MATH 60093 Monte Carlo Modeling

Vaiance Reduction Technique - Antithetic Variates

The method of *antithetic variates* attempts to reduce variance by introducing negative correlation between pairs of observations.

Example 1 *Let U be uniformly distributed over $[0,1]$ then $1-U$ is also uniformly distributed over $[0,1]$. Hence, if we generate a path using as inputs $U_1, U_2, ..., U_n$ and we generate a second path using $1-U_1, 1-U_2, ..., 1-U_n$ then the U_i and $1-U_i$ form an antithetic pairs in the sense that a large value of one is accompanied by a small value of the other. This suggests that an unusually large or small output computed from the first path is balanced off by the value computed from the antithetic path, resulting in a reduction in variance.*

We may apply inverse transformation method to create antithetic pairs for other distribution F. Note that $F^{-1}(U)$ and $F^{-1}(1-U)$ both have distribution F but are antithetic to each other because F^{-1} is a monotone function. For a distribution that is symmetric about the origin, $F^{-1}(U)$ and $F^{-1}(1-U)$ have the same magnitude but opposite signs.

Example 2 *In a simulation driven by independent standard normal random variables, antithetic variates can be implemented by pairing a sequence $Z_1, Z_2, ..., Z_n$ of i.i.d. $N(0,1)$ variables with the sequence $-Z_1, -Z_2, ..., -Z_n$ i.i.d. $N(0,1)$ variables, whether or not they are sampled through the inverse transform method.*

To understand the method of antithetic variates more closely, suppose our objective is to estimate an expectation $E[Y]$ and that using some implementation of antithetic sampling produces a sequence of antithetic pairs $\left(Y_1, \widetilde{Y_1}\right), \left(Y_2, \widetilde{Y_2}\right), ..., \left(Y_n, \widetilde{Y_n}\right)$ where the pair Y_i and $\widetilde{Y}_i$ for each i have the same distribution, and the pairs $\left(Y_1, \widetilde{Y_1}\right), \left(Y_2, \widetilde{Y_2}\right), ..., \left(Y_n, \widetilde{Y_n}\right)$ are i.i.d. The antithetic variates estimator of $E[Y]$ is simply the average of all $2n$ observations

$$\widehat{Y}_{AV} = \frac{1}{2n}\left(\sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} \widetilde{Y}_i\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{Y_i + \widetilde{Y}_i}{2}\right).$$

Therefore, $\widehat{Y}_{AV}$ is the sample mean of the n independent observations

$$\left(\frac{Y_1 + \widetilde{Y_1}}{2}\right), \left(\frac{Y_2 + \widetilde{Y_2}}{2}\right), ..., \left(\frac{Y_n + \widetilde{Y_n}}{2}\right).$$

By Central Limit theorem,

$$\frac{\sqrt{n}\left(\widehat{Y}_{AV} - E[Y]\right)}{\sigma_{AV}^2}$$

converges to $N(0,1)$ in distribution, where

$$\begin{aligned}
\sigma_{AV}^2 &= Var\left(\frac{Y_i + \widetilde{Y}_i}{2}\right) \\
&= \frac{1}{4}\left(Var[Y_i] + Var\left[\widetilde{Y}_i\right] + 2Cov\left[Y_i, \widetilde{Y}_i\right]\right) \\
&= \frac{1}{4}\left(2Var[Y_i] + 2Cov\left[Y_i, \widetilde{Y}_i\right]\right) \\
&= \frac{1}{2}\left(Var[Y_i] + Cov\left[Y_i, \widetilde{Y}_i\right]\right).
\end{aligned}$$

Thus, we need $Cov\left[Y_i, \widetilde{Y}_i\right] < 0$ for the pairs Y_i and $\widetilde{Y}_i$ to reduce variance.

Implementation of this method to options pricing is very simple. For example, consider pricing a European call option. Our simulated payoffs are

$$C_{T,j} = \max\left(0, Se^{\nu T + \sigma\sqrt{T}\epsilon_j} - K\right). \tag{1}$$

We can simulate the payoffs to the option on the perfectly negatively correlated asset as

$$\widetilde{C_{T,j}} = \max\left(0, Se^{\nu T - \sigma\sqrt{T}\epsilon_j} - K\right)$$

where $\epsilon_j \tilde{} N(0,1)$. In other words, we simply replace ϵ_j by $-\epsilon_j$ in the equation (1) for the simulation. We then take the average of the two payoffs as the payoff for that simulation.

Note that, not only do we obtain a much more accurate estimate from M pairs of $\left(C_{T,j}, \widetilde{C_{T,j}}\right)$ than from $2M$ of $C_{T,j}$, but it is also computationally cheaper to generate the pair $\left(C_{T,j}, \widetilde{C_{T,j}}\right)$ than two instances of $C_{T,j}$.

Exercise 3 *By using antithetic variates method, price a one-year maturity, $T = 1$, at-the-money, $K = S_0$, European call option with the current asset price $S_0 = 100$ and the volatility $\sigma = 20\%$. The continuously compounded interest rate is assumed to be $r = 6\%$ per annum. The asset pays a continuous dividend $\delta = 3\%$ per annum. Simulate the price with $N = 10$ time steps and $M = 10000$ simulations. Calculate the standard error of the estimated price. Compare and comment on the standard error of estimated price with the previous (identical) exercise that does not use variance reduction.*