SPACE REGULARITY OF STOCHASTIC HEAT EQUATIONS DRIVEN BY IRREGULAR GAUSSIAN PROCESSES

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Abstract. We study linear stochastic evolution equations driven by various infinite-dimensional Gaussian processes, some of which are more irregular in time than fractional Brownian motion (fBm) with any Hurst parameter H, while others are comparable to fBm with $H < \frac{1}{2}$. Sharp necessary and sufficient conditions for the existence and uniqueness of solutions are presented. Specializing to stochastic heat equations on compact manifolds, especially on the unit circle, sharp Gaussian regularity results are used to determine sufficient conditions for a given fixed function to be an almost-sure modulus of continuity for the solution in space; these sufficient conditions are also proved necessary in highly irregular cases, and are nearly necessary (logarithmic corrections are given) in other cases, including the Hölder scale.

1. Introduction

This article deals with precise existence results for stochastic PDEs driven by arbitrary Gaussian processes, and specializes to stochastic heat equations for sharp spatial regularity results. Since the pioneering work of stochastic analysts in the 1970's and 1980's (see for instance John Walsh's Saint-Flour lecture notes [24] or DaPrato and Zabczyk's textbook [6]), probabilists have investigated the question of how to define the weakest conditions sufficient to guarantee existence and/or regularity of a stochastic PDE's solution. In order to express results that are as sharp as possible, the choice was made by many – including our past and present work – to study the simplest possible problems with some appeal for applications, hence the use of the stochastic heat equation driven by additive noise. The framework of Itō stochastic calculus was deemed most appropriate, implying the study of equations of the form

$$u(t,x) = u(0,x) + \int_0^t \Delta_x u(s,x)\, ds + W(t,x) \tag{1.1}$$

for all $t \geq 0$, and all x in some (e.g. Euclidean) space E, where W is some random field on $\mathbf{R}_+ \times E$. For many years, attention was directed towards the case where W is Brownian motion in its parameter t, and it had been known since Walsh that W need not be a bonafide function for u to exist, as indeed it may be white noise in space while still allowing a solution in an analytically weak form (evolution form

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of DaPrato and Zabczyk), namely for $E = \mathbf{R}$, with $p(t,x) = (2\pi t)^{-1/2} e^{-x^2/(2t)}$,

$$u(t,x) = \int_E p(t,x-y)\, u(0,y)\, dy + \int_E \int_0^t p(t-s,x,y)\, W(ds,dy).$$

In the 1990's the question of precisely how the spatial regularity of W (or lack thereof) effects the solution's was posed. For instance, it was found that the so-called *space-time white noise* of the example above yields a solution with as much regularity in x as Brownian motion, so that more irregular noises should still imply existence, while qualitatively different results should be expected in higher space dimensions. Sufficient conditions were established for various additive and non-linear multiplicative stochastic PDEs: [5], [16], [17], [18], [22]. Some of these papers also covered the issue of spatial Hölder continuity of the solution, with [22] being the first one to supply necessary and sufficient conditions for this property, and its follow-up work [23] providing an indication that extensions to non-Hölder regularity may be possible

As the case of Brownian-based noise for stochastic PDEs was now better understood, the year 2000 saw the emergence of several works focusing on the effect of fractional-Brownian-based models. These are Gaussian noise terms $W(t,x) = B^H(t,x)$ whose behavior in time has non-independent increments and a scaling property in the power $H \in (0,1)$, not simply the Brownian case $H = 1/2$; these are simply infinite-dimensional analogues of scalar fractional Brownian motion (fBm) introduced e.g. in [13]. The difficulty of stochastic integration with respect to such behavior in time (see [7], [1], [2]), made it so that not much progress was possible in the case of nonlinear equations: [8], [10], [11], [14]. For linear equations, however, following the impetus in [22] and [23], necessary and sufficient conditions for existence were established in a wide abstract setting in [20], and, using the stochastic heat equation on the circle, for any scale of regularity, Hölder or not, in [21].

Still, the issue of changing the *time* regularity of the driving noise beyond the fractional Brownian scale has never been addressed. Only recently, in [3] and [4], has the technique for stochastic calculus for highly irregular fBm (low parameter H) been perfected. Stochastic calculus with respect to arbitrarily irregular Gaussian processes was performed in [15].

With the exception of a less than efficient treatment in [15], this article presents the first work in which the time behavior of a SPDE is neither a semimartingale nor of fractional Brownian type. The results proposed herein are systematic and quite sharp. They are formulated in a way (e.g. trace conditions in abstract Hilbert spaces: Theorem 3.3) which is consistent with the older results on semimartingale in the 1990's, and with the new wave of fractional-Brownian-based treatments since 2000.

Using infinite dimensional analogues of the processes defined in [15], we follow the framework in the work [20] to find necessary and sufficient conditions for existence of solutions to infinite-dimensional stochastic evolution equations driven by these arbitrary Gaussian fields. Our new calculation technique appears to be superior to that employed originally in [20], because we establish our existence and uniqueness theorem without needing to assume the existence of a spectral gap and

finite-dimensional kernel, which was used in [20]. Thus our corresponding work appears as an improvement over, as well as a generalization to all time scales of, the results in [20].

We apply the techniques in [21] for sharp Gaussian regularity in order to find necessary and sufficient spatial regularity conditions for the stochastic heat equation on the circle driven by arbitrary Gaussian fields. These very sharp results are owed to the use of Gaussian fields with the corresponding regularity theory introduced by Dudley, Fernique, and Talagrand, which were carefully exploited in our context in [21].

Our article is structured as follows. Preliminaries, and the definition of infinite dimensional Gaussian processes (fields) with arbitrary regularity is given in Section 2. Section 3 gives our existence and uniqueness results. Section 4 gives conditions for specific spatial regularity. While the existence result of Theorem 3.3 is stated in an abstract Hilbert-space setting, we summarized it here, together with our regularity results, using the stochastic heat equation on the unit circle S^1, for illustrative purposes.

Hypothesis: Let $B^\gamma(t,x)$ be a centered Gaussian random field defined for all $(t,x) \in \mathbf{R}_+ \times S^1$. Assume B^γ is homogeneous in x, with canonical metric $\mathbf{E}\left[\left(B^\gamma(1,x) - B^\gamma(1,y)\right)^2\right] = \delta^2(x-y)$. Let q_n be the nth Fourier coefficient of δ^2. Note that if q_n is not summable, the above definition is only formally a function, and B^γ must be understood as being a Schwartz-distribution-valued Gaussian process. Assume B^γ's behavior in time is bounded above as follows:

$$\mathbf{E}\left[\left(B^\gamma(t,0) - B^\gamma(s,0)\right)^2\right] \le \gamma^2(|t-s|),$$

where γ^2 is increasing and concave on $\mathbf{R}_+$, and differentiable except at 0, with $\gamma(0) = 0$. The necessary conditions below are valid specifically if $B^\gamma(t,0)$ can be written as $\int_0^t \sqrt{d\gamma^2/dt}\,(t-s)\,dB(s)$ where B is a standard Brownian motion. The sufficient conditions do not require this form.

Conclusion for existence: [Theorem 3.3]. The stochastic heat equation (1.1) with $W = B^\gamma$ has a unique evolution solution in the sense of DaPrato and Zabczyk, in $L^2\left(\Omega \times \mathbf{R}_+ \times S^1\right)$, if and only if

$$\sum_{n=1}^{\infty} q_n \gamma^2\left(n^{-2}\right) < \infty.$$

Conclusion for regularity: [Theorem 4.2]. Let f be an increasing continuous function on a neighborhood of 0 with $f(0) = 0$, differentiable except at 0. Let

$$\delta_f(r) = \int_0^r f'(s)\left(\log(1/s)\right)^{-1/2} ds$$

The aforementioned solution admits f, up to a non-random constant, as a uniform modulus of continuity almost surely, if

$$\sum_{n=1}^{\infty} q_n \gamma^2\left(n^{-2}\right) h\left(\delta_f^2\left(\frac{1}{n}\right)\right) < \infty$$

for all decreasing continuous functions h on a neighborhood of 0 such that $\int_0 h(r)\,dr < \infty$. This is also an "only if" statement when $f(r) \gg r^H$ for all $H > 0$. Otherwise, the "only if" part holds with $\delta_f\left(n^{-1}\right)$ replaced by $\delta_f\left(n^{-1}\right)\log n$.

The *sufficient* conditions for existence and regularity hold for processes defined on $\mathbf{R}_+ \times \mathbf{R}$ as well, if one simply replaces series by integrals above. Similar sufficient condition results also holds in higher dimensions ($\mathbf{R}^d$ and other compact or non-compact manifolds). We leave exact statements and proofs of these facts out of this article.

2. Preliminaries

2.1. Irregular Gaussian processes. In the remainder of the article, the symbol $\asymp$ denotes commensurability between two functions: f and g are *commensurable* if there exist positive constants c, C such that $cg(x) \le f(x) \le Cg(x)$ for all values of a common variable x.

A continuous centered Gaussian process X on $\mathbf{R}_+$ that starts from the origin at time 0 has a distribution entirely determined by its increments' variance structure, i.e. the canonical metric

$$\delta^2(s,t) = \mathbf{E}\left[(X(t) - X(s))^2\right].$$

The case of Brownian motion $X = W$ is $\delta^2(s,t) = |t-s|$, for fBm $X = B^H$ we have $\delta^2(s,t) = |t-s|^{2H}$. It is well-known that, beyond the scaling property of fBm by which $B^H(ct) = c^H t$ in distribution, fBm admits the function $f(r) = r\log^{1/2}(1/r)$ up to a constant as a uniform modulus of continuity almost surely. The so-called Volterra representation of fBm from standard Brownian motion has the form

$$B^H(t) = \int_0^t K(t,s)\,dW(s), \tag{2.1}$$

where the kernel K has the property that for s away from 0, $K(t,s) \asymp |t-s|^{H-1/2}$.

We consider a class of Gaussian processes with arbitrary correlation between increments, by assuming that a Voterra-type representation holds, where the kernel K can be chosen to be commensurate with some given function that implies a certain type of almost-sure modulus of continuity for B^H. To simplify the presentation, and as the only simple means we have found to ensure that our sufficient conditions are also necessary, we assume that K depends only on the difference $t-s$. All our sufficient conditions proved below hold for any other K bounded above by a given $K(t-s)$; the proof of this fact is left to the reader. The fact that $r^{H-1/2} = \left(d\left(r^{2H}\right)/dr\right)^{1/2}$ is a motivation for the definition that follows. Also note that any Gaussian process starting from 0 that is adapted to a Brownian filtration must be of the Volterra form (2.1) for an appropriate function K.

Let W be a standard Brownian motion on $\mathbf{R}_+$ with respect to the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the filtration $\{\mathcal{F}_t\}_{t\ge 0}$. Assume γ^2 is of class C^2 everywhere in $\mathbf{R}_+$ except at 0 and that $d\gamma^2/dr$ is non-increasing. In [15] it is proved that the

centered Gaussian process

$$B^{\gamma}(t) := \int_0^t \varepsilon(t-s)\, dW(s), \tag{2.2}$$

where

$$\varepsilon(r) := \left(\frac{d\left(\gamma^2\right)}{dr}\right)^{1/2}, \tag{2.3}$$

satisfies the following conditions with respect to $\{\mathcal{F}_t\}_{t\geq 0}$: for any $t \geq 0$

(i) $\delta(s,t) \asymp \gamma(|t-s|)$, where δ is the canonical metric of B^{γ} on $(\mathbf{R}_+)^2$
(ii) $B(0) = 0$.
(iii) B is adapted to a $\{\mathcal{F}_t\}_{t\geq 0}$.

The choice for B^{γ} in (2.2) above is the simplest choice satisfying conditions (i), (ii), and (iii) in terms of applications to stochastic calculus. Again, since any process written as $\int_0^t K(t,s)\, dW(s)$ with $K(t,s) \leq \gamma(|t-s|)$ will satisfy the sufficient condition statements in our theorems below, our work actually covers a very wide class of Gaussian processes, and in particular reaches along the entire regularity scale of Gaussian processes.

The condition that $d\gamma^2/dr$ is non-increasing implies that B^{γ} is more irregular than standard Brownian motion, such as fBm with $H < 1/2$, or specifically $\varepsilon(r) = r^{H-1/2}$, which yields the so-called Liouville process, whose regularity and scaling properties are identical to those of fBm, but which has slightly inhomogeneous increments. The case where B^{γ} is more regular than Brownian motion, such as fBm with $H > 1/2$, uses a slightly different, and considerably easier, calculation. We omit this case. In all cases, condition (i) above implies that up to a constant, $f(r) = \gamma(r)\log^{1/2}(1/r)$ is a uniform modulus of continuity for B^{γ} almost surely, when $\lim_0 f = 0$. When this limit is not 0, one can prove using Gaussian supremum estimation that B^{γ} is almost-surely discontinuous everywhere. Nevertheless, all that we claim below still holds in this extremely irregular case.

2.2. The Wiener integral with respect to B^{γ}. Let $(B^{\gamma}(t))_{t\in[0,T]}$ be the centered Gaussian process defined by its Wiener integral Volterra-type representation as in (2.2) and let f be a deterministic measurable function on $\mathbf{R}_+$. We define the operator $K^* = K^*_{\gamma}$ acting on f by

$$K^*_{\gamma} f(r) := \left[f(r)\,\varepsilon(T-r) + \int_r^T (f(s) - f(r))\,\varepsilon'(s-r)\, ds \right],$$

if it exists. If $K^*_{\gamma} f(\cdot)$ is in $L^2([0,T], dr)$ then we say that f belongs to the space $L^2_{\gamma}([0,T])$, and we denote

$$\|f\|^2_{\gamma} = \left\|K^*_{\gamma} f\right\|^2_{L^2([0,T])} = \int_0^T \left| f(r)\,\varepsilon(T-r) + \int_r^t (f(s) - f(r))\,\varepsilon'(s-r)\, ds \right|^2 dr.$$

This L^2_{γ} is the so-called *canonical* Hilbert space of B^{γ} on $[0,T]$. We will also denote it by $\mathcal{H}$. For any f in $\mathcal{H}$ we define the stochastic integral of f with respect

to B^γ on $[0,T]$ as the Gaussian random variable given by

$$\int_0^T f(t)\, dB^\gamma(t) = \int_0^T K_\gamma^* f(r)\, dW(r).$$

Remark 2.1. One easily sees that if g is a function in L^2_γ then the function $f : r \mapsto g(r)\, 1_{[0,t]}(r)$ is in L^2_γ and

$$K^* f(r) = 1_{[0,t]}(r) \left(\varepsilon(t-r)\, g(r) + \int_r^t [g(s) - g(r)]\, \varepsilon'(s-r)\, ds \right),$$

i.e. $K^* f$ depends on t, not on T.

2.3. Infinite dimensional irregular Gaussian processes. We define an infinite dimensional Gaussian process using the classical approach of [6] for cylindrical Brownian motion (see section 4.3.1 therein). Let U be a real separable Hilbert space. For any fixed complete orthonormal basis $(e_n)_n$ in U and any fixed sequence of positive numbers $(\lambda_n)_n$, even if $\sum_{n\geq 0} \lambda_n = \infty$, we define

$$B^\gamma(t) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} e_n B_n^\gamma(t), \tag{2.4}$$

where B_n^γ are an IID sequence of Gaussian processes with the same distribution as the B^γ in the previous section. This slight abuse of notation will not lead to confusion, since henceforth B^γ denotes an infinite-dimensional process. Observe that for any fixed t the above series converges in $L^2(\Omega \times U)$ if and only if $\sum_{n\geq 0} \lambda_n < \infty$. In the other case, $B^\gamma(t)$ will be a well defined Gaussian process with values in a larger Hilbert space U_1, where the embedding $U \subset U_1$ is continuous, Hilbert-Schmidt. For instance, if $U = L^2([0,1])$, we can use the space of Schwartz tempered distributions for U_1.

To define the Wiener integral with respect to the above infinite dimensional Gaussian process B^γ we consider another real separable Hilbert space V and $(\phi_s)_{s\in T}$ a deterministic function with values in $\mathcal{L}_2(U;V)$, the space of Hilbert-Schmidt linear operators from U to V. The stochastic integral of ϕ with respect to B^γ is defined by

$$\int_0^t \phi_s dB^\gamma(s) = \sum_{n=0}^{\infty} \int_0^t \phi_s e_n dB_n^\gamma(s) = \sum_{n=0}^{\infty} \int_0^t (K^* \phi e_n)_s dB_n(s),$$

where B_n is the standard Brownian motion used to represent B^γ in the Volterra-type representation (2.1). The above sum is finite almost surely, and is indeed a Gaussian random element of V, if and only if it has a finite variance in V, i.e.

$$\sum_n \|K^*(\phi e_n)\|^2_{L^2([0,T],V)} < \infty.$$

3. Linear stochastic equations. Existence of solutions

Consider the setting from the previous section where B^γ a cylindrical Gaussian process (a process defined as above for which $\lambda_n = 1$), and ϕ a linear operator in $\mathcal{L}(U,V)$ that is not necessarily Hilbert-Schmidt.

We will study the existence of solutions for the equation

$$dX(t) = AX(t)dt + \phi dB^{\gamma}(t), \tag{3.1}$$

with boundary condition $X(0) = x \in V$, where $A : Dom(A) \in V \to V$ is the infinitesimal generator of the strongly continuous semigroup $(e^{tA})_{t\in T}$, a self adjoint operator on V.

Remark 3.1. This equation is defined if the integral $\int_0^t \phi dB^{\gamma}$ is defined as a V-valued random variable. Since

$$\begin{aligned}\mathbf{E}\left|\int_0^t \phi dB^{\gamma}(s)\right|_V^2 &= \sum_n \mathbf{E}\left|\int_0^t \phi e_n dB_n^{\gamma}(s)\right|_V^2 \\ &= \sum_n \mathbf{E}\left|\int_0^t dB_n^{\gamma}(s)\right|^2 |\phi e_n|_V^2 = \gamma^2(t)\|\phi\|_{HS}^2,\end{aligned}$$

this occurs if and only if $\phi \in \mathcal{L}_2(U, V)$.

Nevertheless, it is standard practice in stochastic analysis, useful in applications, to consider a weak form of the above differential equation, well-defined even for many non-Hilbert-Schmidt integrands ϕ, depending on the regularity of the operator A.

Definition 3.2. The *mild*, or *evolution*, form of the stochastic differential equation (3.1), with starting point $X(0) = x \in V$, is given as follows: for all $t \in [0, T]$, we have the equality in V almost surely:

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}\phi dB^{\gamma}(s).$$

Since this is an explicit formula for X, we call it the mild, or evolution, solution to (3.1).

Reed and Simon in [19] presented the existence of an uniquely defined projection measure dP_λ on the real line, such that for every $\phi \in V$, $d\langle\phi, P_\lambda\phi\rangle$ is a Borel measure on $\mathbb{R}$, and for every $\phi \in Dom(A)$

$$\langle\phi, A\phi\rangle = \int_{\mathbb{R}} \lambda d\langle\phi, P_\lambda\phi\rangle.$$

The next theorem is a generalization of Theorem 40 in [15].

Theorem 3.3. *Let ε and γ be defined as in Section 2 (see definitions (2.2) and (2.3)). Assume*

$$\frac{|\varepsilon'(x)|}{\varepsilon(x)} \leq \frac{1}{2x} \tag{3.2}$$

and

$$|\varepsilon'(x)| \asymp x^{-\frac{3}{2}} f(x) \tag{3.3}$$

hold, where f is an increasing differentiable function, and let B^{γ} be a cylindrical Gaussian process on the Hilbert space U. Assume $A : Dom(A) \in V \to V$ is a negative, self-adjoint operator on the Hilbert space V. Note that we do not need to assume A is negative-definite.

Then for any fixed $\phi \in \mathcal{L}(U,V)$ there is a unique mild solution $X(t) \in L^2(\Omega, V)$ for the equation

$$dX(t) = AX(t)dt + \phi dB^\gamma(t),\ t \in [0,T];\ X(0) = x \in V$$

if and only if $\phi^* G_H(-A)\phi$ is a trace-class operator, where

$$G_H(\lambda) = \gamma^2\left(\frac{1}{max(\lambda,1)}\right).$$

Here for any integrable function F on $\mathbf{R}_+$, $F(-A)$ is defined by $F(-A)x = \int_0^\infty F(\lambda) P_{-\lambda} x d\lambda$ for all $x \in V$.

Remark 3.4. Our conditions on ε are not restrictive. Since ε^2 is integrable near 0 due to definition (2.2), we can assume without loss of generality that $\varepsilon^2(x)x$ is increasing near 0. This means $2\varepsilon(x)\varepsilon'(x)x + \varepsilon^2(x) \geq 0$, which implies (3.2) for all x near 0.

Remark 3.5. Condition (3.3) signifies how to compare the Volterra kernel with those standard power scale ones, with $\gamma(x) = x^H$, so that $|\varepsilon'(x)| = c_H x^{-3/2} x^H$, or logarithmic scale ones, with $\gamma(x) = \log^{-\beta}(1/x)$, so that indeed $|\varepsilon'(x)| \asymp x^{-3/2}\log^{-\beta}(1/x)$, from which we see that requiring f increasing is not a restriction in any scale.

Proof of the theorem.
Step 1: setup and exact calculations.

Consider the scalar measure μ_n defined as

$$d\mu_n(\lambda) = d\langle \phi e_n, P_\lambda \phi e_n\rangle_V.$$

Denoting

$$I_t = \mathbf{E}|X(t) - e^{tA}x|_V^2,$$

it is sufficient to estimate I_t optimally from above and below. By independence of the components in the definition (2.4) of the infinite-dimensional B^γ, we have

$$\begin{aligned} I_t = \mathbf{E}|X(t) - e^{tA}x|_V^2 &= \mathbf{E}\left|\int_0^t e^{(t-s)A}\phi dB^\gamma(s)\right|_V^2 \\ &= \mathbf{E}\left|\sum_n \int_0^t e^{(t-s)A}\phi e_n dB_n^\gamma(s)\right|_V^2. \end{aligned}$$

Then using the definition of Wiener integration in Section 2.3,

$$\begin{aligned} I_t &= E\left|\sum_n \int_0^T K^*\left[e^{(t-s)A}\phi e_n 1_{[0,t]}(s)\right] dW(s)\right|_V^2 \\ &= \sum_n \int_0^T \left|K_\gamma^*\left[1_{[0,t]} e^{(t-s)A}\phi e_n\right]\right|_V^2 ds \\ &= \sum_n \int_0^t \left|e^{(t-s)A}\phi e_n \varepsilon(t-s) + \int_s^t (e^{(t-r)A}\phi e_n - e^{(t-s)A}\phi e_n)\varepsilon'(r-s)dr\right|_V^2 ds. \end{aligned}$$

For ease of computations we can take $t = 1$. Then, by definition of e^{sA},

$$
\begin{aligned}
&I_1 \qquad (3.4)\\
&= \sum_n \int_0^1 \left| e^{(1-s)A}\phi e_n \varepsilon(1-s) + \int_s^1 (e^{(1-r)A}\phi e_n - e^{(1-s)A}\phi e_n)\varepsilon'(r-s)dr \right|_V^2 ds\\
&= \sum_n \int_0^\infty \int_0^1 \left[e^{-(1-s)\lambda}\varepsilon(1-s) + \int_s^1 (e^{-(1-r)\lambda} - e^{-(1-s)\lambda})\varepsilon'(r-s)dr \right] ds d\mu_n(\lambda)\\
&= \sum_n \int_0^\infty \left\{ \int_0^1 \left[e^{-2\lambda(1-s)}\varepsilon^2(1-s) + \left(\int_s^1 (e^{-\lambda(1-r)} - e^{-\lambda(1-s)})\varepsilon'(r-s)dr \right)^2 \right.\right.\\
&\qquad \left.\left. + 2e^{-\lambda(1-s)}\varepsilon(1-s) \int_s^1 (e^{-\lambda(1-r)} - e^{-\lambda(1-s)})\varepsilon'(r-s)dr \right] ds \right\} d\mu_n(\lambda)\\
&= \sum_n \int_0^\infty I(\lambda) d\mu_n(\lambda), \qquad (3.5)
\end{aligned}
$$

where

$$
\begin{aligned}
I(\lambda) &= \int_0^1 \left[e^{-2\lambda(1-s)}\varepsilon^2(1-s) + \left(\int_s^1 (e^{-\lambda(1-r)} - e^{-\lambda(1-s)})\varepsilon'(r-s)dr \right)^2 \right.\\
&\qquad \left. + 2e^{-\lambda(1-s)}\varepsilon(1-s) \int_s^1 (e^{-\lambda(1-r)} - e^{-\lambda(1-s)})\varepsilon'(r-s)dr \right] ds\\
&= \int_0^1 e^{-2\lambda(1-s)} \left\{ \varepsilon^2(1-s) + \left(\int_s^1 (e^{\lambda(r-s)} - 1)\varepsilon'(r-s)dr \right)^2 \right.\\
&\qquad \left. + 2\varepsilon(1-s) \int_s^1 (e^{-\lambda(r-s)} - 1)\varepsilon'(r-s)dr \right\} ds\\
&= \int_0^1 e^{-2\lambda(1-s)} \left(\varepsilon(1-s) + \int_s^1 (e^{\lambda(r-s)} - 1)\varepsilon'(r-s)dr \right)^2 ds\\
&= \int_0^1 e^{-2\lambda(1-s)} \left(\varepsilon(1-s) + \int_0^{1-s} (e^{u\lambda} - 1)\varepsilon'(u)du \right)^2 ds\\
&= \int_0^1 e^{-2\lambda s} \left(\varepsilon(s) + \int_0^s (e^{u\lambda} - 1)\varepsilon'(u)du \right)^2 ds. \qquad (3.6)
\end{aligned}
$$

The equality before the last one is obtain by the change of variable $u = r - s$ and the last one with the change of variable $s = 1 - s$. Estimation of the formula in (3.6) is non-trivial: indeed the two terms in the square inside $I(\lambda)$ are of opposite signs, since $\varepsilon(r) = \sqrt{d\gamma^2/dr}$ is assumed to be decreasing. In steps 2 and 3 below, we assume $\lambda \geq 1$.

Step 2: upper bound when $\lambda \geq 1$.

We first establish an upper bound on $I(\lambda)$. We observe that

$$
I(\lambda) \leq 2 \left[\int_0^1 e^{-2\lambda s}\varepsilon^2(s)ds + \int_0^1 e^{-2\lambda s} \left(\int_0^s (e^{u\lambda} - 1)\varepsilon'(u)du \right)^2 ds \right]
$$

$$= I_1(\lambda, 1) + I_2(\lambda, 1)$$

This is not a sharp inequality a priori, since it kills the negativity of ε'. Nevertheless, our lower bound below shows that it is actually sharp.

Step 2a: upper bound, 2nd term. We start by analyzing the second term of the inequality

$$\begin{aligned} I_2(\lambda,1) &= \int_0^1 e^{-2\lambda s}\left(\int_0^s (e^{u\lambda}-1)\varepsilon'(u)du\right)^2 ds \\ &\le \int_0^{\frac{1}{\lambda}} e^{-2\lambda s}\left(\int_0^s (e^{u\lambda}-1)\varepsilon'(u)du\right)^2 ds \\ &\quad + \int_{\frac{1}{\lambda}}^1 e^{-2\lambda s} 2\left(\int_0^{\frac{1}{\lambda}} (e^{u\lambda}-1)\varepsilon'(u)du\right)^2 ds \\ &\quad + \int_{\frac{1}{\lambda}}^1 e^{-2\lambda s} 2\left(\int_{\frac{1}{\lambda}}^s (e^{u\lambda}-1)\varepsilon'(u)du\right)^2 ds \\ &:= I_{2,0}(\lambda) + I_{2,1}(\lambda) + I_{2,2}(\lambda), \end{aligned}$$

and now bounding each of these three terms. To control the first term we bound $e^{\lambda r-1}$ above by $C\lambda r$ and $2^{-2\lambda s}$ by 1. The actual value of the constant C below may change from line to line, but never depends on λ.

$$I_{2,0}(\lambda) \le C\int_0^{1/\lambda} \lambda^2 \left(\int_0^s r\,|\varepsilon'(r)|\,dr\right)^2 ds.$$

Keeping in mind the special condition (3.2) we obtain

$$\begin{aligned} I_{2,0}(\lambda) &\le C\lambda^2 \int_0^{\frac{1}{\lambda}} \left(\int_0^s \varepsilon(r)dr\right)^2 ds \\ &\le C\lambda^2 \int_0^{\frac{1}{\lambda}} \left(\int_0^{\frac{1}{\lambda}} \varepsilon(r)dr\right)^2 ds \\ &= C\lambda \left(\int_0^{\frac{1}{\lambda}} \varepsilon(r)dr\right)^2. \end{aligned}$$

For the second term, using the same approximations and inequality as above, we obtain

$$\begin{aligned} I_{2,1}(\lambda) &\le C\int_{1/\lambda}^1 e^{-2s\lambda}\left(\int_0^{1/\lambda} \lambda r\,|\varepsilon'(r)|\,dr\right)^2 ds \\ &\le C\int_{1/\lambda}^1 e^{-2s\lambda}\left(\lambda\int_0^{1/\lambda} \varepsilon(r)\,dr\right)^2 ds \\ &= C\frac{1}{2\lambda}\left(e^{-2}-e^{-2\lambda}\right)\left(\lambda\int_0^{1/\lambda} \varepsilon(r)\,dr\right)^2 \end{aligned}$$

$$\leq C\frac{e^{-2}}{2}\lambda\left(\int_0^{\frac{1}{\lambda}}\varepsilon(r)dr\right)^2$$
$$= C\lambda\left(\int_0^{\frac{1}{\lambda}}\varepsilon(r)dr\right)^2.$$

The last term can be evaluated as in [15]. It was shown that if ε' has the representation $|\varepsilon'(r)| \asymp r^{-3/2}f(r)$ with f differentiable and increasing, and $|\varepsilon'|$ decreasing, then

$$I_{2,2} \leq Cf^2(\frac{1}{\lambda}).$$

We can rewrite this as

$$\begin{aligned}
I_{2,2} &\leq Cf^2(\frac{1}{\lambda}) = C\lambda^3\lambda^{-3}f^2(\frac{1}{\lambda})\\
&= C\frac{1}{\lambda^3}\left(\left(\frac{1}{\lambda}\right)^{-\frac{3}{2}}f(\frac{1}{\lambda})\right)^2 \leq C\frac{1}{\lambda^3}(\varepsilon')^2(\frac{1}{\lambda})\\
&= C\frac{1}{\lambda}\frac{1}{\lambda^2}(\varepsilon')^2(\frac{1}{\lambda}) \leq C\frac{1}{\lambda}\varepsilon^2(\frac{1}{\lambda})\\
&= C\lambda\left[\frac{1}{\lambda}\varepsilon(\frac{1}{\lambda})\right]^2\\
&\leq C\lambda\left(\int_0^{\frac{1}{\lambda}}\varepsilon(r)dr\right)^2,
\end{aligned}$$

where the last inequality was obtain by the monotonicity of ε.

Putting the bounds of the three terms together we obtain

$$I_2(\lambda,1) \leq I_{2,0}(\lambda) + I_{2,1}(\lambda) + I_{2,2}(\lambda) \leq C\lambda\left(\int_0^{\frac{1}{\lambda}}\varepsilon(r)dr\right)^2.$$

Step 2b: upper bound, first term. For the upper bound of the first term in the evaluation of $I(\lambda)$, $I_1(\lambda,1)$, we use the fact that for $s \geq \frac{1}{\lambda}$ we have $e^{2\lambda s} \leq \frac{1}{(\lambda s)^2}$, we use a scalar change of variables and the monotonicity of ε in order to obtain

$$\begin{aligned}
I_1(\lambda,1) &= \int_0^1 e^{-2\lambda s}\varepsilon^2(s)ds \leq \int_0^{\frac{1}{\lambda}}\varepsilon^2(s)ds + \int_{\frac{1}{\lambda}}^1\frac{1}{(\lambda s)^2}\varepsilon^2(s)ds\\
&= \int_0^{\frac{1}{\lambda}}\varepsilon^2(s)ds + \frac{1}{\lambda}\int_1^{\lambda}\frac{1}{u^2}\varepsilon^2(\frac{u}{\lambda})du\\
&\leq \int_0^{\frac{1}{\lambda}}\varepsilon^2(s)ds + \frac{1}{\lambda}\varepsilon^2(\frac{1}{\lambda})\int_1^{\lambda}\frac{1}{u^2}du\\
&\leq \int_0^{\frac{1}{\lambda}}\varepsilon^2(s)ds + \frac{1}{\lambda}\varepsilon^2(\frac{1}{\lambda})\\
&\leq \int_0^{\frac{1}{\lambda}}\varepsilon^2(s)ds + \lambda\left(\int_0^{\frac{1}{\lambda}}\varepsilon(s)ds\right)^2.
\end{aligned}$$

Again using the assumption

$$\frac{|\varepsilon'(x)|}{\varepsilon(x)} \leq \frac{1}{2x},$$

we have

$$\int_0^x \varepsilon(r)\,dr \geq 2\int_0^x |\varepsilon'(r)|\,r dr = -2\varepsilon(x)\,x + 2\int_0^x \varepsilon(r)\,dr,$$

which implies $\int_0^x \varepsilon(r)\,dr \leq 2\varepsilon(x)\,x$ and in particular

$$\lambda \int_0^{\frac{1}{\lambda}} \varepsilon(r)\,dr \leq 2\varepsilon\left(\frac{1}{\lambda}\right).$$

Then using the monotonicity of ε,

$$\gamma^2(\frac{1}{\lambda}) = \int_0^{\frac{1}{\lambda}} \varepsilon^2(r)dr \geq \varepsilon(\frac{1}{\lambda}) \int_0^{\frac{1}{\lambda}} \varepsilon(r)dr \geq C\lambda \left(\int_0^{\frac{1}{\lambda}} \varepsilon(r)dr\right)^2.$$

This completes the upper bound proof since, now

$$I(\lambda) \leq \gamma^2(\frac{1}{\lambda}) + C\lambda \left(\int_0^{\frac{1}{\lambda}} \varepsilon(r)dr\right)^2 \leq \gamma^2(\frac{1}{\lambda}) + C\gamma^2(\frac{1}{\lambda}) = C\gamma^2(\frac{1}{\lambda}).$$

Step 3. Lower bound when $\lambda \geq 1$.

For the lower bound there is a simple strategy. It is certainly true that $1/(2\lambda) \leq 1$, and from (3.6) we obtain the trivial lower bound

$$I(\lambda) \geq \int_0^{\frac{1}{2\lambda}} e^{-2\lambda s} \left(\varepsilon(s) + \int_0^s (e^{u\lambda} - 1)\varepsilon'(u)dr\right)^2 ds.$$

Since the two terms inside the square are of opposite sign, our strategy is to show that the second term (in absolute value) is less than a constant K times the first term, with $K < 1$: if we have

$$\int_0^s (e^{u\lambda} - 1)|\varepsilon'(u)|dr \leq K\varepsilon(s), \tag{3.7}$$

then

$$\varepsilon(s) - \int_0^s (e^{u\lambda} - 1)|\varepsilon'(u)|dr \geq (1-K)\varepsilon(s),$$

which implies, using the fact that ε decreases,

$$\begin{aligned} I(\lambda) &\geq \int_0^{\frac{1}{2\lambda}} e^{-2\lambda s} \left((1-K)\varepsilon(s)\right)^2 ds \\ &= (1-K)\int_0^{\frac{1}{2\lambda}} e^{-2\lambda s}\varepsilon^2(s)ds \\ &\geq (1-K)\,e^{-1}\int_0^{\frac{1}{2\lambda}} \varepsilon^2(s)ds \\ &\geq (1-K)\,e^{-1}\int_0^{\frac{1}{2\lambda}} \varepsilon^2(2s)ds \end{aligned}$$

$$\geq (1-K)\, e^{-1} \int_0^{\frac{1}{\lambda}} \varepsilon^2(s)ds = (1-K)\, e^{-1}\gamma^2(\frac{1}{\lambda}),$$

which is all that is needed for the proof of the lower bound.

To establish this we use again the special condition (3.2). Since ε^2 is integrable at the origin, it holds that $\varepsilon^2(r) = o\left(r^{-1}\right)$ and thus we can also assume without loss of generality, similarly to condition (3.3) (see Remark 3.5), that there exists an increasing function g with with $\varepsilon(r) = r^{-\frac{1}{2}} g(r)$. Thus we get

$$\int_0^s (e^{u\lambda} - 1)|\varepsilon'(u)|dr \leq \frac{\lambda}{2} \int_0^s \frac{e^{u\lambda} - 1}{u\lambda} \varepsilon(u)du$$

Since the function $\frac{e^x - 1}{x}$ is bounded for $x \in [0, \frac{1}{2}]$, by 1 below and $\frac{\sqrt{e}-1}{2}$ above, we get

$$\begin{aligned}
\int_0^s (e^{u\lambda} - 1)|\varepsilon'(u)|dr &\leq \lambda \frac{\sqrt{e}-1}{2\sqrt{2}} \int_0^s \varepsilon(u)du \\
&= \lambda \frac{\sqrt{e}-1}{2\sqrt{2}} \int_0^s u^{-\frac{1}{2}} g(u)du \leq \lambda \frac{\sqrt{e}-1}{2\sqrt{2}} g(s) 2\sqrt{s} \\
&= \frac{\sqrt{e}-1}{\sqrt{2}} \lambda s \varepsilon(s) \\
&\leq \frac{\sqrt{e}-1}{2\sqrt{2}} \varepsilon(s).
\end{aligned}$$

Since $\frac{\sqrt{e}-1}{2\sqrt{2}} < 1$ this completes the proof for the lower bound when $\lambda \geq 1$.

Step 4: conclusion when $\lambda \geq 1$.

From the results of Steps 2 and 3, we have proved that for any $\lambda \geq 1$,

$$I(\lambda) \asymp \gamma^2(\frac{1}{\lambda}) = \gamma^2\left(\frac{1}{\max(\lambda, 1)}\right).$$

Step 5: case $\lambda \in [0, 1]$.

A precise estimate of $I(\lambda)$ is more difficult in this case, but we do not need to have a precise result. Indeed, we only need to show that for all $\lambda \in [0, 1]$,

$$I(\lambda) \asymp \gamma^2\left(\frac{1}{\max(\lambda, 1)}\right) = \gamma^2(1).$$

In other words, we only need to show that $I(\lambda)$ is bounded above and below by positive constants, uniformly in $\lambda \in [0, 1]$.

Using $e^x - 1 \leq ex$ for $x \in [0, 1]$, using $\varepsilon' < 0$, and integrating by parts (using the fact that $\varepsilon(s) = o\left(s^{-1/2}\right)$), the negative term (with the ε') in formula (3.6) is bounded above as

$$\begin{aligned}
\left| \int_0^s \left(e^{u\lambda} - 1\right) \varepsilon'(u)\, du \right|^2 &\leq e^2 \left| \int_0^s u\lambda \varepsilon'(u)\, du \right|^2 \\
&= e^2\lambda^2 \left| \int_0^s (\varepsilon(u) - \varepsilon(s))\, du \right|^2 \\
&\leq e^2 s\lambda^2 \int_0^s \varepsilon^2(u)\, du = e^2 s\lambda^2 \gamma^2(s).
\end{aligned}$$

where we used Jensen's inequality in the last step. Therefore we immediately have the upper bound

$$\begin{aligned} I(\lambda) &\leq \int_0^1 2\varepsilon^2(s)\,ds + \int_0^1 2e^2 s\lambda^2\gamma^2(s)\,ds \\ &\leq \left(2+2e^2\right)\gamma^2(1). \end{aligned}$$

For a lower bound, define

$$f(\lambda,s) = \varepsilon(s) + \int_0^s (e^{u\lambda}-1)\varepsilon'(u)du. \tag{3.8}$$

From (3.6), we see that $I(\lambda) \geq e^{-2}\int_0^1 f^2(\lambda,s)\,ds$. A positive lower on $I(\lambda)$ uniform for all $\lambda \in [0,1]$ now follows from the lemma below.

Step 6: final lemma, and conclusion.

Since the results above, including the next lemma, establish that $I(\lambda) \asymp \gamma^2(1/\max(\lambda,1))$ for all $\lambda \in \mathbf{R}_+$, the theorem follows; indeed we can assert

$$I_1 \asymp \sum_n \int_0^\infty \gamma^2(1/\max(\lambda,1))\,d\mu_n(\lambda) = \operatorname{tr}\left(\phi^* G_H(-A)\phi\right).$$

□

Lemma 3.6. *With $f(\lambda,s)$ as in (3.8),* $\min\left\{\int_0^1 f^2(\lambda,s)\,ds : \lambda\in[0,1]\right\} > 0$.

Proof. First note that f is differentiable with respect to s everywhere except at 0, and that we have

$$\frac{\partial f}{\partial s}(\lambda,s) = \varepsilon'(s)\,e^{s\lambda} < 0$$

so that $f(\lambda,\cdot)$ is decreasing on $(0,1]$. We have $\lim_{s\to 0}\varepsilon(s) = +\infty$, and we proved in Step 5 that

$$\lim_{s\to 0}\left|\int_0^s \left(e^{u\lambda}-1\right)\varepsilon'(u)\,du\right| \leq \lim_{s\to 0} e^2 s\gamma^2(s) = 0.$$

Therefore $\lim_{s\to 0} f(\lambda,s) = +\infty$. Hence for each $\lambda\in[0,1]$, there exists a value $s^*(\lambda)\in(0,1]$ such that $f(\lambda,s)\geq 1$ for all $s\leq s^*(\lambda)$, and define $s^*(\lambda)$ to be maximal such. Note that for those values of λ for which $f(\lambda,s)$ exceeds 1 for all $s\in[0,1]$, this simply means that the corresponding $s^*(\lambda)$'s are all equal to 1. Moreover, we calculate

$$\frac{\partial f}{\partial \lambda}(\lambda,s) = \int_0^s ue^{u\lambda}\varepsilon'(u)\,du < 0,$$

so that s^* is non-increasing. This means that, defining $s^{**} = s^*(1)$ which is strictly positive as noted above, we have for all $s\leq s^{**}$, for all $\lambda\in[0,1]$, $f(\lambda,s)\geq 1$, and we finally obtain

$$\int_0^1 f^2(\lambda,s)\,ds \geq \int_0^{s^{**}} f^2(\lambda,s)\,ds \geq s^{**} > 0,$$

finishing the proof of the lemma. □

4. Functional equations and space regularity

4.1. Evolution equations on manifolds. The abstract framework of Theorem 3.3 is useful in a number of more concrete situations. We will illustrate this point by investigating the so-called additive stochastic heat equation, namely a parabolic stochastic PDE of the form

$$u(t,x) = u_0(x) + \int_0^t \Delta_x u(s,x)\, ds + W(t,x) \tag{4.1}$$

for some Gaussian random field W on the cartesian product of $\mathbf{R}_+ \times M$ where M is a finite-dimensional space where Δ_x can be defined; W could thus range over a wide array of infinite-dimensional versions of our $B^\gamma(t)$ defined by (2.2).

Using again the *evolution* interpretation in the manner of Da Prato and Zabczyk [6], we replace (4.1) by

$$u(t,x) = P_t u_0(x) + \int_0^t P_{t-s} W(ds,\cdot)(x), \tag{4.2}$$

where $(P_t)_{t\geq 0}$ is the semigroup of operators generated by Δ_x. Indeed, a solution to (4.1) also solves (4.2), but the latter is considerably weaker, since the Wiener integral above may exist even if W is not a bonafide function in x.

More specifically, for an arbitrary smooth compact Riemannian manifold M and its Laplace-Beltrami operator Δ_x, let $(\lambda_n, e_n)_{n\in\mathbf{N}}$ be the eigenvalues and eigenfunctions of $-\Delta_x$ which we can arrange in increasing order with $\lambda_0 = 0$ and $\lambda_n > 0$ for all $n > 0$; then under the Riemannian inner product, $\{e_n\}_{n\in\mathbf{N}}$ can be chosen as an orthonormal basis for a Hilbert space of functions V on M, the space of square-integrable functions with respect to the Riemannian volume element.

We use for W the random field B^γ defined formally by

$$B^\gamma(t,x) = \sum_{n\in\mathbf{N}} \sqrt{q_n} e_n(x)\, B^\gamma(t),$$

where $(B_n^\gamma)_{n\in\mathbf{N}}$ is a family of independent copies of our B^γ in (2.2). If $\sum_n q_n$ is finite, this $B^\gamma(t,\cdot)$ is a V-valued Gaussian element, but if $\sum_n q_n$ is infinite, this definition is only formal, and in reality $B^\gamma(t,\cdot)$ is typically distribution-valued. Theorem 3.3 is easier to express in this framework because Δ_x and the spatial covariance of B^γ are both diagonalizable in the basis of V, and also Δ_x has a spectral gap. Theorem 3.3 implies that

$$u(t,x) = P_t u_0(x) + \int_0^t P_{t-s} B^\gamma(ds,\cdot)(x) \tag{4.3}$$

has a solution $u(t,\cdot)$ in V if and only if

$$\sum_{n\in\mathbf{N}} q_n \gamma^2\left(\frac{1}{\lambda_n}\right) < \infty, \tag{4.4}$$

and in this case the solution is a Gaussian random field on $\mathbf{R}_+ \times M$, and is also a V-valued Gaussian stochastic process. Since $\lim_{r\to 0}\gamma(r) = 0$, there are obviously solutions of (4.3) corresponding to non-summable sequences q_n, i.e. to fields B^γ which are not bonafide L^2 functions in x. This is the usual observation for additive

stochastic PDEs, generalized here to all scales of potential irregularity in time, and there seems to be little to gain by singling out the case of summable q_n.

The above development works also for non-compact manifolds, such as $\mathbf{R}^d$, where all above statements can be written using integrals with respect to $n^2 = \lambda \in \mathbf{R}_+$ instead of series over $n \in \mathbf{N}$, but the absence of a spectral gap for the Laplacian forces one to revert to expressions involving $\gamma^2(1/\max(\lambda, 1))$ instead of simply $\gamma^2(1/\lambda)$. We leave these details out.

Returning to the compact case, if condition (4.4) is satisfied more than just barely, one should expect some regularity for the solution in (4.3). We now illustrate this phenomenon in the specific example of the circle.

4.2. Solution regularity on the unit circle. Assume now that $M = S^1$, the circle parametrized by $[0, 2\pi)$. Therefore, it is most convenient to represent the eigenstructure of the Laplace-Beltrami Δ_x by saying that for each $n \in \mathbf{N}$ the eigenvalue $\lambda_n = -n^2$ has two eigenfunctions $e_n(x) = \cos(nx)$ and $f_n(x) = \sin(nx)$. Let now B^γ be as above, and assume in addition that it is homogeneous in the parameter x, a with a given covariance structure Q in space, which means that it can be represented as

$$B^\gamma(t,x) = \int_0^t \varepsilon(t-s)\, W(ds,x),$$

where the Gaussian field W on $\mathbf{R}_+ \times S^1$ has covariance $E[W(t,x)W(s,y)] = Q(x-y)\min(s,t)$. It is then possible to express the decomposition of B^γ in the trigonometric basis of $V = L^2\left(S^1\right)$, i.e. as a Gaussian Fourier series:

$$B^\gamma(t,x) = \sqrt{q_0}B_0^\gamma(t) + \sum_{n=1}^{\infty}\sqrt{q_n}\bar{B}_n^\gamma(t)\sin(nx) + \sum_{n=1}^{\infty}\sqrt{q_n}B_n^\gamma(t)\cos(nx), \quad (4.5)$$

where $(B_n^\gamma)_n$ and $\left(\bar{B}_n^\gamma\right)_n$ are independent families of independent copies of the B^γ in (2.2), and $(q_n)_n$ is a sequence of non-negative numbers. In fact, since Q is a positive definite function on S^1, and thus a member of $L^1\left(S^1\right)$, the values q_n are easily seen to be its Fourier coefficients. Since $\sin(nx)$ and $\cos(nx)$ share the eigenvalue $\exp\left(-n^2t\right)$ for P_t, we can immediately rewrite (4.3), assuming without loss of generality that $u_0 \equiv 0$, that

$$\begin{aligned} u(t,x) = \sqrt{q_0}\int_0^t e^{-(t-s)n^2}\, dB_0^\gamma(s) + \sum_{n=1}^{\infty}\sqrt{q_n}\cos(nx)\int_0^t dB_n^\gamma(s)\, e^{-(t-s)n^2} \\ + \sum_{n=1}^{\infty}\sqrt{q_n}\sin(nx)\int_0^t d\bar{B}_n^\gamma(s)\, e^{-(t-s)n^2} \end{aligned} \quad (4.6)$$

with, by Theorem 3.3, existence and uniqueness holding if and only if

$$\sum_{n\in\mathbf{N}} q_n\gamma^2\left(n^{-2}\right) < \infty. \quad (4.7)$$

More precisely, since the proof of Theorem 3.3 translates here as nothing more than an estimation of the variances of the Gaussian random variables in (4.6) from above and below, and (4.6) clearly shows that $u(t,\cdot)$ is spatially homogeneous, we have actually proved the following.

Corollary 4.1. *When $V = L^2\left(S^1\right)$, with the V-valued Gaussian process B^γ defined in (4.5), and u the solution (4.3) of the stochastic heat equation driven by B^γ, expressed for example in (4.6), for every fixed $t \in [0,1]$, $u(t,\cdot)$ is a homogeneous Gaussian field on S^1 satisfying*

$$u(t,\cdot) = \sum_{n\in\mathbb{Z}} \sqrt{s_n(t)}\left(\cos(n,\cdot)\, G_n + \sin(n,\cdot)\, \bar{G}_n\right),$$

where G and $\bar{G}$ are independent sequences of IID standard normal random variables, and

$$s_n(t) \asymp q_n \gamma^2(n^{-2}).$$

The commensurability constants depend on t, γ and the sequence $\{q_n\}_{n\in\mathbf{N}}$, but not on n.

This corollary is all that is needed to apply the regularity results in Section 3.3 of [21]. We leave the proof of the theorem below, which is no more than bookkeeping, to the reader. It gives sufficient, and largely necessary, conditions for the solution u to have a prescribed almost-sure modulus of continuity in space. Let Y be the Gaussian random field defined on S^1 by

$$Y(x) = \sqrt{q_0} Z_0 + \sum_{n=1}^{\infty} \sqrt{q_n}\gamma\left(n^{-2}\right)\left(Z_n \sin(nx) + \bar{Z}_n \cos(nx)\right), \tag{4.8}$$

where Z and $\bar{Z}$ are independent sequences of IID standard normal r.v.'s. Since Y is clearly a homogeneous Gaussian field on S^1, we can calculate its homogeneous canonical metric function δ_Y:

$$\delta_Y^2(x-y) = \mathbf{E}\left[\left(Y(x) - Y(y)\right)^2\right] = \sum_{n=1}^{\infty} \sqrt{q_n}\gamma\left(n^{-2}\right)\left(1 - \cos\left(n(x-y)\right)\right). \tag{4.9}$$

Consider then the function $f = f_{\delta_Y}$ defined on a neighborhood of 0 via the rule

$$f(\alpha) = f_\delta(\alpha) := \int_0^\infty \delta\left(\min\left(e^{-x^2}, \alpha\right)\right) dx = \int_0^{\delta(\alpha)} \sqrt{\log 1/\check{\delta}(\varepsilon)}\, d\varepsilon,$$

where $\check{\delta}$ is the inverse function of δ. From the work of Fernique [9], which interprets the so-called Entropy upper bound of Dudley (see [12]), we know that the function f, if its limit is 0 at 0, is an almost-sure uniform modulus of continuity for Y, i.e. that

$$\sup\left\{\frac{|Y(x) - Y(y)|}{f(|x-y|)} : x, y \in S^1\right\} \tag{4.10}$$

is almost-surely finite. The following theorem, established exactly like Theorem 4 in [21], shows that f is also an almost-sure uniform modulus of continuity for $u(t,\cdot)$; it gives a way to construct a Y and a $u(t,\cdot)$ which share a given function f as an almost-sure uniform modulus of continuity, by ensuring a convergence condition on the coefficients q_n; it even shows a converse is true in the sense that if $u(t,\cdot)$ has f as an almost-sure uniform modulus of continuity, then the convergence condition should hold.

Theorem 4.2. *Let f be an increasing continuous function on a neighborhood of 0 in $\mathbf{R}_+$, continuously differentiable everywhere except at 0, with $\lim_{0+} f = 0$. Let Y and u be given by (4.8) and (4.6). Let δ_f be given by*

$$\delta_f(\alpha) = f(\alpha)\left(\log(1/\alpha)\right)^{-1/2} - \int_0^\alpha f(r)\left(\log(1/r)\right)^{-3/2}(2r)^{-1}\,dr.$$

It is not necessary to assume that this function δ_f has the form in (4.9). It is, however, positive and increasing.

Sufficient Condition: *Assume that for any continuous, decreasing, differentiable function h on $[0,1]$ with $\int_0^1 h(x)\,dx < \infty$,*

$$\sum_n q_n \gamma^2\left(n^{-2}\right) h\left(\delta_f\left(\frac{1}{n}\right)^2\right) < \infty. \tag{4.11}$$

Then f is an almost-sure uniform modulus of continuity for both Y and $u(t,\cdot)$, in the sense of (4.10).

Necessary condition: sharp case: *When $f(r) \gg r^H$ for any $H > 0$, the converse is true. Namely, assume f is an almost-sure uniform modulus of continuity for Y or for $u(t,\cdot)$; then (4.11) holds.*

Necessary condition: Holder case: *When it is not true that $f(r) \gg r^H$ holds for all $H > 0$, the converse holds up to a logarithmic correction. Namely, assume f is an almost-sure uniform modulus of continuity for Y or for $u(t,\cdot)$; then (4.11) holds with $\delta_f(1/n)$ replaced by $\delta_f(1/n)\log(n)$.*

It should be noted that the canonical metrics of Y and $u(t,\cdot)$ are, up to constants, bounded above by δ_f.

A similar theorem which, instead of Condition (4.11), uses the condition that Y admits f as an almost-sure uniform modulus of continuity, also holds. See Theorem 3 in [21]. We finish this article with some examples of the precision allowed by the above theorem.

4.3. Examples on the unit circle.

4.3.1. *Fractional Brownian scale.* In the fractional Brownian scale, where $\gamma(r) \asymp r^{H_0}$, the logarithmic correction is not visible in the Hölder scale, because the correction needed to make the function $h(r) = 1/r$ integrable at the origin is also logarithmic, and because the ratio of $f(r) = r^{H'}$ over the corresponding δ_f is again in the logarithmic scale. Therefore we can state the following necessary and sufficient condition. The solution u to (4.7) is almost surely H'-Hölder-continuous in x for all $H' < H_1$ if and only if, for all $H' < H_1$,

$$\sum_n q_n n^{-4H_0+2H'} < \infty.$$

For instance, if $\sqrt{q_n} \asymp n^{-1/2-H''}$, we get

$$H'' > H_1 - 2H_0. \tag{4.12}$$

To be more precise, including the logarithmic terms, and using a general γ, to get that u is precisely H_1-Hölder continuous, i.e. to get $f(r) = r^{H_1}$, we see

that $\delta_f(r) \leq r^{H_1} \log^{-1/2}(1/r)$, so it is sufficient to choose q_n such that (using $h(r) = r^{-1} \log^{-1}(1/r) (\log\log)^{-1}(1/r)$),

$$\begin{aligned}\infty &> \sum_n q_n \gamma^2\left(n^{-2}\right) n^{2H_1} \cdot \log n \cdot \log^{-1}\left(n^{2H_1} \log n\right) (\log\log)^{-1}\left(n^{2H_1} \log n\right) \\ &\asymp \sum_n q_n \gamma^2\left(n^{-2}\right) n^{2H_1} (\log\log)^{-1}(n).\end{aligned}$$

Similarly, to obtain a u which has exactly the same regularity in space as the fBm with parameter H_1, we need $f(r) = r^{H_1} \log^{1/2}(1/r)$. Thus we get $\delta_f(r) \leq r^{H_1}$, and we only need to require that,

$$\infty > \sum_n q_n \gamma^2\left(n^{-2}\right) n^{2H_1} \log^{-1}(n) (\log\log)^{-1}(n).$$

Because of the logarithmic correction needed to make the converse work, we can only state, for instance, that if $f(r) = r^{H_1}$ is an almost-sure modulus of continuity for u in space, then for any $\alpha > 1$,

$$\sum_n q_n \gamma^2\left(n^{-2}\right) n^{2H_1} \log^{-2}(n) (\log\log)^{-\alpha}(n) < \infty,$$

and the $\log^{-2} n$ above should be replaced by $\log^{-1} n$ if we only know that u has the same regularity in space as fBm with parameter H_1.

4.3.2. *Logarithmic regularity scale.* The case of Gaussian fields whose almost-sure modulus of continuity is commensurate with $f(r) = \log^{-\beta}(1/r)$ for $\beta > 0$, which we like to call the *logarithmic Brownian scale*, coincides, according to our statements regarding Y in the above theorem, with $\delta_f(r) \asymp \log^{-\beta-1/2}(1/r)$, and coefficients q_n satisfying, up to a triply iterated logarithmic term,

$$\sum_n q_n \gamma^2\left(n^{-2}\right) \log^{2\beta}(n) (\log\log)^{-1}(n) < \infty.$$

More precisely, the above condition is sufficient for u to have $f(r) = \log^{-\beta}(1/r)$ as a uniform modulus of continuity in x, but if the latter holds, then the above series converges if one adds a factor $(\log\log\log)^{-\alpha}(n)$ for any $\alpha > 1$.

When B^γ itself is in the logarithmic Brownian scale in time, meaning $\gamma(r) \asymp \log^{-\beta_0-1/2}(1/r)$ for some $\beta_0 > 0$, we find that $\log^{-\beta_1}(1/r)$ is a uniform modulus of continuity for u in x as soon as

$$\sum_n q_n \log^{2\beta_1-2\beta_0-1}(n) (\log\log)^{-1}(n) < \infty$$

with the condition being necessary if a factor $(\log\log\log)^{-\alpha}(n)$ is added, and indeed the necessary and sufficient condition is simply that

$$\sum_n q_n a_n \log^{2\beta_1-2\beta_0}(n) < \infty$$

for all positive sequences $\{a_n\}_{n\in\mathbf{N}}$ such that $n^{-1}a_n$ is summable. For instance, if we assume that $\sqrt{q_n} \asymp n^{-1/2} \log^{-2\beta''}(n)$, we see that we only need to take

$$\beta'' \geq \beta_1 - \beta_0. \tag{4.13}$$

4.3.3. *Conclusion.* While the last result may seem esoteric, it actually has an important interpretation, when compared to (4.12). First we have the fact that (4.13) is more precise than (4.12) – we have an exact upper bound on β'', not a gap as required in (4.12). But more importantly condition (4.12) indicates that to obtain a H_1-Hölder-continuous solution u, the Hölder-continuity of B^γ in space (measured by H'') has to be combined with B^γ's Hölder-continuity in time (measure by H_0), but that the latter is twice as strong as the former; this is a phenomenon familiar to those who know that for the standard stochastic heat equation with infinite-dimensional Brownian potential (here $\gamma(r) = r^{-1/2}$), when the spatial regularity of B^γ is such that the solution is H-Hölder-continuous in space, then it is only $H/2$-Hölder-continuous in time. The situation in the logarithmic scale is not the same. Condition (4.13) shows that the combined logarithmic continuity of B^γ in space and time are to be compared with equal weights $(\beta_0 + \beta'')$, i.e. without the factor 2 in time, with the solution's logarithmic continuity. In conclusion, the common intuition saying that the stochastic heat equation's regularity is twice as strong in space as it is in time, the factor 2 being due to the quadratic variation of Brownian motion, is misleading. We see here that, in the Hölder scale, the effect of the potential B^γ's time regularity is always twice as heavy as its space regularity, that this appear to be a general property of the heat equation since it has nothing do to with the presence of white-noise in time, as it holds for all $\gamma(r) \asymp r^{H_0}$, not just $H_0 = 1/2$. But on the other hand, the relative strengths of the potential's time regularity is equal, not double, its space regularity, in the logarithmic regularity scale, which means that the type of noise *can* make a difference in how the potential's regularity effects on the heat equation, even though one has to reach to logarithmic regularity to deviate from the familiar rule by which a potential's time regularity effect's the solution twice as strongly as its space regularity.

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