Additive Summable Processes and their Stochastic Integral

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Abstract

We define and study a class of summable processes, called additive summable processes, that is larger than the class used by Dinculeanu and Brooks [D–B].

We relax the definition of a summable processes $X : \Omega \times \mathbb{R}_+ \to E \subset L(F; G)$ by asking for the associated measure $I_X$ to have just an additive extension to the predictable $\sigma$–algebra $\mathcal{P}$, such that each of the measures $(I_X)_z$, for $z \in (L^p_G)^*$, being $\sigma$–additive, rather than having a $\sigma$–additive extension. We define a stochastic integral with respect to such a process and we prove several properties of the integral. After that we show that this class of summable processes contains all processes $X : \Omega \times \mathbb{R}_+ \to E \subset L(F; G)$ with integrable semivariation if $\ell_0 \notin G$.

Introduction

We study the stochastic integral in the case of Banach-valued processes, from a measure-theoretical point of view.

The classical stochastic integration (for real-valued processes) refers only to integrals with respect to semimartingale (Dellacherie and Meyer [DM78]). A similar technique has also been applied by Kunita [Kun70], for Hilbert valued processes, making use of the inner product. A number of technical difficulties emerge for Banach spaces, since the Banach space lacks an inner product.
Vector integration using different approaches were presented in several books by Dinculeanu [Din00], Diestel and Uhl [DU77], and Kussmaul [Kus77]. Brooks and Dinculeanu [BD76] were the first to present a version of integration with respect to a vector measure with finite semivariation. Later, the same authors [BD90] presented a stochastic integral with respect to so-called summable Banach-valued processes.

A Banach-valued process $X$ is called summable if the Doleans-Dade measure $I_X$ defined on the ring generated by the predictable rectangles can be extended to a $\sigma$-additive measure with finite semivariation on the corresponding $\sigma$-algebra $\mathcal{P}$. The summable process $X$ plays the role of the square integrable martingale in the classical theory: a stochastic integral $H \cdot X$ can be defined with respect to $X$ as a cadlag modification of the process $(\int_{[0,t]} H \, dI_X)_{t \geq 0}$ of integrals with respect to $I_X$ such that $\int_{[0,t]} H \, dI_X \in L^p_G$ for every $t \in \mathbb{R}_+$.

In [Din00] Dinculeanu presents a detailed account of the integration theory with respect to these summable processes, from a measure-theoretical point of view.

Our attention turned to a further generalization of the stochastic integral. Besides the processes considered in the classical theory (Hilbert-valued square-integrable martingales and processes with integrable variation), the class of summable processes also includes processes with integrable semivariation, as long as the Banach space $E$ satisfies some restrictions. To get rid of some of these restrictions, we redefine, in Section 2, the notion of summability: now we only require that $I_X$ can be extended to an additive measure on $\mathcal{P}$, but such that each of the measures $(I_X)_z$, for $z \in Z$ a norming space for $L^p_G$, is $\sigma$-additive. With this new notion of summability, called additive summability, the stochastic integral is then defined, in Section 5.1, as before. The rest of Chapter 5 is dedicated to proving the same type of properties of the stochastic integral as in Dinculeanu [Din00], namely measure theoretical properties.

In Section we will prove that there are more additive summable processes than summable processes by reducing the restrictions imposed on the space $E$. 
1 Notations and definitions

Throughout this paper we consider $S$ to be a set and $\mathcal{R}$, $\mathcal{D}$, $\Sigma$ respectively a ring, a $\delta$–ring, a $\sigma$–ring, and a $\sigma$–algebra of subsets of $S$, $E, F, G$ Banach spaces with $E \subset L(F, G)$ continuously, that is, $|x(y)| \leq |x||y|$ for $x \in E$ and $y \in F$; for example, $E = L(\mathbb{R}, E)$. If $M$ is any Banach space, we denote by $|x|$ the norm of an element $x \in M$, by $M_1$ its unit ball of $M$ and by $M^*$ the dual of $M$. A space $Z \subset G^*$ is called a norming space for $G$, if for every $x \in G$ we have

$$|x| = \sup_{z \in Z_1} \langle x, z \rangle.$$ 

If $m : \mathcal{R} \to E \subset L(F, G)$ is an additive measure for every set $A \subset S$ the semivariation of $m$ on $A$ relative to the embedding $E \subset L(F, G)$ (or relative to the pair $(F, G)$) is denoted by $\tilde{m}_{F,G}(A)$ and defined by the equality

$$\tilde{m}_{F,G}(A) = \sup | \sum_{i \in I} m(A_i)x_i|,$$

where the supremum is taken for all finite families $(A_i)_{i \in I}$ of disjoint sets from $\mathcal{R}$ contained in $A$ and all families $(x_i)_{i \in I}$ of elements from $F_1$.

2 Additive summable processes

The framework for this section is a cadlag, adapted process $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$, such that $X_t \in L^p_E$ for every $t \geq 0$ and $1 \leq p < \infty$.

2.1 The Measures $I_X$ and $(I_X)_z$

Let $\mathcal{S}$ be the semiring of predictable rectangles and $I_X : \mathcal{S} \to L^p_E$ the stochastic measure defined by

$$I_X(\{0\} \times A) = 1_A X_0, \text{ for } A \in \mathcal{F}_0$$

and

$$I_X((s, t] \times A) = 1_A (X_t - X_s), \text{ for } A \in \mathcal{F}_s.$$ 

Note that $I_X$ is finitely additive on $\mathcal{S}$ therefore it can be extended uniquely to a finitely additive measure on the ring $\mathcal{R}$ generated by $\mathcal{S}$. We obtain a finitely additive measure $I_X : \mathcal{R} \to L^p_E$ verifying the previous equalities.
Let $Z \subset (L^p_G)^*$ be a norming space for $L^p_G$. For each $z \in Z$ we define a measure $(I_X)_z : \mathcal{R} \to F^*$ by

$$\langle y, (I_X)_z(A) \rangle = \langle I_X(A)y, z \rangle = \int \langle I_X(A)(\omega)y, z(\omega) \rangle dP(\omega), \text{ for } A \in \mathcal{P} \text{ and } y \in F$$

where the bracket in the integral represents the duality between $G$ and $G^*$.

Since $L^p_E \subset L(F, L^p_G)$, we can consider the semivariation of $I_X$ relative to the pair $(F, L^p_G)$. To simplify the notation, we shall write $(I_X)_{F,G}$ instead of $(I_X)_{F,L^p_G}$ and we shall call it the semivariation of $I_X$ relative to $(F, G)$:

### 2.2 Additive Summable Processes

**Definition 1.** We say that $X$ is $p$-additive summable relative to the pair $(F, G)$ if $I_X$ has an additive extension $I_X : \mathcal{P} \to L^p_E$ with finite semivariation relative to $(F, G)$ and such that the measure $(I_X)_z$ is $\sigma$-additive for each $z \in (L^p_G)^*$.

If $p = 1$, we say, simply, that $X$ is additive summable relative to $(F, G)$.

**Remark.** 1) This definition is weaker than the definition of summable processes since here we don’t require the measure $I_X$ to have a $\sigma$-additive extension to $\mathcal{P}$.

2) The problems that might appear if $(I_X)$ is not $\sigma$-additive are convergence problems (most of the convergence theorem are stated for $\sigma$-additive measures and extension problems (the uniqueness of extensions of measures usually requires $\sigma$-additivity).

3) Note that in the paper “The Riesz representation theorem and extension of vector valued additive measures” N. Dinculeanu and B. Bongiorno [BD01] (Theorem 3.7 II) proved that if each of the measures $(I_X)_z$ is $\sigma$-additive and if $I_X : \mathcal{R} \to L^p_E$ has finite semivariation relative to $(F, G)$ then $I_X$ has canonical additive extension $I_X : \mathcal{P} \to (L^p_E)^{**}$ with finite semivariation relative to $(F, (L^p_E)^{**})$ such that for each $z \in (L^p_G)^*$, the measure $(I_X)_z$ is $\sigma$-additive on $\mathcal{P}$ and has finite variation $|((I_X)_z)$.

**Proposition 2.** If $X$ is $p$-additive summable relative to $(\mathbb{R}, E)$ then $X$ is $p$-summable relative to $(\mathbb{R}, E)$.

**Proof.** If $X$ is $p$-additive summable relative to $(\mathbb{R}, E)$ then the measure $I_X$ has an additive extension $I_X : \mathcal{P} \to L^p_E$ with finite semivariation relative to $(\mathbb{R}, E)$. Moreover for each $z \in (L^p_E)^*$ the measure $(I_X)_z$ is $\sigma$-additive.
By Pettis Theorem, the measure $I_X$ is $\sigma$–additive. Hence, the process $X$ is $p$–summable.

### 2.3 The Integral $\int HdI_X$

Let $X$ be a $p$-additive summable process relative to $(F,G)$.

Consider the additive measure $I_X : \mathcal{P} \to L^p_E \subset L(F,L^p_G)$ with bounded semivariation $\bar{I}_{F,G}$ relative to $(F,L^p_G)$ for which each measure $(I_X)_z$ is $\sigma$-additive for every $z \in Z$.

Then we have

$$(\bar{I}_X)_{F,L^p_G} = \sup\{|m_z| : z \in Z, \|z\| \leq 1, z \in (L^p_F)^*\},$$

(See Corollary 23, Section 1.5 [?].)

If $p$ is fixed, to simplify the notation, we can write $\bar{I}_{F,G} = \bar{I}_{F,L^p_G}$.

For any Banach space $D$ we denote by $\mathcal{F}_D(\bar{I}_{F,G})$ or $\mathcal{F}_D(\bar{I}_{F,L^p_G})$ the space of predictable processes $H : \mathbb{R}_+ \times \Omega \to D$ such that

$$\bar{I}_{F,G}(H) = \sup\{\int |H| d(I_X)_z : \|z\|_q \leq 1\} < \infty.$$  

**Definition 3.** Let $D = F$. For any $H \in \mathcal{F}_F(\bar{I}_{F,G})$ We define the integral $\int HdI_X$ to be the mapping $z \mapsto \int Hd(I_X)_z$.

Observe that if $H \in \mathcal{F}_F(\bar{I}_{F,G})$ the integral $\int Hd(I_X)_z$ is defined and is a scalar for each $z \in Z$, hence the mapping $z \mapsto \int Hd(I_X)_z$ is a continuous linear functional on $(L^p_G)^*$. Therefore, $\int HdI_X \in (L^p_G)^*$

\[
\langle \int HdI_X, z \rangle = \int Hd(I_X)_z, \text{ for } z \in Z
\]

and

$$|\int HdI_X| \leq \bar{I}_{F,G}(H).$$

**Theorem 4.** Let $(H^n)_{0 \leq n < \infty}$ be a sequence of elements from $\mathcal{F}_F(G)(X)$ such that $|H^n| \leq |H^0|$ for each $n$ and $H^n \to H$ pointwise. Assume that

(i) $\int H^ndI_X \in L^p_G$ for every $n \geq 1$

and

(ii) The sequence $(\int H^ndI_X)_n$ converges pointwise on $\Omega$, weakly in $G$.

Then
a) \( \int HdI_X \in L^p_G \)

and

b) \( \int H^n dI_X \to \int HdI_X \), in the weak topology of \( L^p_G \), as well as pointwise, weakly in \( G \).

c) If \( (\int n dI_X) \) converges pointwise on \( \Omega \), strongly in \( G \), then

\[
\int H^n dI_X \to \int HdI_X,
\]

strongly in \( L^1_G \).

**Proof.** This theorem was proved in [Din00] under the assumption that \( I_X \) is \( \sigma \)-additive. But, in fact, only the \( \sigma \)-additivity of each of the measures \( (I_X)_\varepsilon \) was used. Hence the same proof remains valid in our case. \( \square \)

### 2.4 The Stochastic Integral \( H \cdot X \)

In this section we define the stochastic integral and we prove that the stochastic integral is a cadlag adapted process.

Let \( H \in \mathcal{F}_{F,G}(X) \). Then, for every \( t \geq 0 \) we have \( 1_{[0,t]}H \in \mathcal{F}_{F,G}(X) \). We denote by \( \int_{[0,t]}HdI_X \) the integral \( \int 1_{[0,t]}HdI_X \). We define

\[
\int_{[0,\infty]}HdI_X := \int_{[0,\infty)}HdI_X = \int HdI_X.
\]

Taking \( Z = (L^p_G)^* \), for each \( H \in \mathcal{F}_{F,G}(X) \) we obtain a family \( (\int_{[0,t]}HdI_X)_{t \in \mathbb{R}^+} \) of elements of \( (L^p_G)^* \).

We restrict ourselves to processes \( H \) for which \( \int_{[0,t]}HdI_X \in L^p_G \) for each \( t \geq 0 \). Since \( L^p_G \) is a set of equivalence classes, \( \int_{[0,t]}HdI_X \) represents an equivalence class. We use the same notation for any random variable in its equivalence class. We are interested to see whether or not the process \( (\int_{[0,t]}HdI_X)_{t \geq 0} \) is adapted and if it admits a cadlag modification.

**Theorem 5.** Let \( X : \mathbb{R} \to E \subset L(F,G) \) be a cadlag, adapted, \( p \)-summable process relative to \( (F,G) \) and \( H \in \mathcal{F}_{F,G}(X) \) such that \( \int_{[0,t]}HdI_X \in L^p_G \) for every \( t \geq 0 \).

Then the process \( (\int_{[0,t]}HdI_X)_{t \geq 0} \) is adapted.
Proof. This is the equivalent of Theorem 10.4 in [Din00] and since in the proof was used the \(\sigma\)-additivity of the measures \((I_X)_z\) rather than \(\sigma\)-additivity of the measure \(I_X\) that proof will work for our case too. \(\square\)

It is not clear that there is a cadlag modification of the previously defined process \((\int_{[0,t]} H dI_X)_t\). Therefore we use the following definition

**Definition 6.** We define \(L^1_{F,G}(X)\) to be the set of processes \(H \in \mathcal{F}_{F,G}(I_X)\) that satisfy the following two conditions:

a) \(\int_{[0,t]} H dI_X \in L^p_G\) for every \(t \in \mathbb{R}_+\);

b) The process \((\int_{[0,t]} H dI_X)_{t \geq 0}\) has a cadlag modification.

The processes \(H \in L^1_{F,G}(X)\) are said to be integrable with respect to \(X\).

If \(H \in L^1_{F,G}(X)\), then any cadlag modification of the process \((\int_{[0,t]} H dI_X)_{t \geq 0}\) is called the stochastic integral of \(H\) with respect to \(X\) and is denoted by \(H \cdot X\) or \(\int H dX\):

\[
(H \cdot X)_t(\omega) = (\int H dX)_t(\omega) = (\int_{[0,t]} H dI_X)(\omega), \text{ a.s.}
\]

Therefore the stochastic integral is defined up to an evanescent process. For \(t = \infty\) we have

\[
(H \cdot X)_\infty = \int_{[0,\infty]} H dI_X = \int_{[0,\infty]} H dI_X = \int H dI_X.
\]

Note that if \(H : \mathbb{R}_+ \times \Omega \to F\) is an \(\mathcal{R}\)-step process then we have

\[
(H \cdot X)_t(\omega) = \int_{[0,t]} H_s(\omega) dX_s(\omega),
\]

that is, the stochastic integral can be computed pathwise.

The next theorem shows that the stochastic integral \(H \cdot X\) is a cadlag process and it is cadlag in \(L^p_G\).

**Theorem 7.** If \(X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)\) is a \(p\)-additive summable process relative to \((F,G)\) and if \(H \in L^1_{F,G}(X)\), then:

a) The stochastic integral \(H \cdot X\) is a cadlag, adapted process.

b) For every \(t \in [0,\infty)\) we have \((H \cdot X)_t \in L^p_G\) and

\[
(H \cdot X)_t^- = \int_{[0,t]} H dI_X, \text{ a.s.}
\]
If \((H \cdot X)_{\infty-} (\omega)\) exists for each \(\omega \in \Omega\), then
\[
(H \cdot X)_{\infty-} = (H \cdot X)_{\infty} = \int H dI_X, \text{ a.s.}
\]

c) The mapping \(t \mapsto (H \cdot X)_t\) is cadlag in \(L_G^1\).

**Proof.** a) Follows from the previous theorem and definition. b) and c) are proved as in theorem 10.7 in [Din00] since there was not used the \(\sigma\)-additivity of \(I_X\) but rather of \((I_X)_z\). \qed

### 2.5 The Stochastic Integral and Stopping Times

Let \(T\) be a stopping time. If \(A \in \mathcal{F}_T\), then the stopping time \(T_A\) is defined by \(T_A(\omega) = T(\omega)\) if \(\omega \in A\) and \(T_A(\omega) = \infty\) if \(\omega \notin A\). With this notation the predictable rectangles \((s, t] \times A\) with \(A \in \mathcal{F}_s\) could be written as stochastic intervals \((s, t_A]\). Another notation we will use is \(I_X([0, T])\) for \(I_X([0, T] \times \Omega)\).

Let \(X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)\) be an additive summable process

**Proposition 8.** For any stopping time \(T\) we have \(X_T \in L_E^p\) and \(I_X[0, T] = X_T\) for \(T\) simple. For any decreasing sequence \((T_n)\) of simple stopping times such that \(T_n \downarrow T\), and for every \(z \in (L_G^p)^*\) we have
\[
\langle I_X([0, T])y, z \rangle = \lim_{n \to \infty} \langle X_{T_n}y, z \rangle,
\]
where the bracket represents the duality between \(L_G^p\) and \((L_G^p)^*\).

**Proof.** Assume first that \(T\) is a simple stopping time of the form
\[
T = \sum_{1 \leq i \leq n} 1_{A_i}t_i,
\]
with \(0 < t_i \leq \infty, t_i \neq t_j\) for \(i \neq j\), \(A_i \in \mathcal{F}_t\) are mutually disjoint and \(\bigcup_{1 \leq i \leq n} A_i = \Omega\). Then \([0, T] = \bigcup_{1 \leq i \leq n}[0, t_i] \times A_i\) is a disjoint union. Hence \(I_X([0, T]) = \sum_i I_X([0, t_i] \times A_i) = \sum_i 1_{A_i}X_{t_i} = X_T\). Since \(I_X : \mathcal{P} \to L_E^p\), we conclude that \(X_T \in L_E^p\).

Next, assume that \((T_n)\) is a sequence of simple stopping times such that \(T_n \downarrow T\). Then \([0, T_n] \downarrow [0, T]\). Since \((I_X)_z\) is \(\sigma\)-additive in \(F^*\), for any \(y \in F\) and \(z \in (L_G^p)^*\), we have
\[
\langle I_X([0, T])y, z \rangle = \langle (I_X)_z([0, T]), y \rangle = \lim_{n \to \infty} \langle (I_X)_z([0, T_n]), y \rangle
\]
\[
\lim_{n \to \infty} \langle I_X([0, T_n])y, z \rangle = \lim_{n \to \infty} \langle X_{T_n}y, z \rangle.
\]

and the relation (4.1) is proven. It remains to prove that \( X_T \in L^p_E \). Since \( X_{T_n}(\omega) \to X_T(\omega) \) it follows that \( X_T \) is \( \mathcal{F} \)-measurable. We will prove that \( |X_{T_n}| \in L^p \) to deduce that \( X_{T_n} \in L^p_G \).

We saw before that for \( y \in F \) and \( z \in (L^p_G)^* \) the sequence \( \langle (I_X)([0, T_n])y, z \rangle \) is convergent hence bounded, i.e.

\[
\sup_n |\langle (I_X)([0, T_n])y, z \rangle| < \infty, \text{ for } y \in F, z \in (L^p_G)^*.
\]

By the Banach-Steinhauss Theorem, we have

\[
\sup_n \|I_X([0, T_n])y\|_{L^p_E} < \infty, \text{ for } y \in F
\]

hence

\[
\sup_n \|I_X([0, T_n])\|_{L^p_E} < \infty.
\]

or \( \sup_n \|X_{T_n}\|_{L^p_E} < \infty \), which is equivalent to \( \sup_n \int |X_{T_n}|^p dP < \infty \). Now \( |X_T|^p = \lim |X_{T_n}|^p = \lim \inf |X_{T_n}|^p \). If we apply Fatou Lemma we get:

\[
\int |X_T|^p dP = \int \lim \inf |X_{T_n}|^p \leq \lim \inf \int |X_{T_n}|^p dP \leq \sup \int |X_{T_n}|^p dP < \infty.
\]

therefore \( X_T \in L^p_G \).

**Proposition 9.** Let \( S \leq T \) be stopping times and \( h : \Omega \to F \) be an \( \mathcal{F}_S \)-measurable, simple random variable. Then for any pair \((T^n)_n, (S^n)_n\) of sequences of simple stopping times, with \( T^n \downarrow T, S^n \downarrow S \), such that \( S^n \leq T^n \) for each \( n \), we have

\[
\langle \int h1_{(S,T]} dI_X, z \rangle = \lim_n \langle h(X_{T^n} - X_{S^n}), z \rangle, \text{ for } z \in (L^p_G)^*,
\]

where the bracket represents the duality between \( L^p_G \) and \( (L^p_G)^* \).

**Proof.** First we prove that there are two sequences \((T^n)\) and \((S^n)\) of simple stopping times such that \( T^n \downarrow T, S^n \downarrow S \) and \( S^n \leq T^n \). In fact, there are two sequences of simple stopping times \( T^n \) and \( P^n \) such that \( P^n \downarrow S \) and \( T^n \downarrow T \). Consider, now, \( S^n = P^n \wedge T^n \). Since \( P^n \) and \( T^n \) are stopping times, \( S^n \) is a stopping time and \( S^n \leq T^n \). On the other hand, observe that \( S \leq S^n \leq P^n \).
and \( \lim P^n = S \). Therefore \( \lim_{n \to \infty} S^n = S \) too. So we have \( S^n \downarrow S \) and \( S^n \leq T^n \).

Now we want to prove (4.2). Assume first \( h = 1_A y \) with \( A \in \mathcal{F}_S \) and \( y \in F \). Then
\[
\int h1_{(S,T]} dI_X = \int 1_A y 1_{(S,T]} dI_X = \int 1_{(S_A,T_A]} y dI_X = I_X((S_A,T_A])y.
\]
For any sequence of simple stopping times \( (T^n) \) and \( (S^n) \) with \( T^n \downarrow T \), \( S^n \downarrow S \) and \( S^n \leq T^n \), we have \( T^n_A \uparrow T_A \) and \( S^n_A \uparrow S_A \). Therefore, applying Proposition 8 for every \( z \in (L_p^p)^* \), we get
\[
\langle \int h1_{(S,T]} dI_X, z \rangle = \langle I_X((S_A,T_A])y, z \rangle = \langle I_X([0,T_A]) - I_X([0,S_A])y, z \rangle = \lim_{n \to \infty} \langle X_{T^n} - X_{S^n}, y \rangle.
\]

Then the equality holds for any \( \mathcal{F}_S \)-step function \( h \).

**Proposition 10.** Let \( S \leq T \) be stopping times and assume that either

(i) \( h : \Omega \to \mathbb{R} \) is a simple, \( \mathcal{F}_S \)-measurable function and \( H \in L^1_{F,G}(X) \),

or

(ii) The measure \( I_X \) is \( \sigma \)-additive, \( h : \Omega \to F \) is a simple, \( \mathcal{F}_S \)-measurable function and \( H \in L^1_{F,E}(X) \).

If \( \int 1_{(S,T]} dI_X \in L^p_G \) in case (i) and \( \int 1_{(S,T]} H dI_X \in L^p_E \) in case (ii) then
\[
\int h1_{(S,T]} H dI_X = h \int 1_{(S,T]} H dI_X.
\]

**Proof.** Assume first hypothesis (i). Let \((T^n)\) and \((S^n)\) be two sequences of simple stopping times such that \( T^n \downarrow T \), \( S^n \downarrow S \) and \( S^n \leq T^n \). Assume \( H = 1_{(s,t] \times A} y \) with \( A \in \mathcal{F}_s \) and \( y \in F \). Then \( T^n \wedge t \downarrow T \wedge t \), \( S^n \wedge s \downarrow S \wedge s \). Let \( z \in (L_p^p)^* \). Then
\[
\langle h \int 1_{(S,T]} H dI_X, z \rangle = \langle \int h1_{A} 1_{(S\vee s,T\wedge t]} dI_X, z \rangle,
\]
where the bracket represents the duality between \( L^p_G \) and \( (L^p_E)^* \). By (4.2), for the simple \( \mathcal{F}_{S\vee s} \)-measurable function \( h1_A y \) and the stopping times \( S \vee s \leq (T \wedge t) \) we have
\[
\langle h \int 1_{(S,T]} H dI_X, z \rangle = \langle \int 1_{(S,T]} H dI_X, h z \rangle = \langle \int 1_{(S\vee s,T\wedge t]} 1_A y dI_X, h z \rangle
\]

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\[= \lim \langle 1_A y(X_{T^n \wedge t} - X_{S^v \wedge s}), h z \rangle \]
\[= \lim \langle h 1_A y(X_{T^n \wedge t} - X_{S^v \wedge s}), z \rangle = \langle \int h 1_A y 1_{(S^{v} \wedge t)} dI_X, z \rangle \]
\[= \langle \int h 1_A y 1_{(s,t)} 1_{(S,T)} dI_X, z \rangle = \langle \int h 1_{(S,T)} dI_X, z \rangle \]

If \( H = 1_{(0) \times A} y \) with \( A \in \mathcal{F}_0 \) and \( y \in F \), since \( 1_{(S,T)} 1_{(0) \times A} = 0 \) we have
\[\langle h \int 1_{(S,T)} H dI_X, z \rangle = 0 = \langle \int h 1_{(S,T)} dI_X, z \rangle.
\]

It follows that for any \( B \in \mathcal{R} \) we have
\[\langle \int h 1_{(S,T)} 1_{B \times} dI_X, z \rangle = \langle h \int 1_{(S,T)} 1_{B \times} dI_X, z \rangle. \quad (*)\]

The class \( \mathcal{M} \) of sets \( B \in \mathcal{P} \) for which the above equality holds for all \( z \in (L^p_G)^* \) is a monotone class: in fact, let \( B_n \) be a monotone sequence of sets from \( \mathcal{M} \) and let \( B = \lim B_n \). For each \( n \) we have
\[\int h 1_{(S,T)} 1_{B_n \times} dI_X, z = \langle h \int 1_{(S,T)} 1_{B_n \times} dI_X, z \rangle.\]

Since \( h 1_{(S,T)} 1_{B_n \times} \) is a sequence of bounded functions converging to \( h 1_{(S,T)} 1_{B \times} \) (\( h \) is a step-function) with \( |h 1_{(S,T)} 1_{B_n \times}| \leq \|h\| \|y\| \), we can apply Lebesgue Theorem and conclude that \( \int h 1_{(S,T)} 1_{B_n \times} dI_X, z \to \int h 1_{(S,T)} 1_{B \times} dI_X, z \). Using the same reasoning we can conclude that \( \int 1_{(S,T)} 1_{B_n \times} dI_X, z \to \int 1_{(S,T)} 1_{B \times} dI_X, z \).

hence we have
\[\langle \int h 1_{(S,T)} 1_{B_n \times} dI_X, z \rangle = \lim_n \langle \int h 1_{(S,T)} 1_{B_n \times} dI_X, z \rangle = \lim_n \langle h \int 1_{(S,T)} 1_{B_n \times} dI_X, z \rangle = \langle h \lim_n \int 1_{(S,T)} 1_{B_n \times} dI_X, z \rangle = \langle h \int 1_{(S,T)} 1_{B \times} dI_X, z \rangle.
\]

Since the class \( \mathcal{M} \) of sets satisfying equality (*) is a monotone class containing \( \mathcal{R} \) we conclude that the equality (*) is satisfied by all \( B \in \mathcal{P} \).

It follows that for any predictable, simple process \( H \) and for each \( z \in (L^p_G)^* \) we have
\[\langle \int h 1_{(S,T)} H dI_X, z \rangle = \langle h \int 1_{(S,T)} H dI_X, z \rangle \quad (**)
\]
Consider now the general case. If \( H \in L^1_{F,G}(X) \), then there is a sequence \( (H^n) \) of simple, predictable processes such that \( H^n \to H \) and \( |H^n| \leq |H| \). We apply Lebesgue’s Theorem and deduce that

\[
\int h 1_{(S,T]} H^n d(I_X)_z \to \int h 1_{(S,T]} H d(I_X)_z, \tag{1}
\]

and

\[
\int 1_{(S,T]} H^n d(I_X)_{hz} \to \int 1_{(S,T]} H d(I_X)_{hz}. \tag{2}
\]

By equality (***) for each \( n \) we have

\[
\int h 1_{(S,T]} H^n d(I_X)_z = \langle \int h 1_{(S,T]} H^n dI_X, z \rangle = \langle h \int 1_{(S,T]} H^n dI_X, z \rangle = \langle \int 1_{(S,T]} H^n dI_X, hz \rangle = \int 1_{(S,T]} H^n d(I_X)_{hz}
\]

By (1) and (2) we deduce that

\[
\int h 1_{(S,T]} H d(I_X)_z = \int 1_{(S,T]} H d(I_X)_{hz},
\]

which is equivalent to

\[
\langle h 1_{(S,T]} H dI_X, z \rangle = \langle 1_{(S,T]} H dI_X, hz \rangle.
\]

We conclude that

\[
\int h 1_{(S,T]} H dI_X = h \int 1_{(X,T]} H dI_X, \ a.e.
\]

Assume now hypothesis (ii). Since the measure \( I_X \) is \( \sigma \)-additive the process \( X \) is summable. Then observe that the assumptions of (ii) are the same as the assumptions in Proposition 11.5 (ii) of [Din00]. Hence

\[
\int h 1_{(S,T]} H dI_X = h \int 1_{(X,T]} H dI_X,
\]

which concludes our proof. \( \square \)
Proposition 11. Let $X : \mathbb{R} \times \Omega \to E \subset L(F,G)$ be a $p$-additive summable process relative to $(F,G)$ and $T$ a stopping time.

a) For every $z \in (L^p_G)^*$ and every $B \in \mathcal{P}$ we have:

$$(I_{X^T})_z(B) = (I_X)_z(B \cap [0,T]).$$

b) The measure $I_{X^T} : \mathcal{R} \to L^p_E$ has finite semivariation relative to $(F,L^p_G)$

c) If $T$ is a simple stopping time then the process $X^T$ is summable.

Proof. a) First we prove that if $T$ and $S$ are simple stopping times then we have $I_X((S,T]) = X_T - X_S$.

Assume that $T$ is a simple stopping time of the form

$$T = \sum_{1 \leq i \leq n} 1_{A_i} t_i,$$

with $0 < t_i \leq \infty$, $t_i \neq t_j$ for $i \neq j$, $A_i \in \mathcal{F}_t$ are mutually disjoint and $\bigcup_{1 \leq i \leq n} A_i = \Omega$. Then $[0,T] = \bigcup_{1 \leq i \leq n} [0,t_i] \times A_i$ is a disjoint union. Hence $I_X([0,T]) = \sum_i I_X([0,t_i] \times A_i) = \sum_i 1_{A_i} X_{t_i} = X_T$. Since $(S,T] = [0,T] - [0,S]$ and $I_X$ is an additive measure, we have $I_X((S,T]) = I_X([0,T]) - I_X([0,S]) = X_T - X_S$.

Next observe that if $T$ is a simple stopping time then $T \wedge t$ is also a simple stopping time. In fact, if $T = \sum_{1 \leq i \leq n} 1_{A_i} t_i$ then $T \wedge t = \sum_{i:t_i < t} 1_{A_i} t_i + \sum_{i:t_i \geq t} 1_{A_i} t$ which is a simple stopping time.

Now we prove that for $B \in \mathcal{R}$ we have

$$I_{X^T}(B) = I_X([0,T] \cap B).$$

In fact, for $A \in \mathcal{F}_0$ we have

$$I_{X^T}([0] \times A) = 1_A X_0 = I_X([0] \times A) = I_X([0,T] \cap ([0] \times A)).$$

For $s < t$ and $A \in \mathcal{F}_s$ we have,

$$I_{X^T}((s,t] \times A) = 1_A (X^T_t - X^T_s) = 1_A (X_{T \wedge t} - X_{T \wedge s})$$

$$= 1_A (I_X((T \wedge s, T \wedge t])) = 1_A \int_{(s,t]} 1_{[0,T]} dI_X$$

$$= \int 1_A 1_{(s,t]} 1_{[0,T]} dI_X = I_X([0,T] \cap ((s,t] \times A)). \quad (*)$$

We used the above Proposition 10 with $h = 1_A$, $(S,T] = (s,t]$ and $H = 1_{[0,T]}$.
Next we consider the general case, with $T$ a stopping time. For $A \in \mathcal{F}_0$ we have

$$I_{X^T}(\{0\} \times A) = 1_A X_0 = I_X(\{0\} \times A) = I_X([0, T] \cap (\{0\} \times A)).$$

Let $y \in F$ and $z \in (L_G^p)^*$. We have

$$\langle (I_{X^T})_z(\{0\} \times A), y \rangle = \langle I_{X^T}(\{0\} \times A) y, z \rangle = \langle I_X([0, T] \cap (\{0\} \times A)) y, z \rangle = \langle (I_X)_z([0, T] \cap (\{0\} \times A)), y \rangle$$

(1)

For $s < t$ and $A \in \mathcal{F}_s$ we have,

$$I_{X^T}((s, t] \times A) = 1_A (X^T_s - X^T_t) = 1_A (X_{T \land t} - X_{T \land s})$$

(2)

Let $T_n$ be a sequence of simple stopping times such that $T_n \downarrow T$. Let $y \in F$ and $z \in (L_G^p)^*$. We have by (2):

$$\langle (I_{X^T})_z((s, t] \times A), y \rangle = \langle I_{X^T}((s, t] \times A) y, z \rangle = \langle 1_A (X_{T \land t} - X_{T \land s}) y, z \rangle$$

$$= \lim_{n \to \infty} \langle 1_A (X_{T_n \land t} - X_{T_n \land s}) y, z \rangle$$

(3)

By (*) we have:

$$\lim_{n \to \infty} \langle 1_A (X_{T_n \land t} - X_{T_n \land s}) y, z \rangle = \lim_{n \to \infty} \langle I_X([0, T_n] \cap ((s, t] \times A) y, z \rangle$$

$$= \lim_{n \to \infty} \langle (I_X)_z([0, T_n] \cap ((s, t] \times A)), y \rangle = \langle (I_X)_z([0, T] \cap ((s, t] \times A)), y \rangle$$

(3)

since $(I_X)_z$ is $\sigma$-additive. By (1) and (3) and the fact that $(I_{X^T})_z$ is $\sigma$-additive we deduce that

$$(I_{X^T})_z(B) = (I_X)_z(B \cap [0, T]), \text{ for all } B \in \mathcal{R} \quad (4)$$

Since $(I_X)_z$ is $\sigma$-additive we deduce that $(I_{X^T})_z$ is $\sigma$-additive, hence it can be extended to a $\sigma$-additive measure on $\mathcal{F}$. Since $(I_{X^T})_z(B) = (I_X)_z(B \cap [0, T])$ for all $B \in \mathcal{R}$ we deduce that

$$(I_{X^T})_z(B) = (I_X)_z(B \cap [0, T]), \text{ for all } B \in \mathcal{F}$$

b) Let $A$ be a set in $\mathcal{R}$. By Proposition 4.15 in [Din00] we have $svar_{F, L_G} I_{X^T}(A) < \infty$ if and only if $var(I_{X^T})_z(A) < \infty$ for each $z \in (L_G^p)^*$. But

$$\sup_{z \in ((L_G^p)^*)_1} var(I_{X^T})_z(A) = \sup_{z \in ((L_G^p)^*)_1} var(I_X)_z(A \cap [0, T])$$

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\[ = \text{svr}_{F,L_G^p} I_X(A \cap [0,T]) < \infty, \]

and Assertion b) is proved.

c) Assume \( T \) is a simple stopping time. By the equality (*) we have
\[ I_X^T(B) = I_X([0,T] \cap B), \quad \text{for } B \in \mathcal{R}. \]

Since \( X \) is \( p \)-additive summable relative to \((F,G)\), \( I_X \) has a canonical additive extension \( I_X : \mathcal{P} \to L_G^p \). The equality
\[ I_X^T(A) = I_X([0,T] \cap A), \quad \text{for } A \in \mathcal{P}, \]
defines an additive extension of \( I_X^T \) to \( \mathcal{P} \). Since the measure \( I_X \) has finite semivariation relative to \((F,L_G^p)\) \((X \) is additive summable), the measure \( I_X^T \) has finite semivariation relative to \((F,L_G^p)\) also. Moreover, for each \( z \in (L_G^p)^* \), by Assertion a), the measure \( (I_X^T)_z \), defined on \( \mathcal{P} \) is \( \sigma \)-additive. Therefore \( X^T \) is additive summable. We have \(|(I_X^T)_z(A)| = |(I_X)_z|([0,T] \cap A)\) for \( A \in \mathcal{P} \) since \(|(I_X)_z| \) is the canonical extension of its restriction on \( \mathcal{R} \). Then \(|(I_X^T)_z| \) is the canonical extension of its restriction to \( \mathcal{R} \). it follows that \( I_X^T \) is the canonical extension of its restriction to \( \mathcal{R} \).

The next theorem gives the relationship between the stopped stochastic integral and the integral of the process \( 1_{[0,T]}H \). The same type of relation was proved in Theorem 11.6 in [Din00].

**Theorem 12.** Let \( H \in L_{F,G}^1(X) \) and let \( T \) be a stopping time. Then \( 1_{[0,T]}H \in L_{F,G}^1(X) \) and
\[ (1_{[0,T]}H) \cdot X = (H \cdot X)^T. \]

**Proof.** Suppose first that \( T \) is a simple stopping time of the form
\[ T = \sum_{1 \leq i \leq n} 1_{A_i} t_i \]
with \( 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq +\infty, \quad A_i \in \mathcal{F}_{t_i} \) mutually disjoint and with union \( \Omega \). Then for \( t \geq 0 \) we have
\[ (H \cdot X)^T_t(\omega) = \sum_{1 \leq i \leq n} (H \cdot X)_{t_i \wedge t}(\omega) 1_{A_i}(\omega). \]
In fact, for $\omega \in \Omega$ there is $1 \leq i \leq n$ such that $\omega \in A_i$. Then $T(\omega) = t_i$, hence
\[
(H \cdot X)^{T_i}(\omega) = (H \cdot X)_{t_i \wedge t}(\omega).
\]
On the other hand
\[
(1_{[0,T]}H) \cdot X(t)(\omega) = \sum_{1 \leq i \leq n} (H \cdot X)_{t_i \wedge t}(\omega)1_{A_i}(\omega).
\]
In fact,
\[
(\int_{[0,t]} 1_{[0,T]}HdI_X)(\omega) = \left( \int_{[0,t]} \sum_{1 \leq i \leq n} 1_{[0,t_i]}1_{A_i}HdI_X \right)(\omega) = \sum_{1 \leq i \leq n} \left( \int_{[0,t]} 1_{A_i}HdI_X \right)(\omega)
\]
\[
= \sum_{1 \leq i \leq n} \left( \int_{[0,\infty]} H1_{A_i}dI_X \right)(\omega) - \sum_{1 \leq i \leq n} \left( \int_{(t_i,\infty]} 1_{A_i}HdI_X \right)(\omega)
\]
\[
= \sum_{1 \leq i \leq n} 1_{A_i}(\omega)\left( \int_{[0,\infty]} HdI_X \right)(\omega) - \sum_{1 \leq i \leq n} 1_{A_i}(\omega)\left( \int_{(t_i,\infty]} HdI_X \right)(\omega)
\]
\[
= \sum_{1 \leq i \leq n} 1_{A_i}(\omega)\left( \int_{[0,t_i \wedge t]} HdI_X \right)(\omega) = \sum_{1 \leq i \leq n} (H \cdot X)_{t_i \wedge t}(\omega)1_{A_i}(\omega),
\]
where the 4th equality is obtained by applying Proposition 10, with $h = 1_{A_i}$.

Hence, for $T$ simple, we have $1_{[0,T]}H \in L^1_{F,G}(X)$ and
\[
(1_{[0,T]}H) \cdot X = (H \cdot X)^{T_i}.
\]

Now choose $T$ arbitrary. Then there is a decreasing sequence $(T_n)$ of simple stopping times, such that $T_n \downarrow T$.

Note first that since $(H \cdot X)$ is cadlag we have
\[
(H \cdot X)^{T_n} \rightarrow (H \cdot X)^{T_i}.
\]
Moreover for $t \geq 0$ we have $1_{[0,T_n \wedge t]}H \downarrow 1_{[0,T_n \wedge t]}H$ pointwise. Since $1_{[0,T_n \wedge t]}H \in L^1_{F,G}(X)$, for each $(z \in L^0_p)$ we have $1_{[0,T_n \wedge t]}H \in L^1_F((I_X)_z)$, hence
\[
\langle \int 1_{[0,T_n \wedge t]}HdI_X, z \rangle = \int 1_{[0,T_n \wedge t]}Hd(I_X)_z \rightarrow \int 1_{[0,T \wedge t]}Hd(I_X)_z = \langle \int 1_{[0,T \wedge t]}HdI_X, z \rangle.
\]
By Theorem 4 we conclude that \( \int 1_{[0,T]} H dI_X = \int_{[0,t]} 1_{[0,T]} H dI_X \in L^p_G \) and

\[
\int 1_{[0,T_n \wedge t]} H dI_X \to \int 1_{[0,T]} H dI_X,
\]
or

\[
\int_{[0,t]} 1_{[0,T_n]} H dI_X \to \int_{[0,t]} 1_{[0,T]} H dI_X.
\]

Since for each \( n \) we have \( 1_{[0,T_n]} H \cdot X_t = (H \cdot X)_t^{T_n} \), by (1) we deduce that \( \int_{[0,t]} 1_{[0,T]} H dI_X = (H \cdot X)_t^T \). Hence the mapping \( t \mapsto \int_{[0,t]} 1_{[0,T]} H dI_X \) is cadlag, from which we conclude that \( 1_{[0,T]} H \in L_{F,G}^1(X) \). Moreover

\[
(1_{[0,T]} H \cdot X)_t = (H \cdot X)_{T \wedge t} = (H \cdot X)_t^T.
\]

The next corollary is a useful particular case of the previous theorem:

**Corollary 13.** For every stopping time \( T \) we have

\[
1_{[0,T]} \cdot X = X^T.
\]

**Proof.** Taking \( H = 1 \in L_{F,G}^1(X) \) and applying Theorem 12 we conclude that \( 1_{[0,T]} \cdot X = X^T \). \( \square \)

The following theorem gives the same type of results as Theorem 11.8 in [Din00].

**Theorem 14.** Let \( S \leq T \) be stopping times and assume that either

(i) \( h : \Omega \to \mathbb{R} \) is a simple, \( \mathcal{F}_S \)-measurable function and \( H \in L_{F,G}^1(X) \),

or

(ii) The measure \( I_X \) is \( \sigma \)-additive, \( h : \Omega \to F \) is a simple, \( \mathcal{F}_S \)-measurable function and \( H \in L_{R,F}^1(X) \).

Then \( 1_{(S,T]} H \) and \( h 1_{(S,T]} H \) are integrable with respect to \( X \) and

\[
(h 1_{(S,T]} H) \cdot X = h[(1_{(S,T]} H) \cdot X].
\]

**Proof.** Note that

\[
1_{(S,T]} H = 1_{[0,T]} H - 1_{[0,S]} H.
\]

Assume first the case (i). Applying Theorem 12 for \( 1_{[0,T]} H \) and \( 1_{[0,S]} H \) we conclude that \( 1_{(S,T]} H \in L_{F,G}^1(X) \).
If for each \( t \geq 0 \) we apply Proposition 10, we obtain

\[
\int_{[0,t]} h1_{(S,T]}H d\mathcal{I}_X = h \int_{[0,t]} 1_{(S,T]}H d\mathcal{I}_X.
\]

Since \( 1_{(S,T]}H \in L^1_{F,G}(X) \) we deduce that \( h1_{(S,T]}H \in L^1_{F,G}(X) \) and

\[
((h1_{(S,T]}H) \cdot X)_t = h((1_{(S,T]}H) \cdot X)_t,
\]

which concluded the proof of case (i). Case (ii) is treated similarly. \( \square \)

2.6 The Integral \( \int H d\mathcal{I}_{X^T} \)

In this section we define the set of processes integrable with respect to the measure \( \mathcal{I}_{X^T} \) with finite semivariation relative to the pair \( (F, L^p_G) \).

Let \( X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G) \) be a cadlag, adapted process and assume \( X \) is \( p \)-additive summable relative to \( (F, L^p_G) \).

Consider the additive measure \( \mathcal{I}_X : \mathcal{P} \to L^p_E \subset L(F, L^p_G) \) with bounded semivariation \( I_{F,G} \) relative to \( (F, L^p_G) \), such that each of the measures \( (\mathcal{I}_X)_z \) with \( z \in (L^p_G)^* \) is \( \sigma \)-additive.

To simplify the notations denote \( m = \mathcal{I}_{X^T} \). We proved in the previous proposition that the measure \( m : \mathcal{R} \to L^p_E \subset L(F, L^p_G) \) has bounded semivariation relative to \( (F, L^p_G) \), on \( \mathcal{R} \), and for each \( z \in (L^p_G)^* \) the measures \( m_z \), is \( \sigma \)-additive. In order for the process \( X^T \) to be additive summable we need the measure \( m : \mathcal{R} \to L^p_E \) to have an extension \( m : \mathcal{P} \to L^p_E \) with finite semivariation and such that each of the measures \( m_z \) with \( z \in (L^p_G)^* \) is \( \sigma \)-additive. Applying Theorem 7 from Bongiorno-Dinculeanu, citeBD2001, the measure \( m \) has a unique canonical extension \( m : \mathcal{P} \to (L^p_E)^{**} \), with bounded semivariation such that for each \( z \in (L^p_G)^* \) the measure \( m_z \), is \( \sigma \)-additive and has bounded variation \( |m_z| \), therefore \( X^T \) is summable.

Then we have

\[
\tilde{m}_{F,L^p_G} = \sup \{|m_z| : z \in (L^p_G)^*, \|z\|_q \leq 1\}.
\]

We denote by \( \mathcal{F}_{F,G}(X^T) \) the space of predictable processes \( H : \mathbb{R}_+ \times \Omega \to F \) such that

\[
\tilde{m}_{F,G}(H) = \tilde{m}_{F,L^p_G}(H) = \sup \left\{ \int |H|d|m_z| : \|z\| \leq 1 \right\} < \infty.
\]
Let $H \in \mathcal{F}_{F,G}(X^T)$; then $H \in L^1_F(|m_z|)$ for every $z \in (L^p_G)^*$, hence the integral $\int Hdm_z$ is defined and is a scalar. The mapping $z \mapsto \int Hdm_z$ is a linear continuous functional on $(L^p_G)^*$, denoted $\int Hdm$. Therefore, $\int Hdm \in (L^p_G)^*$, 

$$\langle \int Hdm, z \rangle = \int Hdm_z, \text{ for } z \in (L^p_G)^*.$$ 

We denote by $L^{1,F,G}(X^T)$ the set of processes $H \in \mathcal{F}_{F,G}(I^T_X)$ satisfying the following two conditions:

a) $\int_{[0,t]} Hdm \in L^p_G$ for every $t \in \mathbb{R}_+$;

b) The process $(\int_{[0,t]} Hdm)_{t \geq 0}$ has a cadlag modification.

**Theorem 15.** Let $X : \mathbb{R} \to E \subset L(F,G)$ be a $p$-additive summable process relative to $(F,G)$ and $T$ a stopping time.

a) We have $H \in \mathcal{F}_{F,G}(X^T)$ iff $1_{[0,T]}H \in \mathcal{F}_{F,G}(X)$ and in this case we have:

$$\int HdI_{X^T} = \int 1_{[0,T]}HdI_X.$$ 

b) We have $H \in L^{1,F,G}(X^T)$ iff $1_{[0,T]}H \in L^{1,F,G}(X)$ and in this case we have:

$$H \cdot X^T = (1_{[0,T]}H) \cdot X.$$ 

If $H \in L^{1,F,G}(X)$, then $H \in L^{1,F,G}(X^T), 1_{[0,T]}H \in L^{1,F,G}(X)$ and 

$$(H \cdot X)^T = H \cdot X^T = (1_{[0,T]}H) \cdot X.$$ 

**Proof.** a) Define $m : \mathcal{R} \to E$ by $m(B) = I_{X^T}(B)$ for $B \in \mathcal{R}$. We proved in Theorem 11 (a) that for every $z \in (L^p_G)^*$ we have 

$$m_z(B) = (I_X)_z(B \cap [0,T]), \text{ for all } B \in \mathcal{R}. \quad (*)$$ 

Since $(I_X)_z((\cdot) \cap [0,T])$ is a $\sigma$–additive measure, with bounded variation on $\mathcal{P}$ satisfying (*) and since $\mathcal{P}$ is the $\sigma$–algebra generated by $\mathcal{R}$, by the uniqueness theorem 7.4 in [Din00] we conclude that 

$$m_z(B) = (I_X)_z(B \cap [0,T]), \text{ for all } B \in \mathcal{P}.$$ 

Let $H \in \mathcal{F}_{F,G}(X^T) = \bigcap_{\|z\|_q \leq 1, z \in (L^p_G)^*} L_F^1(m_z)$. From the previous equality we deduce that 

$$\int Hdm_z = \int 1_{[0,T]}Hd(I_X)_z,$$ 

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therefore
\[ \int HdI_{XT} = \int 1_{[0,T]}HdI_X, \]
and this is the equality in Assertion a).

b) To prove Assertion b) we replace \( H \) with \( 1_{[0,t]}H \) in the previous assertion and deduce that \( 1_{[0,t]}H \in \mathcal{F}_{F,G}(X^T) \) iff \( 1_{[0,t]}1_{[0,T]}H \in \mathcal{F}_{F,G}(X) \). In this case we have
\[ \int_{[0,t]} HdI_{XT} = \int_{[0,t]} 1_{[0,T]}HdI_X. \]
It follows that \( H \in L^1_{F,G}(X^T) \) iff \( 1_{[0,T]}H \in L^1_{F,G}(X) \) and in this case we have
\[ (H \cdot X^T)_t = ((1_{[0,T]}H) \cdot X)_t. \]
If now \( H \in L^1_{F,G}(X) \), then, from Theorem 12 we deduce that \( 1_{[0,T]}H \in L^1_{F,G}(X) \) and
\[ (1_{[0,T]}H) \cdot X = (H \cdot X)^T. \]

\[ \square \]

2.7 Convergence Theorems

Assume \( X \) is \( p \)-additive summable relative to \((F, G)\). In this section we shall present several convergence theorems.

**Lemma 16.** Let \((H^n)\) be a sequence in \(L^1_{F,G}(X)\) and assume that \(H^n \to H\) in \(\mathcal{F}_{F,G}(X)\). Then there is a subsequence \((r_n)\) such that
\[ (H^{r_n} \cdot X)_t \to (H \cdot X)_t = \int_{[0,t]} HdI_X, \text{ a.s., as } n \to \infty, \]
uniformly on every bounded time interval.

**Proof.** Since \(H^n\) is a convergent sequence in \(\mathcal{F}_{F,G}(X)\) there is a subsequence \(H^{r_n}\) of \((H^n)\) such that
\[ \tilde{I}_{F,G}(H^{r_n} - H^{r_{n+1}}) \leq 4^{-n}, \text{ for each } n. \]

Let \(t_0 > 0\). Define the stopping time
\[ u_n = \inf\{t : |(H^{r_n} \cdot X)_t - (H^{r_{n+1}} \cdot X)_t| > 2^{-n}\} \wedge t_0. \]
By Theorem 12 applied to the stopping time $u_n$, we obtain

$$(H^r_n \cdot X)_{u_n} = (H^r_n \cdot X)_{\infty} = ((1_{[0,u_n]}H^r_n) \cdot X)_{\infty} = \int_{[0,u_n]} H^r_n dI_X,$$

hence

$$E(|(H^r_n \cdot X)_{u_n} - (H^r_{n+1} \cdot X)_{u_n}|) = E(|\int_{[0,u_n]} H^r_n dI_X - \int_{[0,u_n]} H^r_{n+1} dI_X|)$$

$$= E(\int_{[0,u_n]} |(H^r_n - H^r_{n+1})dI_X|) = (\int_{[0,u_n]} |(H^r_n - H^r_{n+1})dI_X|_{L^G})$$

$$\leq \int_{[0,u_n]} |(H^r_n - H^r_{n+1})dI_X|_{L^G} \leq \tilde{I}_{F,G}(H^r_n - H^r_{n+1}) \leq 4^{-n}. \quad (*)$$

Using inequality (*) and following the same techniques as in Theorem 12.1 a) in [Dim00] one could show first that the sequence $(H^r_n \cdot X)_t$ is a Cauchy sequence in $L^p_G$ uniformly for $t < t_0$ and then conclude that

$$(H^r_n \cdot X)_t \to \int_{[0,t]} H dI_X,$$

uniformly on every bounded time interval.

**Theorem 17.** Let $(H^n)$ be a sequence from $L^1_{F,G}(X)$ and assume that $H^n \to H$ in $F_{F,G}(X)$. Then:

a) $H \in L^1_{F,G}(X)$.

b) $(H^n \cdot X)_t \to (H \cdot X)_t$, in $L^p_G$, for $t \in [0, \infty]$.

c) There is a subsequence $(r_n)$ such that

$$(H^{r_n} \cdot X)_t \to (H \cdot X)_t, \text{ a.s., as } n \to \infty,$$

uniformly on every bounded time interval.

**Proof.** For every $t \geq 0$ we have $1_{[0,t]}H^n \to 1_{[0,t]}H$ in $F_{F,G}(X)$. Since the integral is continuous, we deduce that

$$(H^n \cdot X)_t = \int_{[0,t]} H^n dI_X \to \int_{[0,t]} H dI_X, \text{ in } (L^p_G)^*.$$  

Since $H^n \in L^1_{F,G}(X)$ we have $\int_{[0,t]} H^n dI_X \in L^p_G$ and

$$(H^n \cdot X)_t \to \int_{[0,t]} H dI_X, \text{ in } L^p_G.$$
From the previous lemma we deduce that there is a subsequence \((H^{r_n})\) such that

\[(H^{r_n} \cdot X)_t \to (H \cdot X)_t, \text{ a.s., as } n \to \infty,\]

uniformly on every bounded time interval. Since \((H^{r_n} \cdot X)\) are cadlag it follows that the limit is also cadlag, hence \(H \in L^1_{F,G}(X)\) which is Assertion a). Hence

\[(H \cdot X)_t = \int_{[0,t]} HdI_X, \text{ a.s.}\]

and therefore \((H^n \cdot X)_t \to (H \cdot X)_t\) in \(L^p_G\), which is Assertion b). Also observe that for the above subsequence \((H^{r_n})\) we have

\[(H^{r_n} \cdot X)_t \to (H \cdot X)_t, \text{ a.s., as } n \to \infty,\]

uniformly on every bounded time interval.

We can restate Theorem 17 as:

**Corollary 18.** \(L^1_{F,G}(X)\) is complete.

Next we state an uniform convergence theorem. Uniform convergence implies convergence in \(L^1_{F,G}(X)\).

**Theorem 19.** Let \((H^n)\) be a sequence from \(\mathcal{F}_{F,G}(X)\). If \(H^n \to H\) pointwise uniformly then \(H \in \mathcal{F}_{F,G}(X)\) and \(H^n \to H\) in \(\mathcal{F}_{F,G}(X)\).

If, in addition, for each \(n\), \(H^n\) is integrable, i.e. \(H^n \in L^1_{F,G}(X)\) then

a) \(H \in L^1_{F,G}(X)\) and \(H^n \to H\) in \(L^1_{F,G}(X)\);

b) For every \(t \in [0,\infty]\) we have \((H^n \cdot X)_t \to (H \cdot X)_t\), in \(L^p_G\).

c) There is a subsequence \((r_n)\) such that \((H^{r_n} \cdot X)_t \to (H \cdot X)_t, \text{ a.s. as } n \to \infty, \text{ uniformly on any bounded interval.}\)

**Proof.** Assertion a) is immediate. Assertions b), c) and d) follow from Theorem 17.

Now we shall state Vitali and Lebesgue-type Convergence Theorems. They are direct consequences of the convergence Theorem 17 and of the uniform convergence Theorem 19.

**Theorem 20.** (Vitali). Let \((H^n)\) be a sequence from \(\mathcal{F}_{F,G}(X)\) and let \(H\) be an \(F\)-valued, predictable process. Assume that
(i) \( \tilde{I}_{F,G}(H^n1_A) \to 0 \) as \( \tilde{I}_{F,G}(A) \to 0 \), uniformly in \( n \) and that any one of the conditions (ii) or (iii) below is true:
(ii) \( H^n \to H \) in \( \tilde{I}_{F,G} \)-measure;
(iii) \( H^n \to H \) pointwise and \( I_{F,L^p_G} \) is uniformly \( \sigma \)-additive (this is the case if \( H^n \) are real-valued, i.e., \( F = \mathbb{R} \)).

Then:
a) \( H \in \mathcal{F}_{F,G}(X) \) and \( H^n \to H \) in \( \mathcal{F}_{F,G}(X) \).
Conversely, if \( H^n, H \in \mathcal{F}_{F,G}(\mathcal{B}, X) \) and \( H^n \to H \) in \( \mathcal{F}_{F,G}(X) \), then conditions (i) and (ii) are satisfied.

Under the hypotheses (i) and (ii) or (iii), assume, in addition, that \( H^n \in L^1_{F,G}(X) \) for each \( n \). Then
b) \( H \in L^1_{F,G}(X) \) and \( H^n \to H \) in \( L^1_{F,G}(X) \);
c) For every \( t \in [0, \infty) \) we have \( (H^n \cdot X)_t \to (H \cdot X)_t \), in \( L^p_G \);
d) There is a subsequence \( (r_n) \) such that \( (H^{r_n} \cdot X)_t \to (H \cdot X)_t \), a.s., as \( n \to \infty \), uniformly on any bounded interval.

Theorem 21. (Lebesgue). Let \( (H^n) \) be a sequence from \( \mathcal{F}_{F,G}(X) \) and let \( H \) be an \( F \)-valued predictable process. Assume that
(i) There is a process \( \phi \in \mathcal{F}_{\mathbb{R}}(\mathcal{B}, I_{F,G}) \) such that
\[
|H^n| \leq \phi \text{ for each } n;
\]
and that any one of the conditions (ii) or (iii) below is true:
(ii) \( H^n \to H \) in \( \tilde{I}_{F,G} \)-measure;
(iii) \( H^n \to H \) pointwise and \( I_{F,L^p_G} \) is uniformly \( \sigma \)-additive (this is the case if \( H^n \) are real-valued, i.e., \( F = \mathbb{R} \)).

Then:
a) \( H \in \mathcal{F}_{F,G}(\mathcal{B}, X) \) and \( H^n \to H \) in \( \mathcal{F}_{F,G}(X) \).
Assume, in addition that \( H^n \in L^1_{F,G}(X) \) for each \( n \). Then
b) \( H \in L^1_{F,G}(X) \) and \( H^n \to H \) in \( L^1_{F,G}(X) \);
c) For every \( t \in [0, \infty) \) we have \( (H^n \cdot X)_t \to (H \cdot X)_t \), in \( L^p_G \);
d) There is a subsequence \( (r_n) \) such that \( (H^{r_n} \cdot X)_t \to (H \cdot X)_t \), a.s., as \( n \to \infty \), uniformly on any bounded interval.

2.8 Summability of the Stochastic Integral

Assume \( X \) is \( p \)-additive summable relative to \( (F,G) \). In this section we are studying the additive summability of the stochastic integral \( H \cdot X \) for \( F \)-valued processes \( H \).
If $H$ is a real valued processes then in order for the stochastic integral $H \cdot X$ to be defined we need each of the measure $(I_X)_z$, for $z \in (L_F^p)^*$, to be $\sigma-$additive, hence the measure $I_X$ would be $\sigma-$additive. Therefore the process $X$ would be summable. In this case the summability of the stochastic integral is proved in Theorem 13.1 of [Din00].

The next theorem shows that if $H$ is $F-$valued then the measure $I_H \cdot X$ is $\sigma-$additive even if $I_X$ is just additive.

Theorem 22. Let $H \in L_{F,G}^1(X)$ be such that $\int_A H dI_X \in L_G^p$ for $A \in \mathcal{P}$. Then the measure $I_{H \cdot X} : \mathcal{R} \to L_G^p$ has a $\sigma-$additive extension $I_{H \cdot X} : \mathcal{P} \to L_G^p$ to $\mathcal{P}$.

Proof. We first note that $H \cdot X : \mathbb{R}_+ \times \Omega \to G = L(\mathbb{R}, G)$ is a cadlag adapted process with $(H \cdot X)_t \in L_G^p$ for $t \geq 0$ (by the definition of $H \cdot X$).

Since $\int_A H dI_X \in L_G^p$ for every $A \in \mathcal{P}$, by Proposition ??, with $m = I_X$ and $g = H$, we deduce that $HI_X$ is $\sigma-$additive on $\mathcal{P}$.

Next we prove that for any predictable rectangle $A \in \mathcal{R}$ we have

$$I_{H \cdot X}(A) = \int_A H dI_X.$$  

In fact, consider first $A = \{0\} \times B$ with $B \in \mathcal{F}_0$. Using Proposition 10 for $h = 1_B$ we have

$$I_{H \cdot X}(\{0\} \times B) = 1_B((H \cdot X)_0) = 1_B \int_{\{0\}} H dI_X$$

$$= \int_{\{0\}} 1_B H dI_X = \int_{\{0\} \times B} H dI_X;$$

Let now $A = (s,t] \times B$ with $B \in \mathcal{F}_s$. Using Proposition 10 for $h = 1_B$ and $(S,T] = (s,t]$ we have

$$I_{H \cdot X}((s,t] \times B) = 1_B((H \cdot X)_t - (H \cdot X)_s)$$

$$= 1_B(\int_{[0,t]} H dI_X - \int_{[0,s]} H dI_X) = 1_B (\int_{(s,t]} H dI_X$$

$$= \int_{(s,t]} 1_B H dI_X = \int_{(s,t] \times B} H dI_X;$$

and the desired equality is proved.
Since the measure $A \mapsto \int_A H dI_X$ is $\sigma-$additive for $A \in \mathcal{P}$ it will follow that $I_{H,X}$ can be extended to a $\sigma$-additive measure on $\mathcal{P}$ by the same equality

$$I_{H,X}(A) = \int_A H dI_X, \text{ for } A \in \mathcal{P}. \quad (2)$$

The next theorem states the summability of the stochastic integral.

**Theorem 23.** Let $H \in L^1_{F,G}(X)$ be such that $\int_A H dI_X \in L^p_G$ for $A \in \mathcal{P}$. Then:

a) $H \cdot X$ is $p$-summable, hence $p$-additive summable relative to $(\mathbb{R}, G)$ and

$$dI_{H,X} = d(HI_X).$$

b) For any predictable process $K \geq 0$ we have

$$(\tilde{I}_{H,X})_{\mathbb{R},G}(K) \leq (\tilde{I}_X)_{F,G}(KH).$$

c) If $K$ is a real-valued predictable process and if $KH \in L^1_{F,G}(X)$, then $K \in L^1_{\mathbb{R},G}(H \cdot X)$ and we have

$$K \cdot (H \cdot X) = (KH) \cdot X.$$  

**Proof.** By Theorem 22 we know that the measure $I_{H,X}$ is $\sigma-$additive. Therefore To prove (a) we only need to show that the extension of $I_{H,X}$ to $\mathcal{P}$ has finite semivariation relative to $(\mathbb{R}, L^p_G)$.

Let $z \in (L^p_G)^*$. From the equality (2) in Theorem 22 we deduce that for every $A \in \mathcal{P}$, and we have

$$(I_{H,X})_z(A) = \langle I_{H,X}(A), z \rangle = \langle \int_A H dI_X, z \rangle = \int_A H d(I_X)_z.$$ 

From this we deduce the inequality

$$|(I_{H,X})_z|(A) \leq \int_A |H||d|(I_X)_z|, \text{ for } A \in \mathcal{P}. \quad (*)$$

Taking the supremum for $z \in (L^p_G)^*$ we obtain

$$\sup\{|(I_{H,X})_z|(A), z \in (L^p_G)^*\} \leq \sup\{\int_A |H||d|(I_X)_z|, z \in (L^p_G)^*\}.$$
\[ \leq \sup \{ \int |1_A H| d((I_X)_z), z \in (L_{E}^p)_{1}\}, \text{ for } A \in \mathcal{P}. \]

Therefore
\[ (\tilde{I}_{H,X})_{R,G}(A) \leq (\tilde{I}_X)_{F,G}(1_A H) < \infty, \text{ for } A \in \mathcal{P}. \]

It follows that \( H \cdot X \) is \( p \)-summable, hence \( p \)-additive summable, relative to \((R, G)\) and this proves Assertion a).

Since the extension to \( \mathcal{P} \) of the measure \( I_{X,H} \) is \( \sigma \)-additive and has finite semivariation b) and c) follow from Theorem 13.1 of [Din00].

### 2.9 Summability Criterion

Let \( Z \subset L_{E}^p \), be any closed subspace norming for \( L_{E}^p \). The next theorem differs from the summability criterion in [Din00] by the fact that the restrictive condition \( c_0 \notin E \) was not imposed. Also note that this theorem does not give us necessary and sufficient conditions for the summability of the process.

**Theorem 24.** Let \( X : \mathbb{R}_+ \times \Omega \to E \) be an adapted, cadlag process such that \( X_t \in L_{E}^p \) for every \( t \geq 0 \). Then the Assertions a)-d) below are equivalent.

a) \( I_X : \mathcal{R} \to L_{E}^p \) has an additive extension \( I_X : \mathcal{P} \to Z^* \) such that for each \( g \in Z \), the real valued measure \( \langle I_X, g \rangle \) is a \( \sigma \)-additive on \( \mathcal{P} \).

b) \( I_X \) is bounded on \( \mathcal{R} \);

c) For every \( g \in Z \), the real valued measure \( \langle I_X, g \rangle \) is bounded on \( \mathcal{R} \);

d) For every \( g \in Z \), the real valued measure \( \langle I_X, g \rangle \) is \( \sigma \)-additive and bounded on \( \mathcal{R} \).

**Proof.** The proof will be done as follows: b) \( \iff \) c) \( \iff \) d) and a) \( \iff \) d).

b) \( \implies \) c) and c) \( \implies \) b) can be proven in the same fashion as in [Din00].

c) \( \implies \) d) Assume c), and let \( g \in Z \). The real valued measure \( \langle I_X, g \rangle \) is defined on \( \mathcal{R} \) by
\[ \langle I_X, g \rangle(A) = \langle I_X(A), g \rangle = \int \langle I_X(A), g \rangle dP, \text{ for } A \in \mathcal{R}. \]

By assumption, \( \langle I_X, g \rangle \) is bounded on \( \mathcal{R} \). We need to prove that the measure \( \langle I_X, g \rangle \) is \( \sigma \)-additive. For that consider, as in [Din00], the real-valued process \( XG = (X_t, G_t)_{t \geq 0} \), where \( G_t = E(g|\mathcal{F}_t) \) for \( t \geq 0 \). Then \( XG : \mathbb{R}_+ \times \Omega \to \mathbb{R} \) is a cadlag, adapted process and it can be proven, using the same techniques as in [Din00] that it is a quasimartingale.
Now, for each \( n \), define the stopping time
\[
T_n(\omega) = \inf\{ t : |X_t| > n \}.
\]
Then \( T_n \uparrow \infty \) and \( |X_t| \leq n \) on \([0, T_n)\). Since \( XG \) is a quasimartingale on \((0, \infty)\), we know that \((XG)_{T_n} \in L^1\) (Proposition A 3.5 in [BD87]: \( XG \) is a quasimartingale on \((0, \infty)\) iff \( XG \) is a quasimartingale on \((0, \infty)\) and \( \sup_t \|XG\|_1 < \infty \).

Moreover,
\[
|(XG)_{T_n}^T| = |(XG)_t|_{1\{t < T_n\}} + |(XG)_{T_n}|_{1\{t \geq T_n\}}
\]
\[
\leq |X_t||G_t|_{1\{t < T_n\}} + |(XG)_{T_n}|_{1\{t \geq T_n\}}
\]
\[
\leq n|G_t|_{1\{t < T_n\}} + |(XG)_{T_n}|_{1\{t \geq T_n\}}.
\]

Besides, since \( G_t = E(g|\mathcal{F}_t) \) it follows that \( G \) is a uniformly integrable martingale.

Next we prove that the family \( \{(XG)_{T_n}^T, T \text{ simple stopping time}\} \) is uniformly integrable.

In fact, note that by inequality (2) we have
\[
\int_{\{(XG)_{T_n}^T > p\}} |(XG)_{T_n}^T| dP
\]
\[
\leq \int_{\{(XG)_{T_n}^T > p\} \cap \{T < T_n\}} n|(XG)_{T_n}^T| dP + \int_{\{(XG)_{T_n}^T > p\} \cap \{T \geq T_n\}} |(XG)_{T_n}^T| dP \tag{3}
\]

Now observe that
\[
\{(XG)_T > p\} \cap \{T < T_n\} = \{(X_T, G_T) > p\} \cap \{T < T_n\}
\]
\[
\subset \{|X_T|G_T > p\} \cap \{T < T_n\} \subset \{p < n|G_T|\} \cap \{T < T_n\} \subset \{p < nG_T\}
\]

Since \( G \) is a uniformly integrable martingale, it is a martingale of class D; from \( n|G_t|_{1\{t < T_n\}} \leq n|G_t| \) we deduce that \( n|G_t|_{1\{t < T_n\}} \) is a martingale of class (D):
\[
\lim_{p \to \infty} \int_{\{n|G_t|_{1\{t < T_n\}} > p\}} n|G_t|_{1\{t < T_n\}} dP \leq \lim_{p \to \infty} \int_{\{n|G_t| > p\}} n|G_t| dP
\]
\[
= n \lim_{p \to \infty} \int_{\{|G_t| > \frac{p}{n}\}} n|G_t| dP = \lim_{\frac{p}{n} \to \infty} \int_{\{n|G_t| > p\}} n|G_t| dP = 0.
\]
Hence there is a $p_{1\epsilon}$ such that for any $p \geq p_{1\epsilon}$ and any simple stopping time $T$ we have

$$\int_{\{(XG)^{T_n}_T > p\} \cap \{T < T_n\}} n|(XG)^{T_n}_T|dP \leq \int_{\{n|G_t| > p\}} n|G_t|dP < \frac{\epsilon}{2} \quad (4)$$

We look now at the second term of the right hand side of the inequality (3).

$$\int_{\{(XG)^{T_n}_T > p\} \cap \{T \geq T_n\}} |(XG)^{T_n}_T|dP \leq \int_{\{(XG)^{T_n}_T > p\}} |(XG)^{T_n}_T|dP$$

Since $(XG)^{T_n}_T \in L^1$, for every $\epsilon > 0$ there is a $p_{2\epsilon} > 0$ such that for every $p \geq p_{2\epsilon}$ we have

$$\int_{\{(XG)^{T_n}_T > p\}} |(XG)^{T_n}_T|dP < \frac{\epsilon}{2} \quad (5)$$

If we put (4) and (5) together we deduce that for every $\epsilon > 0$ there is a $p_{\epsilon} = \max(p_{1\epsilon}, p_{2\epsilon})$ such that for any $p > p_{\epsilon}$ and any $T$ simple stopping time we have

$$\int_{\{(XG)^{T_n}_T > p\}} |(XG)^{T_n}_T|dP < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves the fact that $(XG)^{T_n}_T$ is a quasimartingale of class (D). From Theorem 14.2 of [Din00] we deduce that the Doléans measure $\mu_{(XG)^{T_n}_T}$ associated to the process $(XG)^{T_n}_T$ is $\sigma-$additive and has bounded variation on $\mathcal{R}$, hence it can be extended to a $\sigma$-additive measure with bounded variations on $\mathcal{P}$ (Theorem 7.4 b) of [Din00]).

Next we show that for any $B \in \mathcal{P}$ we have

$$\mu_{(XG)^{T_n}_T}(B) = \mu_{XG}(B \cap [0, T_n]).$$

In fact, for $A \in \mathcal{F}_0$ we have

$$\mu_{(XG)^{T_n}_T}(\{0\} \times A) = \mu_{XG}(((\{0\} \times A) \cap [0, T_n]).$$

and for $(s, t] \times A$ with $A \in \mathcal{F}_s$ we have

$$\mu_{(XG)^{T_n}_T}((s, t] \times A) = E(1_A((XG)^{T_n}_t - (XG)^{T_n}_s)) = \mu_{XG}(((s, t] \times A) \cap [0, T_n]),$$

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which proves our equality. Hence the measure $\mu_{XG}$ is $\sigma$-additive on the $\sigma$-ring $\mathcal{P} \cap [0, T_n]$ for each $n$, hence it is $\sigma$-additive on the ring $\mathcal{B} = \bigcup_{1 \leq n < \infty} \mathcal{P} \cap [0, T_n]$.

Next we observe that $\mu_{XG}$ is bounded on $\mathcal{R}$, therefore it has bounded variation on $\mathcal{R}$ which implies that the measure defined on $\mathcal{B} \cap \mathcal{R}$ is $\sigma$-additive and has bounded variation. Since $\mathcal{B} \cap \mathcal{R}$ generates $\mathcal{P}$, by Theorem 7.4 b) of [Din00], $\mu_{XG}$ can be extended to a $\sigma$-additive measure with bounded variation on $\mathcal{P}$.

Since $\langle I_X, g \rangle = \mu_{XG}$, it follows that $\langle I_X, g \rangle$ is bounded and $\sigma$-additive on $\mathcal{R}$, thus d) holds. The implication $d) \Rightarrow c)$ is evident.

$\Rightarrow d)$ is evident since for each $g \in Z$, the measure $\langle I_X, g \rangle$ is $\sigma$-additive on $\mathcal{P}$ and since any $\sigma$-additive measure on a $\sigma$-algebra is bounded we conclude that for $g \in Z$, the measure $\langle I_X, g \rangle$ is bounded on $\mathcal{P}$ hence on $\mathcal{R}$.

Next we prove $d) \Rightarrow a)$. Assume d) is true. Then the real valued measure $\langle I_X, g \rangle$ is $\sigma$-additive and bounded on $\mathcal{R}$. Since we proved that $b) \iff c) \iff d)$ we deduce from (1) that

$$|\langle I_X, g \rangle(A)| \leq M\|g\| \text{ for all } A \in \mathcal{R}$$

where $M = \sup\{|I_X(A)| : A \in \mathcal{R}\}$. By Proposition 2.16 of [Din00] it follows that

the measure $\langle I_X(\cdot), g \rangle$ has bounded variation $|\langle I_X, g \rangle|(\cdot)$ satisfying

$$|\langle I_X, g \rangle|(A) \leq 2M\|g\|, \text{ for } A \in \mathcal{R}.$$ 

Applying Proposition 4.15 in [Din00] we deduce that $\tilde{I}_{X_{\mathbb{R},E}}$ is bounded. By Theorem 3.7 b) of [BD01] we conclude that the measure $I_X : \mathcal{R} \rightarrow L^p_E$ has an additive extension $I_X : \mathcal{P} \rightarrow Z^{**}$ to $\mathcal{P}$ such that for each $g \in Z$, the real valued measure $\langle I_X, g \rangle$ is a $\sigma$-additive on $\mathcal{P}$ which is Assertion a).

### 3 Examples of Additive Summable Processes

**Definition 25.** Let $X : \mathbb{R}_+ \times \Omega \rightarrow E$ be an $E$-valued process. We say that $X$ has finite variation, if for each $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ has finite variation on each interval $[0, t]$. If $1 \leq p < \infty$, the process $X$ has $p$-integrable variation if the total variation $|X|_\infty = \var(X, \mathbb{R}_+)$ is $p$-integrable.
Definition 26. We define the variation process $|X|$ by

$$|X|_{t}(\omega) = \text{var}(X_{t}(\omega), (-\infty, t])$$

for $t \in \mathbb{R}$ and $\omega \in \Omega,$

where $X_{t} = 0$ for $t < 0.$

Noting that if $m : \mathcal{D} \to E \subset L(F, G)$ is a $\sigma$-additive measure then for each $z \in G^{*}$, the measure $m_{z} : \mathcal{D} \to F^{*}$ is $\sigma-$additive, we deduce that, if the process $X$ is summable, then it is also additive summable. Hence the following theorem is a direct consequence of Theorem 19.13 in [Din00]

**Theorem 27.** Let $X : \mathbb{R}_{+} \times \Omega \to E$ be a cadlag, adapted process with integrable variation $|X|$. Then $X$ is 1-additive summable relative to any embedding $E \subset L(F, G)$.

**Proof.** If $m : \mathcal{D} \to E \subset L(F, G)$ is a $\sigma$-additive measure then for each $z \in G^{*}$, the measure $m_{z} : \mathcal{D} \to F^{*}$ is $\sigma-$additive. We deduce that, if the process $X$ is summable, then it is additive summable. Hence applying Theorem 19.13 b) in [Din00] we conclude our proof. \( \square \)

### 3.1 Processes with Integrable Semivariation

**Definition 28.** We define the semivariation process of $X$ relative to $(F, G)$ by

$$\tilde{X}_{t}(\omega) = \text{svar}_{F,G}(X_{t}(\omega), (-\infty, t])$$

for $t \in \mathbb{R}$ and $\omega \in \Omega,$

where $X_{t} = 0$ for $t < 0.$

**Definition 29.** The total semivariation of $X$ is defined by

$$\tilde{X}_{\infty}(\omega) = \sup_{t \geq 0} \tilde{X}_{t}(\omega) = \text{svar}_{F,G}(X_{t}(\omega), \mathbb{R}),$$

for $\omega \in \Omega.$

**Definition 30.** Let $X : \mathbb{R}_{+} \times \Omega \to E \subset L(F, G)$. The process $X$ is said to have finite semivariation relative to $(F, G)$, if for every $\omega \in \Omega$, the path $t \mapsto X_{t}(\omega)$ has finite semivariation relative to $(F, G)$ on each interval $(-\infty, t]$. The process $X$ is said to have $p$-integrable semivariation if $\tilde{X}_{F,G}$ if the total semivariation $(\tilde{X}_{F,G})_{\infty}$ belongs to $L^{p}$.

**Remark:** If $X : \mathbb{R}_{+} \times \Omega \to E \subset L(F, G)$ is a process and $z \in G^{*}$ we define, the process $X_{z} : \mathbb{R}_{+} \times \Omega \to F^{*}$ by

$$\langle x, (X_{z})_{t}(\omega) \rangle = \langle X_{t}(\omega)x, z \rangle,$$

for $x \in F, t \in \mathbb{R}_{+}$ and $\omega \in \Omega.$
For fixed $t \geq 0$, we consider the function $X_t : \omega \mapsto X_t(\omega)$ from $\Omega$ into $E \subset L(F, G)$ and for $z \in G^*$ we define $(X_t)_z : \Omega \to F^*$ by the equality

$$\langle x, (X_t)_z(\omega) \rangle = \langle X_t(\omega)x, z \rangle, \text{ for } \omega \in \Omega, \text{ and } x \in F.$$ 

It follows that

$$(X_t)_z(\omega) = X_z(t)(\omega), \text{ for } t \in \mathbb{R}_+ \text{ and } \omega \in \Omega.$$ 

The semivariation $\tilde{X}$ can be computed in terms of the variation of the processes $X_z$:

$$\tilde{X}_t(\omega) = \sup_{z \in G^*_t} |X_z|_t(\omega).$$

If $X$ has finite semivariation $\tilde{X}$, then each $X_z$ has finite variation $|X_z|$. 

The following theorem is an improvement over the Theorem 21.12 in [Din00], where it was supposed that $c_0 \not\in E$ and $c_0 \not\in G$.

**Theorem 31.** Assume $c_0 \not\in G$. Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$ be a cadlag, adapted process with $p$-integrable semivariation relative to $(\mathbb{R}, E)$ and relative to $(F, G)$. Then $X$ is $p$-additive summable relative to $(F, G)$.

**Proof.** First we present the sketch of the proof, after which we prove all the details.

The prove goes as follows:

1) First we will show that

$$I_X(A)(\omega) = m_X(\omega)(A(\omega)), \text{ for } A \in \mathcal{R} \text{ and } \omega \in \Omega,$$ 

where $A(\omega) = \{t; (t, \omega) \in A\}$ and $X(\omega)$ is $X(\omega)$. For the definition of the measure $m_X(\omega)$ see Section 2.2.

2) Then we will prove that the measure $m_X(\omega)$ has an additive extension to $\mathcal{B}(\mathbb{R}_+)$, with bounded semivariation relative to $(F, G)$ and such that for every $g \in G^*$ the measure $(m_X(\omega))_g$ is $\sigma-$additive.

3) Next we prove that the function $\omega \mapsto m_X(\omega)(M(\omega))$ belongs to $L^p_E$ for all $M \in \mathcal{P}$.

4) Then we show that the extension of the measure $I_X$ to $\mathcal{P}$ has bounded semivariation relative to $(F, L^p_G)$.

5) Finally we show that for each $z \in (L^p_G)^*$ the measure $(I_X)_z : \mathcal{P} \to F^*$ is $\sigma-$additive.

6) We conclude that the process $X$ is $p-$additive summable.
Now we prove each step in detail.

1) First we prove (*) for predictable rectangles. Let \( A = \{0\} \times B \) with \( B \in \mathcal{F}_0 \). Then we have
\[
I_X(\{0\} \times B)(\omega) = 1_B(\omega)X_0(\omega) = \int 1_{\{0\} \times B}(s, \omega) dX_s(\omega) = m_X(\omega)(A(\omega)).
\]
Now let \( A = (s, t] \times B \) with \( B \in \mathcal{F}_s \). In this case we also obtain
\[
I_X((s, t]\times B)(\omega) = 1_B(\omega)(X_t(\omega) - X_s(\omega)) = \int 1_{(s, t]\times B}(p, \omega) dX_p(\omega) = m_X(\omega)(A(\omega)).
\]
Since both \( I_X(A)(\omega) \) and \( m_X(\omega)(A(\omega)) \) are additive we conclude that the equality (*) is true for \( A \in \mathcal{R} \).

2) Since \( X \) has \( p \)-integrable semivariation relative to \( (F, G) \) we infer that \( (X_{F,G})_\infty(\omega) < \infty \) a.s. If we redefine \( X_t(\omega) = 0 \) for those \( \omega \) for which \( (X_{F,G})_\infty(\omega) = \infty \) we obtain a process still denoted \( X \) with bounded semivariation. In this case for each \( \omega \in \Omega \) the function \( t \mapsto X_t(\omega) \) is right continuous and with bounded semivariation. By Theorem ?? we deduce that the measure \( m_X(\omega) \) can be extended to an additive measure \( m_X(\omega) : \mathcal{B}(\mathbb{R}_+) \rightarrow E \subset L(F, G) \), with bounded semivariation relative to \( (F, G) \) and such that for every \( g \in G^* \) the measure \( (m_X(\omega))_g : \mathcal{B}(\mathbb{R}_+) \rightarrow F^* \) is \( \sigma \)-additive.

3) Since \( X \) has \( p \)-integrable semivariation relative to \( (F, G) \), for each \( t \geq 0 \) we have \( X_t \in L^p_E \). Hence, by step 1, the function \( \omega \mapsto m_X(\omega)(M_0(\omega)) \) belongs to \( L^p_E \) for all \( M \in \mathcal{R} \). To prove that \( \omega \mapsto m_X(\omega)(M(\omega)) \) belongs to \( L^p_E \) for all \( M \in \mathcal{P} \) we will use the Monotone Class Theorem. We will prove that the set \( \mathcal{P}_0 \) of all sets \( M \in \mathcal{P} \) for which the affirmation is true is a monotone class, containing \( \mathcal{R} \), hence equal to \( \mathcal{P} \). In fact, let \( M_n \) be a monotone sequence from \( \mathcal{P}_0 \) converging to \( M \). By assumption, for each \( n \) the function \( \omega \mapsto m_X(\omega)(M_n(\omega)) \) belongs to \( L^p_E \) and for each \( \omega \) the sequence \( (M_n(\omega)) \) is monotone in \( \mathcal{B}(\mathbb{R}_+) \) and has limit \( M(\omega) \). Moreover \( \|m_X(\omega)(M_n(\omega))\| \leq m_X(\omega)(\mathbb{R}_+ \times \Omega) = X_\infty(\omega) \), which is \( p \)-integrable. By Lebesgue’s Theorem we deduce that the mapping \( \omega \mapsto m_X(\omega)(M(\omega)) \) belongs to \( L^p_E \), hence \( M \in \mathcal{P}_0 \). Therefore \( \mathcal{P}_0 \) is a monotone class.

4) We use the equality (*) to extend \( I_X \) to the whole \( \mathcal{P} \), by
\[
I_X(A)(\omega) = m_X(\omega)(A(\omega)), \quad \text{for } A \in \mathcal{P}.
\]
Let \( A \in \mathcal{P} \), \( (A_i)_{i \in I} \) be a finite family of disjoint sets from \( \mathcal{P} \) contained in \( A \), and \( (x_i)_{i \in I} \) a family of elements from \( F \) with \( |x_i| \leq 1 \). Then we have
\[
\| \sum I_X(A_i)x_i \|_p = E(\| \sum I_X(A_i)(\omega)x_i \|_p^p)
\]
= E(\left|\sum m_{X(\omega)}(A_i(\omega))x_i\right|^p) \leq E(\left|\sum (m_{X(\omega)}(A(\omega))\right|^p) \\
= \left\|\sum (m_{X(\omega)}(A(\omega)))\right\|^p = \left\|\sum (X_{FG}(A(\omega)))\right\|^p \leq \left\|\sum (X_{FG})\right\|^p < \infty.

Taking the supremum over all the families \((A_i)\) and \((x_i)\) as above, we deduce
\((\bar{I}_X)^{FG}_p \leq \left\|\sum (X_{FG})\right\|^p < \infty.\)

5) Let \(z \in (L^p_G)^*\) and \(x \in F\). Then \(z(\omega) \in G^*\) and for each set \(M \in \mathcal{P}\) we have

\[
\langle (I_X)_z(M), x \rangle = \langle I_X(M) x, z \rangle = E(\langle I_X(M)(\omega) x, z(\omega) \rangle) \\
= E(\langle (m_{X(\omega)})(M(\omega)) x, z(\omega) \rangle) = E(\langle (m_{X(\omega)})(z(\omega))(M(\omega)), x \rangle).
\]

(3)

By step we conclude that the measure \((I_X)_z\) is \(\sigma\)-additive for each \(z \in (L^p_G)^*\).

6) By the definition in step 4,

\[I_X(A)(\omega) = m_{X(\omega)}(A(\omega)), \text{ for } A \in \mathcal{P} \text{ and } \omega \in \Omega,\]

and by steps 2 and 3 we conclude that the measure \(I_X\) has an additive extension \(\bar{I}_X : \mathcal{P} \to L^p_E\). By step 5 the measure \((I_X)_z\) is \(\sigma\)-additive for each \(z \in (L^p_G)^*\). By step 4 this extension has bounded semivariation. Therefore the process \(X\) is \(p\)-additive summable.

The following theorem gives sufficient conditions for two processes to be indistinguishable. For the proof see [Din00], Corollary 21.10 b’.

**Theorem 32.** ([Din00]21.10b’) Assume \(c_0 \not\in E\) and let \(A, B : \mathbb{R}_+ \times \Omega \to E\) be two predictable processes with integrable semivariation relative to \((\mathbb{R}, E)\). If for every stopping time \(T\) we have \(E(A_\infty - A_T) = E(B_\infty - B_T)\), then \(A\) and \(B\) are indistinguishable.

The next theorem gives examples of processes with locally integrable variation or semivariation. For the proof see [Din00], Theorems 22.15 and 22.16.

**Theorem 33.** ([Din00]22.15,16) Assume \(X\) is right continuous and has finite variation \(|X|\) (resp. finite semivariation \(X_{FG}\)). If \(X\) is either predictable or a local martingale, then \(X\) has locally integrable variation \(|X|\) (resp. locally integrable semivariation \(X_{FG}\)).

**Proposition 34.** Let \(X : \mathbb{R}_+ \times \Omega \to E\) be a process with finite variation. If \(X\) has locally integrable semivariation \(X_{FG}\), then \(X\) has locally integrable variation.
Proof. Assume $X$ has locally integrable semivariation $\tilde{X}$ relative to $(\mathbb{R}, E)$. Then there is an increasing sequence $S_n$ of stopping times with $S_n \uparrow \infty$ such that $E(\tilde{X}_{S_n}) < \infty$ for each $n$. For each $n$ define the stopping times $T_n$ by $T_n = S_n \wedge \inf\{t | |X|_t \geq n\}$. It follows that $|X|_{T_n} \leq n$. Since $X$ has finite variation, by Proposition 6 we have $\Delta |X_{T_n}| = |\Delta X_{T_n}| \leq \tilde{X}_{T_n}$. From $\Delta |X_{T_n}| = |X|_{T_n} - |X|_{T_n-}$ we deduce that $|X|_{T_n} = |X|_{T_n-} + \Delta |X_{T_n}| \leq n + \tilde{X}_{T_n}$; therefore $E(|X|_{T_n}) \leq n + E(\tilde{X}_{T_n}) < \infty$; hence $X$ has locally integrable variation. \qed

References


