2.4(a)

(i) Follows from (ii).

(ii) Follows from 2.4.2(a).

(iii) First, $\mathrm{E}\,|(q/p)^{S_n}| = ((q/p)p + (p/q)q)^n = 1 < \infty$. Likewise $\mathrm{E}((q/p)^{S_{n+1}}|S_0, \ldots, S_n) = (q/p)^{S_n}$.

(iv) It is easy to check that

$$\mathrm{E}|S_n^2 - 2\mu n S_n + (n\mu)^2 - n\sigma^2| \le n^2 + 2\mu n^2 + n^2\mu^2 + n\sigma^2 < \infty.$$

Now

$$\begin{aligned}
&\mathrm{E}(S_{n+1}^2 - 2\mu(n+1)S_{n+1} - (n+1)\sigma^2 + (n+1)^2\mu^2 \mid S_0, \ldots, S_n) \\
&\quad = S_n^2 + 2\mu S_n + 1 - 2\mu(n+1)(S_n + \mu) - n\sigma^2 - \sigma^2 \\
&\quad + n^2\mu^2 + 2n\mu^2 + \mu^2 \\
&\quad = S_n^2 - 2\mu n S_n + (n\mu)^2 - n\sigma^2
\end{aligned}$$

as required.

2.4(b) By 2.1(a), $\mathrm{E}T < \infty$ and $\mathrm{var}T < \infty$. Assume $p \neq q$ (you can do the case $p = q$).

(i) $(q/p)^{S_{n \wedge T}}$ is uniformly bounded, so by optional stopping,

$$1 = \mathrm{E}\left(\frac{q}{p}\right)^{S_T} = \mathrm{P}(A)\left(\frac{q}{p}\right)^a + (1 - \mathrm{P}(A))\left(\frac{q}{p}\right)^b,$$

which gives $\mathrm{P}(A)$ and $\mathrm{P}(B)$.

(ii) Since $\mathrm{E}T < \infty$, we can use dominated convergence for the martingale $S_n - n(p-q)$, which gives

$$0 = \mathrm{E}S_T - (p-q)\mathrm{E}T = a\mathrm{P}(A) + b(1 - \mathrm{P}(A)) - (p-q)\mathrm{E}T.$$

Solve this to obtain $\mathrm{E}T$.

(iii) Likewise, using the final martingale in Exercise (a),

$$a^2\mathrm{P}(A) + b^2(1 - \mathrm{P}(A)) - 2\mu\mathrm{E}(TS_T) + (p-q)^2\mathrm{E}T^2 - \sigma^2\mathrm{E}T = 0.$$

As $a \to \infty$, $a^2\mathrm{P}(A) \to 0$; $\mathrm{P}(B) \to 1$; $\mathrm{E}T \to -b/(q-p)$; $\mathrm{E}(TS_T) \to -b^2/(q-p)$. After some work, $\mathrm{var}T = -4pqb\,(q-p)^{-3}$.