has unpredictable increments. The fact that $H_{t_{i-1}}$ is $I_{t_{i-1}}$-adapted means $H_{t_{i-1}}$ are "constants" given $I_{t_{i-1}}$. Then, increments in Z_{t_i} will be uncorrelated with $H_{t_{i-1}}$ as well. Using these observations, we can calculate

$$E_{t_0}[M_{t_k}] = M_{t_0} + E_{t_0}\left[\sum_{i=1}^{k} E_{t_{i-1}}[H_{t_{i-1}}(Z_{t_i} - Z_{t_{i-1}})]\right]. \tag{106}$$

But increments in Z_{t_i} are unpredictable as of time t_{i-1}.[19] Also, $H_{t_{i-1}}$ is I_t-adapted. This means we can move the $E_{t_{i-1}}[\cdot]$ operator "inside" to get

$$H_{t_{i-1}} E_{t_{i-1}}\left[Z_{t_i} - Z_{t_{i-1}}\right] = 0.$$

This implies

$$E_{t_0}\left[M_{t_k}\right] = M_{t_0}. \tag{107}$$

M_t thus has the martingale property.

[18] We remind the reader that this means, given the information in $I_{t_{i-1}}$, that the value of $H_{t_{i-1}}$ will be known exactly.

[19] Remember that $E_{t_0}[E_{t_{i-1}}[\cdot]] = E_{t_0}[\cdot]$.