Additive Summable Processes and their Stochastic Integral

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Abstract

We define and study a class of summable processes,called additive summable processes, that is larger than the class used by Dinculeanu and Brooks [D–B].

We relax the definition of a summable processes $X : \Omega\times\mathbb{R}_+ \to E \subset L(F,G)$ by asking for the associated measure I_X to have just an additive extension to the predictable $\sigma-$algebra $\mathcal{P}$, such that each of the measures $(I_X)_z$, for $z \in (L^p_G)^*$, being $\sigma-$additive, rather than having a $\sigma-$additive extension. We define a stochastic integral with respect to such a process and we prove several properties of the integral. After that we show that this class of summable processes contains all processes $X : \Omega \times \mathbb{R}_+ \to E \subset L(F,G)$ with integrable semivariation if $c_0 \notin G$.

Introduction

We study the stochastic integral in the case of Banach-valued processes, from a measure-theoretical point of view.

The classical stochastic integration (for real-valued processes) refers only to integrals with respect to semimartingale (Dellacherie and Meyer [DM78]). A similar technique has also been applied by Kunita [Kun70], for Hilbert valued processes, making use of the inner product. A number of technical difficulties emerge for Banach spaces, since the Banach space lacks an inner product.

Vector integration using different approaches were presented in several books by Dinculeanu [Din00], Diestel and Uhl [DU77], and Kussmaul [Kus77]. Brooks and Dinculeanu [BD76] were the first to present a version of integration with respect to a vector measure with finite semivariation. Later, the same authors [BD90] presented a stochastic integral with respect to so-called summable Banach-valued processes.

A Banach-valued process X is called *summable* if the Doleans-Dade measure I_X defined on the ring generated by the predictable rectangles can be extended to a σ-additive measure with finite semivariation on the corresponding σ-algebra $\mathcal{P}$. The summable process X plays the role of the square integrable martingale in the classical theory: a stochastic integral $H \cdot X$ can be defined with respect to X as a cadlag modification of the process $\left(\int_{[o,t]} H\, dI_X\right)_{t\geq 0}$ of integrals with respect to I_X such that $\int_{[0,t]} H dI_X \in L^p_G$ for every $t \in \mathbb{R}_+$.

In [Din00] Dinculeanu presents a detailed account of the integration theory with respect to these summable processes, from a measure-theoretical point of view.

Our attention turned to a further generalization of the stochastic integral. Besides the processes considered in the classical theory (Hilbert-valued square-integrable martingales and processes with integrable variation), the class of summable processes also includes processes with integrable *semivariation*, as long as the Banach space E satisfies some restrictions. To get rid of some of these restrictions, we redefine, in Section 2, the notion of summability: now we only require that I_X can be extended to an *additive* measure on $\mathcal{P}$, but such that each of the measures $(I_X)_z$, for $z \in Z$ a norming space for L^p_G, is σ-additive. With this new notion of summability, called additive summability, the stochastic integral is then defined, in Section 5.1, as before. The rest of Chapter 5 is dedicated to proving the same type of properties of the stochastic integral as in Dinculeanu [Din00], namely measure theoretical properties.

In Section we will prove that there are more additive summable processes than summable processes by reducing the restrictions imposed on the space E.

1 Notations and definitions

Throughout this paper we consider S to be a set and $\mathcal{R}$, $\mathcal{D}$, Σ respectively a ring, a $\delta-$ring, a $\sigma-$ring, and a $\sigma-$algebra of subsets of S, E, F, G Banach spaces with $E \subset L(F,G)$ continuously, that is, $|x(y)| \leq |x||y|$ for $x \in E$ and $y \in F$; for example, $E = L(\mathbb{R}, E)$. If M is any Banach space, we denote by $|x|$ the norm of an element $x \in M$, by M_1 its unit ball of M and by M^* the dual of M. A space $Z \subset G^*$ is called a *norming space for* G, if for every $x \in G$ we have

$$|x| = \sup_{z \in Z_1} |\langle x, z \rangle|.$$

If $m : \mathcal{R} \to E \subset L(F,G)$ is an additive measure for every set $A \subset S$ the semivariation of m on A relative to the embedding $E \subset L(F,G)$ (or relative to the pair (F,G)) is denoted by $\tilde{m}_{F,G}(A)$ and defined by the equality

$$\tilde{m}_{F,G}(A) = \sup |\sum_{i \in I} m(A_i)x_i|,$$

where the supremum is taken for all finite families $(A_i)_{i \in I}$ of disjoint sets from $\mathcal{R}$ contained in A and all families $(x_i)_{i \in I}$ of elements from F_1.

2 Additive summable processes

The framework for this section is a cadlag, adapted process $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$, such that $X_t \in L^p_E$ for every $t \geq 0$ and $1 \leq p < \infty$.

2.1 The Measures I_X and $(I_X)_z$

Let $\mathcal{S}$ be the semiring of predictable rectangles and $I_X : \mathcal{S} \to L^p_E$ the stochastic measure defined by

$$I_X(\{0\} \times A) = 1_A X_0, \text{ for } A \in \mathcal{F}_0$$

and

$$I_X((s,t] \times A) = 1_A(X_t - X_s), \text{ for } A \in \mathcal{F}_s.$$

Note that I_X is finitely additive on $\mathcal{S}$ therefore it can be extended uniquely to a finitely additive measure on the ring $\mathcal{R}$ generated by $\mathcal{S}$. We obtain a finitely additive measure $I_X : \mathcal{R} \to L^p_E$ verifying the previous equalities.

Let $Z \subset (L^p_G)^*$ be a norming space for L^p_G. For each $z \in Z$ we define a measure $(I_X)_z$, $(I_X)_z : \mathcal{R} \to F^*$ by

$$\langle y, (I_X)_z(A)\rangle = \langle I_X(A)y, z\rangle = \int \langle I_X(A)(\omega)y, z(\omega)\rangle dP(\omega), \text{ for } A \in \mathcal{P} \text{ and } y \in F$$

where the bracket in the integral represents the duality between G and G^*.

Since $L^p_E \subset L(F, L^p_G)$, we can consider the semivariation of I_X relative to the pair (F, L^p_G). To simplify the notation, we shall write $(\tilde{I}_X)_{F,G}$ instead of $(\tilde{I}_X)_{F,L^p_G}$ and we shall call it the semivariation of I_X relative to (F, G):

2.2 Additive Summable Processes

Definition 1. We say that X is p-additive summable relative to the pair (F, G) if I_X has an additive extension $I_X : \mathcal{P} \to L^p_E$ with finite semivariation relative to (F, G) and such that the measure $(I_X)_z$ is σ-additive for each $z \in (L^p_G)^*$.

If $p = 1$, we say, simply, that X is additive summable relative to (F, G).

Remark. 1) This definition is weaker that the definition of summable processes since here we don't require the measure I_X to have a $\sigma-$additive extension to $\mathcal{P}$.

2) The problems that might appear if (I_X) is not $\sigma-$additive are convergence problems (most of the convergence theorem are stated for $\sigma-$additive measures and extension problems (the uniqueness of extensions of measures usually requires $\sigma-$additivity).

3) Note that in the paper "The Riesz representation theorem and extension of vector valued additive measures" N. Dinculeanu and B. Bongiorno [BD01] (Theorem 3.7 II) proved that if each of the measures $(I_X)_z$ is σ-additive and if $I_X : \mathcal{R} \to L^p_E$ has finite semivariation relative to (F, G) then I_X has canonical additive extension $I_X : \mathcal{P} \to (L^p_E)^{**}$ with finite semivariation relative to $(F, (L^p_E)^{**})$ such that for each $z \in (L^p_G)^*$, the measure $(I_X)_z$ is $\sigma-$additive on $\mathcal{P}$ and has finite variation $|(I_X)_z|$.

Proposition 2. *If X is $p-$additive summable relative to $(\mathbb{R}, E)$ then X is $p-$summable relative to $(\mathbb{R}, E)$.*

Proof. If X is $p-$additive summable relative to $(\mathbb{R}, E)$ then the measure I_X has an additive extension $I_X : \mathcal{P} \to L^p_E$ with finite semivariation relative to $(\mathbb{R}, E)$. Moreover for each $z \in (L^p_E)^*$ the measure $(I_X)_z$ is σ-additive.

By Pettis Theorem, the measure I_X is $\sigma-$additive. Hence, the process X is $p-$summable. $\square$

2.3 The Integral $\int H dI_X$

Let X be a p-additive summable process relative to (F, G).

Consider the additive measure $I_X : \mathcal{P} \to L^p_E \subset L(F, L^p_G)$ with bounded semivariation $\tilde{I}_{F,G}$ relative to (F, L^p_G) for which each measure $(I_X)_z$ is σ-additive for every $z \in Z$.

Then we have

$$(\tilde{I}_X)_{F,L^p_G} = \sup\{|m_z| : z \in Z, \|z\| \leq 1, z \in (L^p_F)^*\},$$

(See Corollary 23, Section 1.5 [**?**].)

If p is fixed, to simplify the notation, we can write $\tilde{I}_{F,G} = \tilde{I}_{F,L^p_G}$.

For any Banach space D we denote by $\mathcal{F}_D(\tilde{I}_{F,G})$ or $\mathcal{F}_D(\tilde{I}_{F,L^p_G})$ the space of predictable processes $H : \mathbb{R}_+ \times \Omega \to D$ such that

$$\tilde{I}_{F,G}(H) = \sup\{\int |H| d|(I_X)_z| : \|z\|_q \leq 1\} < \infty.$$

Definition 3. Let $D = F$. For any $H \in \mathcal{F}_F(\tilde{I}_{F,G})$ We define the integral $\int H dI_X$ to be the mapping $z \mapsto \int H d(I_X)_z$.

Observe that if $H \in \mathcal{F}_{F,G}(X) := \mathcal{F}_F(\tilde{I}_{F,G})$ the integral $\int H d(I_X)_z$ is defined and is a scalar for each $z \in Z$, hence the mapping $z \mapsto \int H d(I_X)_z$ is a continuous linear functional on $(L^p_G)^*$ Therefore, $\int H dI_X \in (L^p_G)^{**}$

$$\langle \int H dI_X, z \rangle = \int H d(I_X)_z, \text{ for } z \in Z$$

and

$$|\int H dI_X| \leq \tilde{I}_{F,G}(H).$$

Theorem 4. *Let $(H^n)_{0 \leq n < \infty}$ be a sequence of elements from $\mathcal{F}_{F,G}(X)$ such that $|H^n| \leq |H^0|$ for each n and $H^n \to H$ pointwise. Assume that*
(i) $\int H^n dI_X \in L^p_G$ *for every* $n \geq 1$
and
(ii) *The sequence* $(\int H^n dI_X)_n$ *converges pointwise on* Ω, *weakly in* G.
Then

a) $\int H dI_X \in L^p_G$
and
b) $\int H^n dI_X \to \int H dI_X$, *in the weak topology of* L^p_G, *as well as pointwise, weakly in* G.
c) *If* $(\int H^n dI_X)_n$ *converges pointwise on* Ω, *strongly in* G, *then*

$$\int H^n dI_X \to \int H dI_X,$$

strongly in L^1_G.

Proof. This theorem was proved in [Din00] under the assumtion that I_X is $\sigma-$additive. But, in fact, only the σ-additivity of each of the measures $(I_X)_z$ was used. hence the same proof remains valid in our case. □

2.4 The Stochastic Integral $H \cdot X$

In this section we define the stochastic integral and we prove that the stochastic integral is a cadlag adapted process.

Let $H \in \mathcal{F}_{F,G}(X)$. Then, for every $t \geq 0$ we have $1_{[0,t]}H \in \mathcal{F}_{F,G}(X)$. We denote by $\int_{[0,t]} H dI_X$ the integral $\int 1_{[0,t]} H dI_X$. We define

$$\int_{[0,\infty]} H dI_X := \int_{[0,\infty)} H dI_X = \int H dI_X.$$

Taking $Z = (L^p_G)^*$, for each $H \in \mathcal{F}_{F,G}(X)$ we obtain a family $(\int_{[0,t]} H dI_X)_{t \in \mathbb{R}_+}$ of elements of $(L^p_G)^{**}$.

We restrict ourselves to processes H for which $\int_{[0,t]} H dI_X \in L^p_G$ for each $t \geq 0$. Since L^p_G is a set of equivalence classes, $\int_{[0,t]} H dI_X$ represents an equivalence class. We use the same notation for any random variable in its equivalence class. We are interested to see whether or not the process $(\int_{[0,t]} H dI_X)_{t\geq 0}$ is adapted and if it admits a cadlag modification.

Theorem 5. *Let* $X : \mathbb{R} \to E \subset L(F, G)$ *be a cadlag, adapted, p-summable process relative to* (F, G) *and* $H \in \mathcal{F}_{F,G}(X)$ *such that* $\int_{[0,t]} H dI_X \in L^p_G$ *for every* $t \geq 0$.

Then the process $(\int_{[0,t]} H dI_X)_{t\geq 0}$ *is adapted.*

Proof. This is the equivalent of Theorem 10.4 in [Din00] and since in the proof was used the σ-additivity of the measures $(I_X)_z$ rather than σ-additivity of the measure I_X that proof will work for our case too. $\square$

It is not clear that there is a cadlag modification of the previously defined process $(\int_{[0,t]} HdI_X)_t$. Therefore we use the following definition

Definition 6. We define $L^1_{F,G}(X)$ to be the set of processes $H \in \mathcal{F}_{F,G}(I_X)$ that satisfy the following two conditions:
a) $\int_{[0,t]} HdI_X \in L^p_G$ for every $t \in \mathbb{R}_+$;
b) The process $(\int_{[0,t]} HdI_X)_{t\geq 0}$ has a cadlag modification.
The processes $H \in L^1_{F,G}(X)$ are said to be integrable with respect to X.

If $H \in L^1_{F,G}(X)$, then any cadlag modification of the process $(\int_{[0,t]} HdI_X)_{t\geq 0}$ is called the stochastic integral of H with respect to X and is denoted by $H \cdot X$ or $\int HdX$:

$$(H \cdot X)_t(\omega) = (\int HdX)_t(\omega) = (\int_{[0,t]} HdI_X)(\omega), \text{ a.s.}$$

Therefore the stochastic integral is defined up to an evanescent process. For $t = \infty$ we have

$$(H \cdot X)_\infty = \int_{[0,\infty]} HdI_X = \int_{[0,\infty)} HdI_X = \int HdI_X.$$

Note that if $H : \mathbb{R}_+ \times \Omega \to F$ is an $\mathcal{R}$-step process then we have

$$(H \cdot X)_t(\omega) = \int_{[0,t]} H_s(\omega) dX_s(\omega),$$

that is, the stochastic integral can be computed pathwise.

The next theorem shows that the stochastic integral $H \cdot X$ is a cadlag process and it is cadlag in L^p_G.

Theorem 7. *If $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$ is a p-additive summable process relative to (F,G) and if $H \in L^1_{F,G}(X)$, then:* a) *The stochastic integral $H \cdot X$ is a cadlag, adapted process.*
b) *For every $t \in [0,\infty)$ we have $(H \cdot X)_{t-} \in L^p_G$ and*

$$(H \cdot X)_{t-} = \int_{[0,t)} HdI_X, \text{ a.s.}$$

If $(H \cdot X)_{\infty-}(\omega)$ *exists for each* $\omega \in \Omega$*, then*

$$(H \cdot X)_{\infty-} = (H \cdot X)_{\infty} = \int H dI_X, \ a.s.$$

c) *The mapping* $t \mapsto (H \cdot X)_t$ *is cadlag in* L^1_G*.*

Proof. a) Follows from the previous theorem and definition. b) and c) are proved as in theorem 10.7 in [Din00] since there was not used the σ-additivity of I_X but rather of $(I_X)_z$. ☐

2.5 The Stochastic Integral and Stopping Times

Let T be a stopping time. If $A \in \mathcal{F}_T$, then the stopping time T_A is defined by $T_A(\omega) = T(\omega)$ if $\omega \in A$ and $T_A(\omega) = \infty$ if $\omega \notin A$. With this notation the predictable rectangles $(s,t] \times A$ with $A \in \mathcal{F}_s$ could be written as stochastic intervals $(s_A, t_A]$. Another notation we will use is $I_X[0,T]$ for $I_X([0,T] \times \Omega$.

Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$ be an additive summable process

Proposition 8. *For any stopping time* T *we have* $X_T \in L^p_E$ *and* $I_X[0,T] = X_T$ *for* T *simple. For any decreasing sequence* (T_n) *of simple stopping times such that* $T_n \downarrow T$*, and for every* $z \in (L^p_G)^*$ *we have*

$$\langle I_X([0,T])y, z\rangle = \lim_n \langle X_{T_n} y, z\rangle, \tag{1}$$

where the bracket represents the duality between L^p_G *and* $(L^p_G)^*$*.*

Proof. Assume first that T is a simple stopping time of the form

$$T = \sum_{1 \le i \le n} 1_{A_i} t_i,$$

with $0 < t_i \le \infty$, $t_i \ne t_j$ for $i \ne j$, $A_i \in \mathcal{F}_{t_i}$ are mutually disjoint and $\bigcup_{1 \le i \le n} A_i = \Omega$. Then $[0,T] = \bigcup_{1 \le i \le n} [0,t_i] \times A_i$ is a disjoint union. Hence $I_X([0,T]) = \sum_i I_X([0,t_i] \times A_i) = \sum_i 1_{A_i} X_{t_i} = X_T$. Since $I_X : \mathcal{P} \to L^p_E$, we conclude that $X_T \in L^p_E$.

Next, assume that (T_n) is a sequence of simple stopping times such that $T_n \downarrow T$. Then $[0,T_n] \downarrow [0,T]$. Since $(I_X)_z$ is σ-additive in F^*, for any $y \in F$ and $z \in (L^p_G)^*$, we have

$$\langle I_X([0,T])y, z\rangle = \langle (I_X)_z([0,T]), y\rangle = \lim_{n\to\infty} \langle (I_X)_z([0,T_n]), y\rangle$$

$$= \lim_{n\to\infty} \langle I_X([0,T_n]y, z\rangle = \lim_{n\to\infty} \langle X_{T_n} y, z\rangle.$$

and the relation (4.1) is proven. It remains to prove that $X_T \in L^p_E$. Since $X_{T_n}(\omega) \to X_T(\omega)$ it follows that X_T is $\mathcal{F}-$measurable. We will prove that $|X_{T_n}| \in L^p$ to deduce that $X_{T_n} \in L^p_G$.

We saw before that for $y \in F$ and $z \in (L^p_G)^*$ the sequence $\langle (I_X)([0,T_n])y, z\rangle$ is convergent hence bounded, i.e.

$$\sup_n |\langle (I_X)([0,T_n])y, z\rangle| < \infty, \text{ for } y \in F, z \in (L^p_G)^*.$$

By the Banach-Steinhauss Theorem, we have

$$\sup_n \|I_X([0,T_n]y\|_{L^p_G} < \infty, \text{ for } y \in F$$

hence

$$\sup_n \|I_X([0,T_n]\|_{L^p_E} < \infty.$$

or $\sup_n \|X_{T_n}]\|_{L^p_E} < \infty$, which is equivalent to $\sup_n \int |X_{T_n}|^p dP < \infty$. Now $|X_T|^p = \lim |X_{T_n}|^p = \liminf |X_{T_n}|^p$. If we apply Fatou Lemma we get:

$$\int |X_T|^p dP = \int \liminf |X_{T_n}|^p \leq \liminf \int |X_{T_n}|^p dP \leq \sup \int |X_{T_n}|^p dP < \infty.$$

therefore $X_T \in L^p_G$. $\square$

Proposition 9. *Let $S \leq T$ be stopping times and $h : \Omega \to F$ be an $\mathcal{F}_S$-measurable, simple random variable. Then for any pair $(T^n)_n$, $(S^n)_n$ of sequences of simple stopping times, with $T^n \downarrow T, S^n \downarrow S$, such that $S^n \leq T^n$ for each n, we have*

$$\langle \int h1_{(S,T]}\, dI_X, z\rangle = \lim_n \langle h(X_{T^n} - X_{S^n}), z\rangle, \textit{ for } z \in (L^p_G)^*, \tag{2}$$

where the bracket represents the duality between L^p_G and $(L^p_G)^*$.

Proof. First we prove that there are two sequences (T^n) and (S^n) of simple stopping times such that $T^n \downarrow T, S^n \downarrow S$ and $S^n \leq T^n$. In fact, there are two sequences of simple stopping times T^n and P^n such that $P^n \downarrow S$ and $T^n \downarrow T$. Consider, now, $S^n = P^n \wedge T^n$. Since P^n and T^n are stopping times, S^n is a stopping time and $S^n \leq T^n$. On the other hand, observe that $S \leq S^n \leq P^n$

and $\lim P^n = S$. Therefore $lim_{n\to\infty} S^n = S$ too. So we have $S^n \downarrow S$ and $S^n \le T^n$.

Now we want to prove (4.2). Assume first $h = 1_A y$ with $A \in \mathcal{F}_S$ and $y \in F$. Then

$$\int h 1_{(S,T]}\, dI_X = \int 1_A y 1_{(S,T]}\, dI_X = \int 1_{(S_A,T_A]} y\, dI_X = I_X((S_A,T_A])y.$$

For any sequence of simple stopping times (T^n) and (S^n) with $T^n \downarrow T$, $S^n \downarrow S$ and $S^n \le T^n$, we have $T^n_A \downarrow T_A$ and $S^n_A \downarrow S_A$. Therefore, applying Proposition 8 for every $z \in (L^p_G)^*$, we get

$$\begin{aligned}\langle \int h 1_{(S,T]}\, dI_X, z\rangle &= \langle I_X((S_A,T_A])y, z\rangle = \langle [I_X([0,T_A]) - I_X([0,S_A])]y, z\rangle \\ &= \lim_{n\to\infty} \langle X_{T^n_A} y, z\rangle - \lim_{n\to\infty} \langle X_{S^n_A} y, z\rangle = \lim_{n\to\infty} \langle (X_{T^n_A} - X_{S^n_A})y, z\rangle \\ &= \lim_{n\to\infty} \langle 1_A (X_{T^n} - S_{X^n})y, z\rangle = \lim_{n\to\infty} \langle h(X_{T^n} - X_{S^n}), z\rangle\end{aligned}$$

Then the equality holds for any $\mathcal{F}_S$-step function h. □

Proposition 10. *Let $S \le T$ be stopping times and assume that either*
(i) $h : \Omega \to \mathbb{R}$ *is a simple, $\mathcal{F}_S$-measurable function and $H \in L^1_{F,G}(X)$,*
or
(ii) *The measure I_X is $\sigma-$additive, $h : \Omega \to F$ is a simple, $\mathcal{F}_S$-measurable function and $H \in L^1_{\mathbb{R},E}(X)$.*

If $\int 1_{(S,T]} H\, dI_X \in L^p_G$ in case (i) and $\int 1_{(S,T]} H\, dI_X \in L^p_E$ in case (ii) then

$$\int h 1_{(S,T]} H\, dI_X = h \int 1_{(S,T]} H\, dI_X.$$

Proof. Assume first hypothesis (i). Let (T^n) and (S^n) be two sequences of simple stopping times such that $T^n \downarrow T$, $S^n \downarrow S$ and $S^n \le T^n$. Assume $H = 1_{(s,t]\times A} y$ with $A \in \mathcal{F}_s$ and $y \in F$. Then $T^n \wedge t \downarrow T \wedge t, S^n \wedge s \downarrow S \wedge s$. Let $z \in (L^p_G)^*$. Then

$$\langle \int h 1_{(S,T]} H\, dI_X, z\rangle = \langle \int h 1_A y 1_{(S\vee s, T\wedge t]}\, dI_X, z\rangle,$$

where the bracket represents the duality between L^p_G and $(L^p_G)^*$. By (4.2), for the simple $\mathcal{F}_{S\vee s}$-measurable function $h 1_A y$ and the stopping times $(S \vee s) \le (T \wedge t)$ we have

$$\langle h \int 1_{(S,T]} H\, dI_X, z\rangle = \langle \int 1_{(S,T]} H dI_X, hz\rangle = \langle \int 1_{(S\vee s, T\wedge t]} 1_A y dI_X, hz\rangle$$

$$= \lim \langle 1_A y (X_{T^n \wedge t} - X_{S^n \vee s}), hz \rangle$$
$$= \lim \langle h 1_A y (X_{T^n \wedge t} - X_{S^n \vee s}), z \rangle = \langle \int h 1_A y 1_{(S \vee s, T \wedge t]} dI_X, z \rangle$$
$$= \langle \int h 1_A y 1_{(s,t]} 1_{(S,T]} dI_X, z \rangle = \langle \int h H 1_{(S,T]} dI_X, z \rangle$$

If $H = 1_{\{0\} \times A} y$ with $A \in \mathcal{F}_0$ and $y \in F$, since $1_{(S,T]} 1_{\{0\} \times A} = 0$ we have

$$\langle h \int 1_{(S,T]} H \, dI_X, z \rangle = 0 = \langle \int h H 1_{(S,T]} dI_X, z \rangle.$$

It follows that for any $B \in \mathcal{R}$ we have

$$\langle \int h 1_{(S,T]} 1_B y \, dI_X, z \rangle = \langle h \int 1_{(S,T]} 1_B y \, dI_X, z \rangle. \qquad (*)$$

The class $\mathcal{M}$ of sets $B \in \mathcal{P}$ for which the above equality holds for all $z \in (L^p_G)^*$ is a monotone class: in fact, let B_n be a monotone sequence of sets from $\mathcal{M}$ and let $B = \lim B_n$. For each n we have

$$\int h 1_{(S,T]} 1_{B_n} y d(I_X)_z = \langle h \int 1_{(S,T]} 1_{B_n} y dI_X, z \rangle.$$

Since $h 1_{(S,T]} 1_{B_n} y$ is a sequence of bounded functions converging to $h 1_{(S,T]} 1_B y$ (h is a step-function) with $|h 1_{(S,T]} 1_{B_n} y| \leq |h||y|$, we can apply Lebesgue Theorem and conclude that $\int h 1_{(S,T]} 1_{B_n} y d(I_X)_z \to \int h 1_{(S,T]} 1_B y d(I_X)_z$. Using the same reasoning we can conclude that $\int 1_{(S,T]} 1_{B_n} y d(I_X)_{hz} \to \int 1_{(S,T]} 1_B y d(I_X)_{hz}$. hence we have

$$\langle \int h 1_{(S,T]} 1_B y \, dI_X, z \rangle = \lim_n \langle \int h 1_{(S,T]} 1_{B_n} y \, dI_X, z \rangle = \lim_n \langle h \int 1_{(S,T]} 1_{B_n} y \, dI_X, z \rangle$$
$$= \langle h \lim_n \int 1_{(S,T]} 1_{B_n} y \, dI_X, z \rangle = \langle h \int 1_{(S,T]} 1_B y \, dI_X, z \rangle$$

Since the class $\mathcal{M}$ of sets satisfying equality (*) is a monotone class containing $\mathcal{R}$ we conclude that the equality (*) is satisfied by all $B \in \mathcal{P}$.

It follows that for any predictable, simple process H and for each $z \in (L^p_G)^*$ we have

$$\langle \int h 1_{(S,T]} H \, dI_X, z \rangle = \langle h \int 1_{(S,T]} H \, dI_X, z \rangle \qquad (**)$$

Consider now the general case. If $H \in L^1_{F,G}(X)$, then there is a sequence (H^n) of simple, predictable processes such that $H^n \to H$ and $|H^n| \leq |H|$. We apply Lebesgue's Theorem and deduce that

$$\int h 1_{(S,T]} H^n \, d(I_X)_z \to \int h 1_{(S,T]} H \, d(I_X)_z, \tag{1}$$

and

$$\int 1_{(S,T]} H^n \, d(I_X)_{hz} \to \int 1_{(S,T]} H \, d(I_X)_{hz}. \tag{2}$$

By equality (**) for each n we have

$$\begin{aligned} \int h 1_{(S,T]} H^n \, d(I_X)_z = \langle \int h 1_{(S,T]} H^n \, dI_X, z \rangle &= \langle h \int 1_{(S,T]} H^n \, dI_X, z \rangle \\ = \langle \int 1_{(S,T]} H^n \, dI_X, hz \rangle &= \int 1_{(S,T]} H^n \, d(I_X)_{hz} \end{aligned}$$

By (1) and (2) we deduce that

$$\int h 1_{(S,T]} H \, d(I_X)_z = \int 1_{(S,T]} H \, d(I_X)_{hz},$$

which is equivalent to

$$\langle \int h 1_{(S,T]} H \, dI_X, z \rangle = \langle \int 1_{(S,T]} H \, dI_X, hz \rangle.$$

We conclude that

$$\int h 1_{(S,T]} H \, dI_X = h \int 1_{(X,T]} H \, dI_X, \text{ a.e.}$$

Assume now hypothesis (ii). Since the measure I_X is $\sigma-$additive the process X is summable. Then observe that the assumptions of (ii) are the same as the assumptions in Proposition 11.5 (ii) of [Din00]. Hence

$$\int h 1_{(S,T]} H \, dI_X = h \int 1_{(X,T]} H \, dI_X,$$

which concludes our proof. □

Proposition 11. *Let $X : \mathbb{R} \times \Omega \to E \subset L(F, G)$ be a p-additive summable process relative to (F, G) and T a stopping time.*
a) For every $z \in (L^p_G)^*$ and every $B \in \mathcal{P}$ we have:

$$(I_{X^T})_z(B) = (I_X)_z(B \cap [0, T]).$$

b) The measure $I_{X^T} : \mathcal{R} \to L^p_E$ has finite semivariation relative to (F, L^p_G)
c) If T is a simple stopping time then the process X^T is summable.

Proof. a) First we prove that if T and S are simple stopping times then we have $I_X((S, T]) = X_T - X_S$.

Assume that T is a simple stopping time of the form

$$T = \sum_{1 \le i \le n} 1_{A_i} t_i,$$

with $0 < t_i \le \infty$, $t_i \ne t_j$ for $i \ne j$, $A_i \in \mathcal{F}_{t_i}$ are mutually disjoint and $\bigcup_{1 \le i \le n} A_i = \Omega$. Then $[0, T] = \bigcup_{1 \le i \le n} [0, t_i] \times A_i$ is a disjoint union. Hence $I_X([0, T]) = \sum_i I_X([0, t_i] \times A_i) = \sum_i 1_{A_i} X_{t_i} = X_T$. Since $(S, T] = [0, T] - [0, S]$ and I_X is an additive measure, we have $I_X((S, T]) = I_X([0, T]) - I_X([0, S]) = X_T - X_S$.

Next observe that if T is a simple stopping time then $T \wedge t$ is also a simple stopping time. In fact, if $T = \sum_{1 \le i \le n} 1_{A_i} t_i$ then $T \wedge t = \sum_{i: t_i < t} 1_{A_i} t_i + \sum_{i: t_i \ge t} 1_{A_i} t$ which is a simple stopping time.

Now we prove that for $B \in \mathcal{R}$ we have

$$I_{X^T}(B) = I_X([0, T] \cap B).$$

In fact, for $A \in \mathcal{F}_0$ we have

$$I_{X^T}(\{0\} \times A) = 1_A X_0 = I_X(\{0\} \times A) = I_X([0, T] \cap (\{0\} \times A)).$$

For $s < t$ and $A \in \mathcal{F}_s$ we have,

$$\begin{aligned} I_{X^T}((s, t] \times A) &= 1_A(X^T_t - X^T_s) = 1_A(X_{T \wedge t} - X_{T \wedge s}) \\ &= 1_A(I_X((T \wedge s, T \wedge t]) = 1_A \int 1_{(s,t]} 1_{[0,T]} dI_X \\ &= \int 1_A 1_{(s,t]} 1_{[0,T]} dI_X = I_X([0, T] \cap ((s, t] \times A)). \qquad (*) \end{aligned}$$

We used the above Proposition 10 with $h = 1_A$, $(S, T] = (s, t]$ and $H = 1_{[0,T]}$.

Next we consider the general case, with T a stopping time.

For $A \in \mathcal{F}_0$ we have

$$I_{X^T}(\{0\} \times A) = 1_A X_0 = I_X(\{0\} \times A) = I_X([0,T] \cap (\{0\} \times A)).$$

Let $y \in F$ and $z \in (L^p_G)^*$. We have

$$\begin{aligned} &\langle (I_{X^T})_z(\{0\} \times A), y\rangle = \langle I_{X^T}(\{0\} \times A) y, z\rangle \\ =&\langle I_X([0,T] \cap (\{0\} \times A))y, z\rangle = \langle (I_X)_z([0,T] \cap (\{0\} \times A)), y\rangle \end{aligned} \tag{1}$$

For $s < t$ and $A \in \mathcal{F}_s$ we have,

$$I_{X^T}((s,t] \times A) = 1_A(X^T_t - X^T_s) = 1_A(X_{T\wedge t} - X_{T\wedge s}) \tag{2}$$

Let T_n be a sequence of simple stopping times such that $T_n \downarrow T$. Let $y \in F$ and $z \in (L^p_G)^*$. We have by (2):

$$\begin{aligned} \langle (I_{X^T})_z((s,t] \times A), y\rangle &= \langle I_{X^T}((s,t] \times A)y, z\rangle = \langle 1_A(X_{T\wedge t} - X_{T\wedge s})y, z\rangle \\ &= \lim_{n\to\infty} \langle 1_A(X_{T_n\wedge t} - X_{T_n\wedge s})y, z\rangle. \end{aligned}$$

By (*) we have:

$$\begin{aligned} &\lim_{n\to\infty} \langle 1_A(X_{T_n\wedge t} - X_{T_n\wedge s})y, z\rangle = \lim_{n\to\infty} \langle I_X([0,T_n] \cap ((s,t] \times A))y, z\rangle \\ =& \lim_{n\to\infty} \langle (I_X)_z([0,T_n] \cap ((s,t] \times A)), y\rangle = \langle (I_X)_z([0,T] \cap ((s,t] \times A)), y\rangle \end{aligned} \tag{3}$$

since $(I_X)_z$ is σ-additive. By (1) and (3) and the fact that $(I_{X^T})_z$ is σ-additive we deduce that

$$(I_{X^T})_z(B) = (I_X)_z(B \cap [0,T]), \text{ for all } B \in \mathcal{R} \tag{4}$$

Since $(I_X)_z$ is σ-additive we deduce that $(I_{X^T})_z$ is σ-additive, hence it can be extended to a σ-additive measure on $\mathcal{P}$. Since $(I_{X^T})_z(B) = (I_X)_z(B \cap [0,T])$ for all $B \in \mathcal{R}$ we deduce that

$$(I_{X^T})_z(B) = (I_X)_z(B \cap [0,T]), \text{ for all } B \in \mathcal{P},$$

b) Let A be a set in $\mathcal{R}$. By Proposition 4.15 in [Din00] we have $svar_{F,L^p_G} I_{X^T}(A) < \infty$ if and only if $var(I_{X^T})_z(A) < \infty$ for each $z \in (L^p_G)^*$. But

$$\sup_{z\in((L^p_G)^*)_1} var(I_{X^T})_z(A) = \sup_{z\in((L^p_G)^*)_1} var(I_X)_z(A \cap [0,T])$$

$$= svar_{F,L^p_G} I_X(A \cap [0,T]) < \infty,$$

and Assertion b) is proved.
c) Assume T is a simple stopping time. By the equality (*) we have

$$I_{X^T}(B) = I_X([0,T] \cap B), \text{ for } B \in \mathcal{R}.$$

Since X is p-additive summable relative to (F,G), I_X has a canonical additive extension $I_X : \mathcal{P} \to L^p_G$. The equality

$$I_{X^T}(A) = I_X([0,T] \cap A), \text{ for } A \in \mathcal{P},$$

defines an additive extension of I_{X^T} to $\mathcal{P}$. Since the measure I_X has finite semivariation relative to (F, L^p_G) (X is additive summable), the measure I_{X^T} has finite semivariation relative to (F, L^p_G) also. Moreover, for each $z \in (L^p_G)^*$, by Assertion a), the measure $(I_{X^T})_z$ defined on $\mathcal{P}$ is $\sigma-$additive. Therefore X^T is additive summable. We have $|(I_{X^T})_z|(A) = |(I_X)_z|([0,T] \cap A)$ for $A \in \mathcal{P}$ since $|(I_X)_z|$ is the canonical extension of its restriction on $\mathcal{R}$. Then $|(I_{X^T})_z|$ is the canonical extension of its restriction to $\mathcal{R}$. it follows that I_{X^T} is the canonical extension of its restriction to $\mathcal{R}$. □

The next theorem gives the relationship between the stopped stochastic integral and the integral of the process $1_{[0,T]}H$. The same type of relation was proved in Theorem 11.6 in [Din00].

Theorem 12. *Let $H \in L^1_{F,G}(X)$ and let T be a stopping time. Then $1_{[0,T]}H \in L^1_{F,G}(X)$ and*

$$(1_{[0,T]}H) \cdot X = (H \cdot X)^T.$$

Proof. Suppose first that T is a simple stopping time of the form

$$T = \sum_{1 \le i \le n} 1_{A_i} t_i$$

with $0 \le t_1 \le t_2 \le \ldots t_n \le +\infty$, $A_i \in \mathcal{F}_{t_i}$ mutually disjoint and with union Ω. Then for $t \ge 0$ we have

$$(H \cdot X)^T_t(\omega) = \sum_{1 \le i \le n} (H \cdot X)_{t_i \wedge t}(\omega) 1_{A_i}(\omega).$$

In fact, for $\omega \in \Omega$ there is $1 \leq i \leq n$ such that $\omega \in A_i$. Then $T(\omega) = t_i$, hence

$$(H \cdot X)_t^T(\omega) = (H \cdot X)_{t_i \wedge t}(\omega).$$

On the other hand

$$(1_{[0,T]}H) \cdot X)_t(\omega) = \sum_{1 \leq i \leq n} (H \cdot X)_{t_i \wedge t}(\omega) 1_{A_i}(\omega).$$

In fact,

$$\begin{aligned}
(\int_{[0,t]} 1_{[0,T]} H dI_X)(\omega) &= (\int_{[0,t]} \sum_{1 \leq i \leq n} 1_{[0,t_i]} 1_{A_i} H dI_X)(\omega) = \sum_{1 \leq i \leq n} (\int_{[0,t_i \wedge t]} 1_{A_i} H dI_X)(\omega) \\
&= \sum_{1 \leq i \leq n} (\int_{[0,\infty]} H 1_{A_i} dI_X)(\omega) - \sum_{1 \leq i \leq n} (\int_{(t_i \wedge t,\infty]} 1_{A_i} H dI_X)(\omega) \\
&= (\int_{[0,\infty]} H dI_X)(\omega) - \sum_{1 \leq i \leq n} 1_{A_i}(\omega) (\int_{(t_i,\infty]} H dI_X)(\omega) \\
&= \sum_{1 \leq i \leq n} 1_{A_i}(\omega) (\int_{[0,\infty]} H dI_X)(\omega) - \sum_{1 \leq i \leq n} 1_{A_i}(\omega) (\int_{(t_i,\infty]} H dI_X)(\omega) \\
&= \sum_{1 \leq i \leq n} 1_{A_i}(\omega) (\int_{[0,t_i \wedge t]} H dI_X)(\omega) = \sum_{1 \leq i \leq n} (H \cdot X)_{t_i \wedge t}(\omega) 1_{A_i}(\omega),
\end{aligned}$$

where the 4th equality is obtained by applying Proposition 10, with $h = 1_{A_i}$.

Hence, for T simple, we have$1_{[0,T]}H \in L^1_{F,G}(X)$ and

$$(1_{[0,T]}H) \cdot X = (H \cdot X)^T.$$

Now choose T arbitrary. Then there is a decreasing sequence (T_n) of simple stopping times, such that $T_n \downarrow T$.

Note first that since $(H \cdot X)$ is cadlag we have

$$(H \cdot X)^{T_n} \to (H \cdot X)^T. \tag{1}$$

Moreover for $t \geq 0$ we have $1_{[0,T_n \wedge t]}H \downarrow 1_{[0,T_n \wedge t]}H$ pointwise. Since $1_{[0,T_n \wedge t]}H \in L^1_{F,G}(X)$, for each $(z \in L^p_G)^*$ we have $1_{[0,T_n \wedge t]}H \in L^1_F(|(I_X)_z|)$, hence

$$\langle \int 1_{[0,T_n \wedge t]} H dI_X, z \rangle = \int 1_{[0,T_n \wedge t]} H d(I_X)_z \to \int 1_{[0,T \wedge t]} H d(I_X)_z = \langle \int 1_{[0,T \wedge t]} H dI_X, z \rangle.$$

By Theorem 4 we conclude that $\int 1_{[0,T\wedge t]} H dI_X = \int_{[0,t]} 1_{[0,T]} H dI_X \in L^p_G$ and

$$\int 1_{[0,T_n\wedge t]} H dI_X \to \int 1_{[0,T\wedge t]} H dI_X,$$

or

$$\int_{[0,t]} 1_{[0,T_n]} H dI_X \to \int_{[0,t]} 1_{[0,T]} H dI_X.$$

Since for each n we have $(1_{[0,T_n]} H \cdot X)_t = (H \cdot X)^{T_n}_t$, by (1) we deduce that $\int_{[0,t]} 1_{[0,T]} H dI_X = (H \cdot X)^T_t$. Hence the mapping $t \mapsto \int_{[0,t]} 1_{[0,T]} H dI_X$ is cadlag, from which we conclude that $1_{[0,T]} H \in L^1_{F,G}(X)$. Moreover

$$(1_{[0,T]} H \cdot X)_t = (H \cdot X)_{T\wedge t} = (H \cdot X)^T_t.$$

□

The next corollary is a useful particular case of the previous theorem:

Corollary 13. *For every stopping time T we have*

$$1_{[0,T]} \cdot X = X^T.$$

Proof. Taking $H = 1 \in L^1_{F,G}(X)$ and applying Theorem 12 we conclude that $1_{[0,T]} \cdot X = X^T$. □

The following theorem gives the same type of results as Theorem 11.8 in[Din00].

Theorem 14. *Let $S \le T$ be stopping times and assume that either*
(i) *$h : \Omega \to \mathbb{R}$ is a simple, $\mathcal{F}_S$-measurable function and $H \in L^1_{F,G}(X)$,*
or
(ii) *The measure I_X is $\sigma-$additive, $h : \Omega \to F$ is a simple, $\mathcal{F}_S$-measurable function and $H \in L^1_{\mathbb{R},E}(X)$.*
Then $1_{(S,T]} H$ and $h 1_{(S,T]} H$ are integrable with respect to X and

$$(h 1_{(S,T]} H) \cdot X = h[(1_{(S,T]} H) \cdot X].$$

Proof. Note that

$$1_{(S,T]} H = 1_{[0,T]} H - 1_{[0,S]} H$$

Assume first the case (i). Applying Theorem 12 for $1_{[0,T]} H$ and $1_{[0,S]} H$ we conclude that $1_{(S,T]} H \in L^1_{F,G}(X)$.

If for each $t \geq 0$ we apply Proposition 10, we obtain

$$\int_{[0,t]} h1_{(S,T]} H dI_X = h \int_{[0,t]} 1_{(S,T]} H dI_X.$$

Since $1_{(S,T]} H \in L^1_{F,G}(X)$ we deduce that $h1_{(S,T]} H \in L^1_{F,G}(X)$ and

$$((h1_{(S,T]} H) \cdot X)_t = h((1_{(S,T]} H) \cdot X)_t,$$

which concludeds the proof of case (i). Case (ii) is treated similarly. □

2.6 The Integral $\int H dI_{X^T}$

In this section we define the set of processes integrable with respect to the measure I_{X^T} with finite semivariation relative to the pair (F, L^p_G).

Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$ be a cadlag, adapted process and assume X is p-additive summable relative to (F, G).

Consider the additive measure $I_X : \mathcal{P} \to L^p_E \subset L(F, L^p_G)$ with bounded semivariation $\tilde{I}_{F,G}$ relative to (F, L^p_G), such that each of the measures $(I_X)_z$ with $z \in (L^p_G)^*$ is σ-additive.

To simplify the notations denote $m = I_{X^T}$. We proved in the previous proposition that the measure $m : \mathcal{R} \to L^p_E \subset L(F, L^p_G)$ has bounded semivariation relative to (F, L^p_G), on $\mathcal{R}$, and for each $z \in (L^p_G)^*$ the measures m_z, is σ-additive. In order for the process X^T to be additive summable we need the measure $m : \mathcal{R} \to L^p_E$ to have an extension $m : \mathcal{P} \to L^p_E$ with finite semivariation and such that each of the measures m_z with $z \in (L^p_G)^*$ is $\sigma-$additive. Applying Theorem 7 from Bongiorno–Dinculeanu, citeBD2001, the measure m has a unique canonical extension $m : \mathcal{P} \to (L^p_E)^{**}$, with bounded semivariation such that for each $z \in (L^p_G)^*$ the measure m_z, is σ-additive and has bounded variation $|m_z|$, therefore X^T is summable.

Then we have

$$\tilde{m}_{F,L^p_G} = \sup\{|m_z| : z \in (L^p_G)^*, \|z\|_q \leq 1\}.$$

We denote by $\mathcal{F}_{F,G}(X^T)$ the space of predictable processes $H : \mathbb{R}_+ \times \Omega \to F$ such that

$$\tilde{m}_{F,G}(H) = \tilde{m}_{F,L^p_G}(H) = \sup\{\int |H| d|m_z| : \|z\| \leq 1\} < \infty.$$

Let $H \in \mathcal{F}_{F,G}(X^T)$; then $H \in L^1_F(|m_z|)$ for every $z \in (L^p_G)^*$, hence the integral $\int H dm_z$ is defined and is a scalar. The mapping $z \mapsto \int H dm_z$ is a linear continuous functional on $(L^p_G)^*$, denoted $\int H dm$. Therefore, $\int H dm \in (L^p_G)^{**}$,

$$\langle \int H dm, z \rangle = \int H dm_z, \text{ for } z \in (L^p_G)^*.$$

We denote by $L^1_{F,G}(X^T)$ the set of processes $H \in \mathcal{F}_{F,G}(I^T_X)$ satisfying the following two conditions:

a) $\int_{[0,t]} H dm \in L^p_G$ for every $t \in \mathbb{R}_+$;

b) The process $(\int_{[0,t]} H dm)_{t \geq 0}$ has a cadlag modification.

Theorem 15. *Let $X : \mathbb{R} \to E \subset L(F,G)$ be a p-additive summable process relative to (F,G) and T a stopping time.*

a) We have $H \in \mathcal{F}_{F,G}(X^T)$ iff $1_{[0,T]}H \in \mathcal{F}_{F,G}(X)$ and in this case we have:

$$\int H dI_{X^T} = \int 1_{[0,T]} H dI_X.$$

b) We have $H \in L^1_{F,G}(X^T)$ iff $1_{[0,T]}H \in L^1_{F,G}(X)$ and in this case we have:

$$H \cdot X^T = (1_{[0,T]}H) \cdot X.$$

If $H \in L^1_{F,G}(X)$, then $H \in L^1_{F,G}(X^T), 1_{[0,T]}H \in L^1_{F,G}(X)$ and

$$(H \cdot X)^T = H \cdot X^T = (1_{[0,T]}H) \cdot X.$$

Proof. a) Define $m : \mathcal{R} \to E$ by $m(B) = I_{X^T}(B)$ for $B \in \mathcal{R}$. We proved in Theorem 11 (a) that for every $z \in (L^p_G)^*$ we have

$$m_z(B) = (I_X)_z(B \cap [0,T]), \text{ for all } B \in \mathcal{R}. \qquad (*)$$

Since $(I_X)_z((\cdot) \cap [0,T])$ is a $\sigma-$additive measure, with bounded variation on $\mathcal{P}$ satisfying (*) and since $\mathcal{P}$ is the $\sigma-$algebra generated by $\mathcal{R}$, by the uniqueness theorem 7.4 in [Din00] we conclude that

$$m_z(B) = (I_X)_z(B \cap [0,T]), \text{ for all } B \in \mathcal{P}.$$

Let $H \in \mathcal{F}_{F,G}(X^T) = \bigcap_{\|z\|_q \leq 1, z \in (L^p_G)^*} L^1_F(m_z)$. From the previous equality we deduce that

$$\int H dm_z = \int 1_{[0,T]} H d(I_X)_z,$$

therefore

$$\int H dI_{X^T} = \int 1_{[0,T]} H dI_X,$$

and this is the equality in Assertion a).
b) To prove Assertion b) we replace H with $1_{[0,t]}H$ in the previous assertion and deduce that $1_{[0,t]}H \in \mathcal{F}_{F,G}(X^T)$ iff $1_{[0,t]}1_{[0,T]}H \in \mathcal{F}_{F,G}(X)$ and in this case we have

$$\int_{[0,t]} H dI_{X^T} = \int_{[0,t]} 1_{[0,T]} H dI_X.$$

It follows that $H \in L^1_{F,G}(X^T)$ iff $1_{[0,T]}H \in L^1_{F,G}(X)$ and in this case we have

$$(H \cdot X^T)_t = ((1_{[0,T]}H) \cdot X)_t.$$

If now $H \in L^1_{F,G}(X)$, then, from Theorem 12 we deduce that
$1_{[0,T]}H \in L^1_{F,G}(X)$ and

$$(1_{[0,T]}H) \cdot X = (H \cdot X)^T.$$

□

2.7 Convergence Theorems

Assume X is p-additive summable relative to (F, G). In this section we shall present several convergence theorems.

Lemma 16. *Let (H^n) be a sequence in $L^1_{F,G}(X)$ and assume that $H^n \to H$ in $\mathcal{F}_{F,G}(X)$. Then there is a subsequence (r_n) such that*

$$(H^{r_n} \cdot X)_t \to (H \cdot X)_t = \int_{[0,t]} H dI_X, \text{ a.s., as } n \to \infty,$$

uniformly on every bounded time interval.

Proof. Since H^n is a convergent sequence in $\mathcal{F}_{F,G}(X)$ there is a subsequence H^{r_n} of (H^n) such that

$$\tilde{I}_{F,G}(H^{r_n} - H^{r_{n+1}}) \leq 4^{-n}, \text{ for each } n.$$

Let $t_0 > 0$. Define the stopping time

$$u_n = \inf\{t : |(H^{r_n} \cdot X)_t - (H^{r_{n+1}} \cdot X)_t| > 2^{-n}\} \wedge t_0.$$

By Theorem 12 applied to the stopping time u_n, we obtain

$$(H^{r_n} \cdot X)_{u_n} = (H^{r_n} \cdot X)^{u_n}_{\infty} = ((1_{[0,u_n]} H^{r_n}) \cdot X)_{\infty} = \int_{[0,u_n]} H^{r_n} dI_X,$$

hence

$$\begin{aligned} E(|(H^{r_n} \cdot X)_{u_n} - (H^{r_{n+1}} \cdot X)_{u_n}|) &= E(|\int_{[0,u_n]} H^{r_n} dI_X - \int_{[0,u_n]} H^{r_{n+1}} dI_X|) \\ = E(|\int_{[0,u_n]} ((H^{r_n} - H^{r_{n+1}}) dI_X|) &= (\| \int_{[0,u_n]} (H^{r_n} - H^{r_{n+1}}) dI_X \|_{L^1_G} \\ \le \| \int_{[0,u_n]} (H^{r_n} - H^{r_{n+1}}) dI_X \|_{L^p_G} &\le \tilde{I}_{F,G}(H^{r_n} - H^{r_{n+1}}) \le 4^{-n}. \qquad (*) \end{aligned}$$

Using inequality (*) and following the same techniques as in Theorem 12.1 a) in [Din00] one could show first that the sequence $(H^{r_n} \cdot X)_t$ is a Cauchy sequence in L^p_G uniformly for $t < t_0$ and then conclude that

$$(H^{r_n} \cdot X)_t \to \int_{[0,t]} H dI_X,$$

uniformly on every bounded time interval. □

Theorem 17. *Let (H^n) be a sequence from $L^1_{F,G}(X)$ and assume that $H^n \to H$ in $\mathcal{F}_{F,G}(X)$. Then:*
a) *$H \in L^1_{F,G}(X)$.*
b) *$(H^n \cdot X)_t \to (H \cdot X)_t$, in L^p_G, for $t \in [0, \infty]$.*
c) *There is a subsequence (r_n) such that*

$$(H^{r_n} \cdot X)_t \to (H \cdot X)_t, \text{ a.s., as } n \to \infty,$$

uniformly on every bounded time interval.

Proof. For every $t \ge 0$ we have $1_{[0,t]} H^n \to 1_{[0,t]} H$ in $\mathcal{F}_{F,G}(X)$. Since the integral is continuous, we deduce that

$$(H^n \cdot X)_t = \int_{[0,t]} H^n dI_X \to \int_{[0,t]} H dI_X, \text{ in } (L^p_{G^*})^*.$$

Since $H^n \in L^1_{F,G}(X)$ we have $\int_{[0,t]} H^n dI_X \in L^p_G$ and

$$(H^n \cdot X)_t \to \int_{[0,t]} H dI_X, \text{ in } L^p_G.$$

From the previous lemma we deduce that there is a subsequence (H^{r_n}) such that

$$(H^{r_n} \cdot X)_t \to (H \cdot X)_t, \text{ a.s., as } n \to \infty,$$

uniformly on every bounded time interval. Since $(H^{r_n} \cdot X)$ are cadlag it follows that the limit is also cadlag, hence $H \in L^1_{F,G}(X)$ which is Assertion a). Hence

$$(H \cdot X)_t = \int_{[0,t]} H dI_X, \text{ a.s.}$$

and therefore $(H^n \cdot X)_t \to (H \cdot X)_t$, in L^p_G, which is Assertion b). Also observe that for the above susequence (H^{r_n}) we have

$$(H^{r_n} \cdot X)_t \to (H \cdot X)_t, \text{ a.s., as } n \to \infty,$$

uniformly on every bounded time interval. □

We can restate Theorem 17 as:

Corollary 18. *$L^1_{F,G}(X)$ is complete.*

Next we state an uniform convergence theorem. Uniform convergence implies convergence in $L^1_{F,G}(X)$.

Theorem 19. *Let (H^n) be a sequence from $\mathcal{F}_{F,G}(X)$. If $H^n \to H$ pointwise uniformly then $H \in \mathcal{F}_{F,G}(X)$ and $H^n \to H$ in $\mathcal{F}_{F,G}(X)$.*

If, in addition, for each n, H^n is integrable, i.e. $H^n \in L^1_{F,G}(X)$ then
a) *$H \in L^1_{F,G}(X)$ and $H^n \to H$ in $L^1_{F,G}(X)$;*
b) *For every $t \in [0, \infty]$ we have $(H^n \cdot X)_t \to (H \cdot X)_t$, in L^p_G.*
c) *There is a subsequence (r_n) such that $(H^{r_n} \cdot X)_t \to (H \cdot X)_t$, a.s. as $n \to \infty$, uniformly on any bounded interval.*

Proof. Assertion a) is immediate. Assertions b), c) and d) follow from Theorem 17. □

Now we shall state Vitali and Lebesgue-type Convergence Theorems. They are direct consequences of the convergence Theorem 17 and of the uniform convergence Theorem 19.

Theorem 20. (Vitali). *Let (H^n) be a sequence from $\mathcal{F}_{F,G}(X)$ and let H be an F-valued, predictable process. Assume that*

(i) $\tilde{I}_{F,G}(H^n 1_A) \to 0$ *as* $\tilde{I}_{F,G}(A) \to 0$, *uniformly in* n
and that any one of the conditions (ii) *or* (iii) *below is true:*
(ii) $H^n \to H$ *in* $\tilde{I}_{F,G}$*-measure;*
(iii) $H^n \to H$ *pointwise and* $I_{F,(L^p_G)^*}$ *is uniformly* σ*-additive (this is the case if* H^n *are real-valued, i.e.,* $F = \mathbb{R}$*).*

Then:

a) $H \in \mathcal{F}_{F,G}(X)$ *and* $H^n \to H$ *in* $\mathcal{F}_{F,G}(X)$.

Conversely, if $H^n, H \in \mathcal{F}_{F,G}(\mathcal{B}, X)$ *and* $H^n \to H$ *in* $\mathcal{F}_{F,G}(X)$*, then conditions* (i) *and* (ii) *are satisfied.*

Under the hypotheses (i) *and* (ii) *or* (iii)*, assume, in addition, that* $H^n \in L^1_{F,G}(X)$ *for each* n*. Then*

b) $H \in L^1_{F,G}(X)$ *and* $H^n \to H$ *in* $L^1_{F,G}(X)$*;*
c) *For every* $t \in [0, \infty]$ *we have* $(H^n \cdot X)_t \to (H \cdot X)_t$*, in* L^p_G*;*
d) *There is a subsequence* (r_n) *such that* $(H^{r_n} \cdot X)_t \to (H \cdot X)_t$*, a.s., as* $n \to \infty$*, uniformly on any bounded interval.*

Theorem 21. (Lebesgue). *Let* (H^n) *be a sequence from* $\mathcal{F}_{F,G}(X)$ *and let* H *be an* F*-valued predictable process. Assume that*
(i) *There is a process* $\phi \in \mathcal{F}_{\mathbb{R}}(\mathcal{B}, I_{F,G})$ *such that*

$$|H^n| \leq \phi \text{ for each } n;$$

and that any one of the conditions (ii) *or* (iii) *below is true:*
(ii) $H^n \to H$ *in* $\tilde{I}_{F,G}$*-measure;*
(iii) $H^n \to H$ *pointwise and* $I_{F,L^q_{G^*}}$ *is uniformly* σ*-additive (this is the case if* H^n *are real valued, i.e.,* $F = \mathbb{R}$*).*

Then:

a) $H \in \mathcal{F}_{F,G}(\mathcal{B}, X)$ *and* $H^n \to H$ *in* $\mathcal{F}_{F,G}(X)$.

Assume, in addition that $H^n \in L^1_{F,G}(X)$ *for each* n*. Then*

b) $H \in L^1_{F,G}(X)$ *and* $H^n \to H$ *in* $L^1_{F,G}(X)$*;*
c) *For every* $t \in [0, \infty]$ *we have* $(H^n \cdot X)_t \to (H \cdot X)_t$*, in* L^p_G*;*
d) *There is a subsequence* (r_n) *such that* $(H^{r_n} \cdot X)_t \to (H \cdot X)_t$*, a.s., as* $n \to \infty$*, uniformly on any bounded interval.*

2.8 Summability of the Stochastic Integral

Assume X is p-additive summable relative to (F, G). In this section we are studying the additive summability of the stochastic integral $H \cdot X$ for $F-$valued processes H.

If H is a real valued processes then in order for the stochastic integral $H \cdot X$ to be defined we need each of the measure $(I_X)_z$, for $z \in (L^P_E)^*$, to be $\sigma-$additve, hence the measure I_X would be $\sigma-$additive. Therefore the process X would be summable. In this case the summability of the stochastic integral is proved in Theorem 13.1 of [Din00].

The next theorem shows that if H is $F-$valued then the measure $I_{H\cdot X}$ is $\sigma-$additive even if I_X is just additive.

Theorem 22. *Let $H \in L^1_{F,G}(X)$ be such that $\int_A HdI_X \in L^p_G$ for $A \in \mathcal{P}$. Then the measure $I_{H\cdot X} : \mathcal{R} \to L^p_G$ has a $\sigma-$additive extension $I_{H\cdot X} : \mathcal{P} \to L^p_G$ to $\mathcal{P}$.*

Proof. We first note that $H \cdot X : \mathbb{R}_+ \times \Omega \to G = L(\mathbb{R}, G)$ is a cadlag adapted process with $(H \cdot X)_t \in L^p_G$ for $t \geq 0$ (by the definition of $H \cdot X$).

Since $\int_A HdI_X \in L^p_G$ for every $A \in \mathcal{P}$, by Proposition **??**, with $m = I_X$ and $g = H$, we deduce that HI_X is $\sigma-$additive on $\mathcal{P}$.

Next we prove that for any predictable rectangle $A \in \mathcal{R}$ we have

$$I_{H\cdot X}(A) = \int_A HdI_X. \tag{1}$$

In fact, consider first $A = \{0\} \times B$ with $B \in \mathcal{F}_0$. Using Proposition 10 for $h = 1_B$ we have

$$\begin{aligned} I_{H\cdot X}(\{0\} \times B) = 1_B((H \cdot X)_0) &= 1_B \int_{\{0\}} HdI_X \\ &= \int_{\{0\}} 1_B HdI_X = \int_{\{0\}\times B} HdI_X; \end{aligned}$$

Let now $A = (s, t] \times B$ with $B \in \mathcal{F}_s$. Using Proposition 10 for $h = 1_B$ and $(S, T] = (s, t]$ we have

$$\begin{aligned} &I_{H\cdot X}((s, t] \times B) = 1_B((H \cdot X)_t - (H \cdot X)_s) \\ =&1_B(\int_{[0,t]} HdI_X - \int_{[0,s]} HdI_X) = 1_B \int_{(s,t]} HdI_X \\ =& \int_{(s,t]} 1_B HdI_X = \int_{(s,t]\times B} HdI_X; \end{aligned}$$

and the desired equality is proved.

Since the measure $A \mapsto \int_A H dI_X$ is $\sigma-$additive for $A \in \mathcal{P}$ it will follow that $I_{H\cdot X}$ can be extended to a σ-additive measure on $\mathcal{P}$ by the same equality

$$I_{H\cdot X}(A) = \int_A H dI_X, \text{ for } A \in \mathcal{P}. \tag{2}$$

□

The next theorem states the summability of the stochastic integral.

Theorem 23. *Let $H \in L^1_{F,G}(X)$ be such that $\int_A H dI_X \in L^p_G$ for $A \in \mathcal{P}$. Then:*

a) *$H \cdot X$ is p-summable, hence p-additive summable relative to* $(\mathbb{R}, G)$ *and*

$$dI_{H\cdot X} = d(HI_X).$$

b) *For any predictable process $K \geq 0$ we have*

$$(\tilde{I}_{H\cdot X})_{\mathbb{R},G}(K) \leq (\tilde{I}_X)_{F,G}(KH).$$

c) *If K is a real-valued predictable process and if $KH \in L^1_{F,G}(X)$, then $K \in L^1_{\mathbb{R},G}(H \cdot X)$ and we have*

$$K \cdot (H \cdot X) = (KH) \cdot X.$$

Proof. By Theorem 22 we know that the measure $I_{H\cdot X}$ is $\sigma-$additive. Therefore To prove (a) we only need to show that the extension of $I_{H\cdot X}$ to $\mathcal{P}$ has finite semivariation relative to $(\mathbb{R}, L^p_G)$.

Let $z \in (L^p_G)^*$. From the equality (2) in Theorem 22 we deduce that for every $A \in \mathcal{P}$, and we have

$$(I_{H\cdot X})_z(A)\rangle = \langle I_{H\cdot X}(A), z\rangle = \langle \int_A H dI_X, z\rangle = \int_A H d(I_X)_z.$$

From this we deduce the inequality

$$|(I_{H\cdot X})_z|(A) \leq \int_A |H| d|(I_X)_z|, \text{ for } A \in \mathcal{P}. \tag{*}$$

Taking the supremum for $z \in (L^P_G)^*_1$ we obtain

$$\sup\{|(I_{H\cdot X})_z|(A), z \in (L^P_G)^*_1\} \leq \sup\{\int_A |H| d|(I_X)_z|, z \in (L^P_G)^*_1\}$$

$$\leq \sup\{\int |1_A H| d|(I_X)_z|, z \in (L^P_G)^*_1\}, \text{ for } A \in \mathcal{P}.$$

Therefore

$$(\tilde{I}_{H\cdot X})_{\mathbb{R},G}(A) \leq (\tilde{I}_X)_{F,G}(1_A H) < \infty, \text{ for } A \in \mathcal{P}.$$

It follows that $H \cdot X$ is p-summable, hence $p-$additive summable, relative to $(\mathbb{R}, G)$ and this proves Assertion a).

Since the extension to ¶ of the measure $I_{X\cdot H}$ is $\sigma-$additive and has finite semivariation b) and c) follow from Theorem 13.1 of [Din00].

2.9 Summability Criterion

Let $Z \subset L^q_{E^*}$ be any closed subspace norming for L^p_E. The next theorem differs from the summability criterion in [Din00] by the fact that the restrictive condition $c_0 \notin E$ was not imposed. Also note that this theorem does not give us necessary and sufficient conditions for the sumability of the precess.

Theorem 24. *Let $X : \mathbb{R}_+ \times \Omega \to E$ be an adapted, cadlag process such that $X_t \in L^p_E$ for every $t \geq 0$. Then the Assertions* a)–d) *below are equivalent.*

a) *$I_X : \mathcal{R} \to L^p_E$ has an additive extension $I_X : \mathcal{P} \to Z^*$ such that for each $g \in Z$, the real valued measure $\langle I_X, g\rangle$ is a $\sigma-$additive on $\mathcal{P}$.*

b) *I_X is bounded on $\mathcal{R}$;*

c) *For every $g \in Z$, the real valued measure $\langle I_X, g\rangle$ is bounded on $\mathcal{R}$;*

d) *For every $g \in Z$, the real valued measure $\langle I_X, g\rangle$ is $\sigma-$additive and bounded on $\mathcal{R}$.*

Proof. The proof will be done as follows: b) $\Longleftrightarrow$ c) $\Longleftrightarrow$ d) and a) $\Longleftrightarrow$ d).

b) $\Longrightarrow$ c) and c) $\Longrightarrow$ b) can be proven in the same fashion as in [Din00].

c) $\Longrightarrow$ d) Assume c), and let $g \in Z$. The real valued measure $\langle I_X, g\rangle$ is defined on $\mathcal{R}$ by

$$\langle I_X, g\rangle(A) = \langle I_X(A), g\rangle = \int \langle I_X(A), g\rangle dP, \text{ for } A \in \mathcal{R}.$$

By assumption, $\langle I_X, g\rangle$ is bounded on $\mathcal{R}$. We need to prove that the measure $\langle I_X, g\rangle$ is $\sigma-$ additive. For that consider, as in [Din00], the real-valued process $XG = (\langle X_t, G_t\rangle)_{t\geq 0}$, where $G_t = E(g|\mathcal{F}_t)$ for $t \geq 0$. Then $XG : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a cadlag, adapted process and it can be proven, using the same techniques as in [Din00] that it is a quasimartingale.

Now, for each n, define the stopping time

$$T_n(\omega) = \inf\{t : |X_t| > n\}.$$

Then $T_n \uparrow \infty$ and $|X_t| \leq n$ on $[0, T_n)$. Since XG is a quasimartingale on $(0, \infty]$, we know that $(XG)_{T_n} \in L^1$ (Proposition A 3.5 in [BD87]: XG is a quasimartingale on $(0, \infty]$ iff XG is a quasimartingale on $(0, \infty)$ and $\sup_t \|XG\|_1 < \infty$.)

Moreover,

$$\begin{aligned} |(XG)_t^{T_n}| &= |(XG)_t| 1_{\{t<T_n\}} + |(XG)_{T_n}| 1_{\{t \geq T_n\}} \\ &\leq |X_t||G_t| 1_{\{t<T_n\}} + |(XG)_{T_n}| 1_{\{t \geq T_n\}} \\ &\leq n|G_t| 1_{\{t<T_n\}} + |(XG)_{T_n}| 1_{\{t \geq T_n\}}. \end{aligned} \tag{2}$$

Besides, since $G_t = E(g|\mathcal{F}_t)$ it follows that G is a uniformly integrable martingale.

Next we prove that the family $\{(XG)_T^{T_n}, T \text{ simple stopping time}\}$ is uniformly integrable.

In fact, note that by inequality (2) we have

$$\begin{aligned} &\int_{\{|(XG)_T^{T_n}|>p\}} |(XG)_T^{T_n}| dP \\ &\leq \int_{\{|(XG)_T^{T_n}|>p\} \cap \{T<T_n\}} n|(XG)_T^{T_n}| dP + \int_{\{|(XG)_T^{T_n}|>p\} \cap \{T \geq T_n\}} |(XG)_{T_n}| dP \end{aligned} \tag{3}$$

Now observe that

$$\begin{aligned} &\{|(XG)_T| > p\} \cap \{T < T_n\} = \{|\langle X_T, G_T\rangle| > p\} \cap \{T < T_n\} \\ &\subset \{|X|_T |G|_T > p\} \cap \{T < T_n\} \subset \{p < n|G_T|\} \cap \{T < T_n\} \subset \{p < nG_T\} \end{aligned}$$

Since G is a uniformly integrable martingale, it is a martingale of class D; from $n|G_t| 1_{\{t<T_n\}} \leq n|G_t|$ we deduce that $n|G_t| 1_{\{t<T_n\}}$ is a martingale of class (D):

$$\begin{aligned} &\lim_{p\to\infty} \int_{\{n|G_t|1_{\{t<T_n\}}>p\}} n|G_t| 1_{\{t<T_n\}} dP \leq \lim_{p\to\infty} \int_{\{n|G_t|>p\}} n|G_t| dP \\ &= n \lim_{p\to\infty} \int_{\{|G_t|>\frac{p}{n}\}} n|G_t| dP = \lim_{\frac{p}{n}\to\infty} \int_{\{n|G_t|>p\}} n|G_t| dP = 0. \end{aligned}$$

Hence there is a $p_{1\epsilon}$ such that for any $p \geq p_{1\epsilon}$ and any simple stopping time T we have

$$\int_{\{|(XG)_T^{T_n}|>p\}\cap\{T<T_n\}} n|(XG)_T^{T_n}|dP \leq \int_{\{n|G_t|>p\}} n|G_t|dP < \frac{\epsilon}{2} \tag{4}$$

We look now at the second term of the right hand side of the inequality (3).

$$\int_{\{|(XG)_T^{T_n}|>p\}\cap\{T\geq T_n\}} |(XG)_{T_n}|dP \leq \int_{\{|(XG)_{T_n}|>p\}} |(XG)_{T_n}|dP$$

Since $(XG)_{T_n} \in L^1$, for every $\epsilon > 0$ there is a $p_{2\epsilon} > 0$ such that for every $p \geq p_{2\epsilon}$ we have

$$\int_{\{|(XG)_T^{T_n}|>p\}} |(XG)_{T_n}|dP < \frac{\epsilon}{2} \tag{5}$$

If we put (4) and (5) together we deduce that for every $\epsilon > 0$ there is a $p_\epsilon = \max(p_{1\epsilon}, p_{2\epsilon})$ such that for any $p > p_\epsilon$ and any T simple stopping time we have

$$\int_{\{|(XG)_T^{T_n}|>p\}} |(XG)_T^{T_n}|dP < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves the fact that $(XG)^{T_n}$ is a quasimartingale of class (D). From Theorem 14.2 of [Din00] we deduce that the Doléans measure $\mu_{(XG)^{T_n}}$ associated to the process $(XG)^{T_n}$ is $\sigma-$additive and has bounded variation on $\mathcal{R}$, hence it can be extended to a σ-additive measure with bounded variations on $\mathcal{P}$ (Theorem 7.4 b) of [Din00]).

Next we show that for any $B \in \mathcal{P}$ we have

$$\mu_{(XG)^{T_n}}(B) = \mu_{XG}(B \cap [0, T_n]).$$

In fact, for $A \in \mathcal{F}_0$ we have

$$\mu_{(XG)^{T_n}}(\{0\} \times A) = \mu_{XG}((\{0\} \times A) \cap [0, T_n]).$$

and for $(s, t] \times A$ with $A \in \mathcal{F}_s$ we have

$$\mu_{(XG)^{T_n}}((s,t] \times A) = E(1_A((XG)_t^{T_n} - (XG)_s^{T_n})) = \mu_{XG}(((s,t] \times A) \cap [0, T_n]),$$

which proves our equality. Hence the measure μ_{XG} is σ-additive on the σ-ring $\mathcal{P} \cap [0, T_n]$ for each n, hence it is σ-additive on the ring

$$\mathcal{B} = \bigcup_{1 \leq n < \infty} \mathcal{P} \cap [0, T_n].$$

Next we observe that μ_{XG} is bounded on $\mathcal{R}$, therefore it has bounded variation on $\mathcal{R}$ which implies that the measure defined on $\mathcal{B} \cap \mathcal{R}$ is $\sigma-$additive and has bounded variation. Since $\mathcal{B} \cap \mathcal{R}$ generates $\mathcal{P}$, by Theorem 7.4 b) of [Din00], μ_{XG} can be extended to a σ-additive measure with bounded variation on $\mathcal{P}$.

Since $\langle I_X, g \rangle = \mu_{XG}$, it follows that $\langle I_X, g \rangle$ is bounded and σ-additive on $\mathcal{R}$, thus d) holds. The implication d)$\Longrightarrow$c) is evident.

a) $\Longrightarrow$ d) is evident since for each $g \in Z$, the measure $\langle I_X, g \rangle$ is σ-additive on $\mathcal{P}$ and since any $\sigma-$additive measure on a $\sigma-$algebra is bounded we conclude that for $g \in Z$, the measure $\langle I_X, g \rangle$ is bounded on $\mathcal{P}$ hence on $\mathcal{R}$.

Next we prove d)$\Longrightarrow$a). Assume d) is true. Then the real valued measure $\langle I_X, g \rangle$ is $\sigma-$additive and bounded on $\mathcal{R}$. Since we proved that b) $\Longleftrightarrow$ c) $\Longleftrightarrow$ d) we deduce from (1) that

$$|\langle I_X, g \rangle (A)| \leq M \|g\| \text{ for all } A \in \mathcal{R}$$

where $M = \sup\{|I_X(A)| : A \in \mathcal{R}\}$. By Proposition 2.16 of [Din00] it follows that

the measure $\langle I_X(\cdot), g \rangle$ has bounded variation $|\langle I_X, g \rangle|(\cdot)$ satisfying

$$|\langle I_X, g \rangle|(A) \leq 2M\|g\|, \text{ for } A \in \mathcal{R}.$$

Applying Proposition 4.15 in [Din00] we deduce that $\tilde{I}_{X\mathbb{R},E}$ is bounded. By Theorem 3.7 b) of [BD01] we conclude that the measure $I_X : \mathcal{R} \to L^p_E$ has an additive extension $I_X : \mathcal{P} \to Z^{**}$ to $\mathcal{P}$ such that for each $g \in Z$, the real valued measure $\langle I_X, g \rangle$ is a $\sigma-$additive on $\mathcal{P}$ which is Assertion a). □

3 Examples of Additive Summable Processes

Definition 25. Let $X : \mathbb{R}_+ \times \Omega \to E$ be an E-valued process. We say that X has finite variation, if for each $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ has finite variation on each interval $[0, t]$. If $1 \leq p < \infty$, the process X has p-integrable variation if the total variation $|X|_\infty = \mathrm{var}(X, \mathbb{R}_+)$ is p-integrable.

Definition 26. We define the variation process $|X|$ by

$$|X|_t(\omega) = var(X.(\omega), (-\infty, t]), \text{ for } t \in \mathbb{R} \text{ and } \omega \in \Omega,$$

where $X_t = 0$ for $t < 0$.

Noting that if $m : \mathcal{D} \to E \subset L(F, G)$ is a σ-additive measure then for each $z \in G^*$, the measure $m_z : \mathcal{D} \to F^*$ is $\sigma-$additive, we deduce that, if the process X is summable, then it is also additive summable. Hence the following theorem is a direct consequence of Theorem 19.13 in [Din00]

Theorem 27. *Let $X : \mathbb{R}_+ \times \Omega \to E$ be a cadlag, adapted process with integrable variation $|X|$. Then X is 1–additive summable relative to any embedding $E \subset L(F, G)$.*

Proof. If $m : \mathcal{D} \to E \subset L(F, G)$ is a σ-additive measure then for each $z \in G^*$, the measure $m_z : \mathcal{D} \to F^*$ is $\sigma-$additive. We deduce that, if the process X is summable, then it is additive summable. Hence applying Theorem 19.13 b) in [Din00] we conclude our proof. □

3.1 Processes with Integrable Semivariation

Definition 28. We define the semivariation process of X relative to (F, G) by

$$\tilde{X}_t(\omega) = svar_{F,G}(X.(\omega),\ (-\infty, t]), \text{ for } t \in \mathbb{R} \text{ and } \omega \in \Omega,$$

where $X_t = 0$ for $t < 0$.

Definition 29. The total semivariation of X is defined by

$$\tilde{X}_\infty(\omega) = \sup_{t \geq 0} \tilde{X}_t(\omega) = svar_{F,G}(X.(\omega), \mathbb{R}), \text{ for } \omega \in \Omega.$$

Definition 30. Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$. The process X is said to have finite semivariation relative to (F, G), if for every $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ has finite semivariation relative to (F, G) on each interval $(-\infty, t]$. The process X is said to have p-integrable semivariation $\tilde{X}_{F,G}$ if the total semivariation $(\tilde{X}_{F,G})_\infty$ belongs to L^p.

Remark: If $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$ is a process and $z \in G^*$ we define, the process $X_z : \mathbb{R}_+ \times \Omega \to F^*$ by

$$\langle x, (X_z)_t(\omega)\rangle = \langle X_t(\omega)x, z\rangle, \text{ for } x \in F, t \in \mathbb{R}_+ \text{ and } \omega \in \Omega.$$

For fixed $t \geq 0$, we consider the function $X_t : \omega \mapsto X_t(\omega)$ from Ω into $E \subset L(F, G)$ and for $z \in G^*$ we define $(X_t)_z : \Omega \to F^*$ by the equality

$$\langle x, (X_t)_z(\omega) \rangle = \langle X_t(\omega)x, z \rangle, \text{ for } \omega \in \Omega, \text{ and } x \in F.$$

It follows that

$$(X_t)_z(\omega) = (X_z)_t(\omega), \text{ for } t \in \mathbb{R}_+ \text{ and } \omega \in \Omega.$$

The semivariation $\tilde{X}$ can be computed in terms of the variation of the processes X_z:

$$\tilde{X}_t(\omega) = \sup_{z \in G_1^*} |X_z|_t(\omega).$$

If X has finite semivariation $\tilde{X}$, then each X_z has finite variation $|X_z|$.

The following theorem is an improvement over the Theorem 21.12 in [Din00], where it was supposed that $c_0 \not\in E$ and $c_0 \not\in G$.

Theorem 31. *Assume $c_0 \not\subset G$. Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$ be a cadlag, adapted process with p-integrable semivariation relative to $(\mathbb{R}, E)$ and relative to (F, G). Then X is p-additive summable relative to (F, G)*

Proof. First we present the sketch of the proof, after which we prove all the details.

The prove goes as follows:

1) First we will show that

$$I_X(A)(\omega) = m_{X(\omega)}(A(\omega)), \text{ for } A \in \mathcal{R} \text{ and } \omega \in \Omega, \tag{*}$$

where $A(\omega) = \{t; (t, \omega) \in A\}$ and $X(\omega)$ is $X.(\omega)$. For the definition of the measure $m_{X(\omega)}$ see Section 2.2.

2) Then we will prove that the measure $m_{X(\omega)}$ has an additive extension to $\mathcal{B}(\mathbb{R}_+)$, with bounded semivariation relative to (F, G) and such that for every $g \in G^*$ the measure $(m_{X(\omega)})_g$ is $\sigma-$additive.

3) Next we prove that the function $\omega \mapsto m_{X(\omega)}(M(\omega))$ belongs to L^p_E for all $M \in \mathcal{P}$.

4) Then we show that the extension of the measure I_X to $\mathcal{P}$ has bounded semivariation relative to (F, L^p_G).

5) Finally we show that for each $z \in (L^p_G)^*$ the measure $(I_X)_z : \mathcal{P} \to F^*$ is $\sigma-$additive.

6) We conclude that the process X is $p-$additive summable.

Now we prove each step in detail.

1) First we prove (*) for predictable rectangles. Let $A = \{0\} \times B$ with $B \in \mathcal{F}_0$. Then we have

$$I_X(\{0\} \times B)(\omega) = 1_B(\omega)X_0(\omega) = \int 1_{\{0\}\times B}(s,\omega)dX_s(\omega) = m_{X(\omega)}(A(\omega)).$$

Now let $A = (s,t] \times B$ with $B \in \mathcal{F}_s$. In this case we also obtain

$$I_X((s,t]\times B)(\omega) = 1_B(\omega)(X_t(\omega)-X_s(\omega)) = \int 1_{(s,t]\times B}(p,\omega)dX_p(\omega) = m_{X(\omega)}(A(\omega)).$$

Since both $I_X(A)(\omega)$ and $m_{X(\omega)}(A(\omega))$ are additive we conclude that the equality (*) is true for $A \in \mathcal{R}$.

2) Since X has p-integrable semivariation relative to (F,G) we infer that $(\tilde{X}_{F,G})_\infty(\omega) < \infty$ a.s. If we redefine $X_t(\omega) = 0$ for those ω for which $(\tilde{X}_{F,G})_\infty(\omega) = \infty$ we obtain a process still denoted X with bounded semivariation. In this case for each $\omega \in \Omega$ the function $t \mapsto X_t(\omega)$ is right continuous and with bounded semivariation. By Theorem **??** we deduce that the measure $m_{X(\omega)}$ can be extended to an additive measure $m_{X(\omega)} : \mathcal{B}(\mathbb{R}_+) \to E \subset L(F,G)$, with bounded semivariation relative to (F,G) and such that for every $g \in G^*$ the measure $(m_{X(\omega)})_g : \mathcal{B}(\mathbb{R}_+) \to F^*$ is $\sigma-$additive.

3) Since X has p-integrable semivariation relative to (F,G), for each $t \geq 0$ we have $X_t \in L^p_E$. Hence, by step 1, the function $\omega \mapsto m_{X(\omega)}(M(\omega))$ belongs to L^p_E for all $M \in \mathcal{R}$. To prove that $\omega \mapsto m_{X(\omega)}(M(\omega))$ belongs to L^p_E for all $M \in \mathcal{P}$ we will use the Monotone Class Theorem. We will prove that the set $\mathcal{P}_0$ of all sets $M \in \mathcal{P}$ for which the affirmation is true is a monotone class, containing $\mathcal{R}$, hence equal to $\mathcal{P}$. In fact, let M_n be a monotone sequence from $\mathcal{P}_0$ converging to M. By assumption, for each n the function $\omega \mapsto m_{X(\omega)}(M_n(\omega))$ belongs to L^p_E and for each ω the sequence $(M_n(\omega))$ is monotone in $\mathcal{B}(\mathbb{R}_+)$ and has limit $M(\omega)$. Moreover $|m_{X(\omega)}(M_n(\omega))| \leq \tilde{m}_{X(\omega)}(\mathbb{R}_+ \times \Omega) = \tilde{X}_\infty(\omega)$, which is p-integrable. By Lebesgue's Theorem we deduce that the mapping $\omega \mapsto m_{X(\omega)}(M(\omega))$ belongs to L^p_E, hence $M \in \mathcal{P}_0$. Therefore $\mathcal{P}_0$ is a monotone class.

4) We use the equality (*) to extend I_X to the whole $\mathcal{P}$, by

$$I_X(A)(\omega) = m_{X(\omega)}(A(\omega)), \text{ for } A \in \mathcal{P}.$$

Let $A \in \mathcal{P}$, $(A_i)_{i\in I}$ be a finite family of disjoint sets from $\mathcal{P}$ contained in A, and $(x_i)_{i\in I}$ a family of elements from F with $|x_i| \leq 1$. Then we have

$$\| \sum I_X(A_i)x_i\|_p^p = E(| \sum I_X(A_i)(\omega)x_i|^p)$$

$$= E(|\sum m_{X(\omega)}(A_i(\omega))x_i|^p) \leq E(|(\tilde{m}_{X(\omega)})_{F,G}(A(\omega))|^p)$$
$$= \|(\tilde{m}_{X(\omega)})_{F,G}(A(\omega))\|_p^p = \|\tilde{X}_{F,G}(A(\omega))\|_p^p \leq \|(\tilde{X}_{F,G})_\infty\|_p^p < \infty.$$

Taking the supremum over all the families (A_i) and (x_i) as above, we deduce $(\tilde{I}_X)_{F,L^p_G} \leq \|(\tilde{X}_{F,G})\|_p < \infty$.

5) Let $z \in (L^p_G)^*$ and $x \in F$. Then $z(\omega) \in G^*$ and for each set $M \in \mathcal{P}$ we have

$$\begin{aligned}\langle (I_X)_z(M), x\rangle &= \langle I_X(M)x, z\rangle = E(\langle I_X(M)(\omega)x, z(\omega)\rangle)\\ &= E(\langle m_{X(\omega)}(M(\omega))x, z(\omega)\rangle) = E(\langle (m_{X(\omega)})_{z(\omega)}(M(\omega)), x\rangle).\end{aligned} \quad (3)$$

By step we conclude that the measure $(I_X)_z$ is $\sigma-$additive for each $z \in (L^p_G)^*$.

6) By the definition in step 4,

$$I_X(A)(\omega) = m_{X(\omega)}(A(\omega)), \text{ for } A \in \mathcal{P} \text{ and } \omega \in \Omega,$$

and by steps 2 and 3 we conclude that the measure I_X has an additive extension $I_X : \mathcal{P} \to L^p_E$. By step 5 the measure $(I_X)_z$ is $\sigma-$additive for each $z \in (L^p_G)^*$. By step 4 this extension has bounded semivariation. Therefore the process X is p-additive summable. □

The following theorem gives sufficient conditions for two processes to be indistinguishable. For the proof see [Din00], Corollary 21.10 b').

Theorem 32. *([Din00]21.10b')) Assume $c_0 \not\subset E$ and let $A, B : \mathbb{R}_+ \times \Omega \to E$ be two predictable processes with integrable semivariation relative to $(\mathbb{R}, E)$. If for every stopping time T we have $E(A_\infty - A_T) = E(B_\infty - B_T)$, then A and B are indistinguishable.*

The next theorem gives examples of processes with locally integrable variation or semivariation. For the proof see [Din00], Theorems 22.15 and 22.16.

Theorem 33. *([Din00]22.15,16) Assume X is right continuous and has finite variation $|X|$ (resp. finite semivariation $\tilde{X}_{F,G}$). If X is either predictable or a local martingale, then X has locally integrable variation $|X|$ (resp. locally integrable semivariation $\tilde{X}_{F,G}$).*

Proposition 34. *Let $X : \mathbb{R}_+ \times \Omega \to E$ be a process with finite variation. If X has locally integrable semivariation $\tilde{X}_{\mathbb{R},E}$, then X has locally integrable variation.*

Proof. Assume X has locally integrable semivariation $\tilde{X}$ relative to $(\mathbb{R}, E)$. Then there is an increasing sequence S_n of stopping times with $S_n \uparrow \infty$ such that $E(\tilde{X}_{S_n}) < \infty$ for each n. For each n define the stopping times T_n by $T_n = S_n \wedge \inf\{t |\, |X|_t \geq n\}$. It follows that $|X|_{T_n-} \leq n$. Since X has finite variation, by Proposition 6 we have $\Delta|X_{T_n}| = |\Delta X_{T_n}| \leq \tilde{X}_{T_n}$. From $\Delta|X|_{T_n} = |X|_{T_n} - |X|_{T_n-}$ we deduce that $|X|_{T_n} = |X|_{T_n-} + \Delta|X_{T_n}| \leq n + \tilde{X}_{T_n}$; Therefore $E(|X|_{T_n}) \leq n + E(\tilde{X}_{T_n}) < \infty$; hence X has locally integrable variation. $\square$

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