LOCALLY INTEGRABLE PROCESSES WITH RESPECT TO LOCALLY ADDITIVE SUMMABLE PROCESSES

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Abstract. In [MD] we defined and studied a class of summable processes, called *additive summable processes*, that is larger than the class previously studied by Dinculeanu and Brooks [D–B]. We also defined a stochastic integral with respect to an additive summable process and proved several properties of the integral. In this article we consider examples of processes that are integrable with respect to an additive summable process or locally integrable with respect to a locally additive summable processes. In particular, we show that if X is a locally additive summable process, then X_- is integrable with respect to X. This is essential, for example, in proving an Ito formula for locally additive summable processes.

1. Introduction

The larger context in which this article, which is a continuation of [MD], could be viewed is stochastic integration for Banach-valued processes, studied from a measure-theoretical point of view.

Classical stochastic integration (for real-valued processes) considers integrals with respect to semimartingales (Dellacherie and Meyer [DM78]). Similar techniques were applied by Kunita [Kun70] to the case of Hilbert-valued processes; however, this approach cannot be easily adapted to the case of Banach spaces, since it relies on using the inner product.

Dinculeanu [Din00], Diestel and Uhl [DU77], and Kussmaul [Kus77], present detailed accounts of different approaches to vector integration. Brooks and Dinculeanu [BD76] were the first to introduce a version of integration with respect to a vector measure with finite semivariation. A few years later, the same authors presented a stochastic integral with respect to so-called summable Banach-valued processes.

A Banach-valued process X is called *summable* if the Doleans-Dade measure I_X defined on the ring generated by the predictable rectangles can be extended to a σ-additive measure with finite semivariation on the corresponding σ-algebra $\mathcal{P}$.

In [Din00] Dinculeanu develops the theory of integration with respect to a summable process from a measure-theoretical point of view. In this case, the summable process X plays the role played by the square-integrable martingale in the classical

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theory: a stochastic integral $H \cdot X$ with respect to X is defined as a cadlag modification of the process $\left(\int_{[o,t]} H\, dI_X\right)_{t\geq 0}$ of integrals with respect to I_X such that $\int_{[0,t]} H dI_X \in L^p_G$ for every $t \in \mathbb{R}_+$.

The class of summable processes includes all the processes considered in the classical theory (Hilbert-valued square-integrable martingales and processes with integrable variation), but it also includes processes with integrable *semivariation* (see the definition below), as long as the co-domain Banach space E satisfies some restrictions.

In [MD] we considered a further generalization of the stochastic integral, in which we extend the notion of summability to a larger class of processes, called *additive summable*, with the goal of eliminating some of the restrictions on the space E. Additive summability (see Section 2 below for details) is obtained by relaxing the definition of summability by requiring that I_X be extendible to an additive (rather than σ-additive) measure on $\mathcal{P}$, but in such a way that each of the measures $(I_X)_z$, for $z \in Z$ (a norming space for L^p_G) is σ-additive. Using additive summability instead of summability, we defined stochastic integration in the same way, and proved many basic properties of the integral and of its stopped version. We also showed that the class of additive summable processes is strictly richer than the class of summable processes. In [MD] as well as in this paper, the difficulty in proving results similar to those in [Din00] arises from the fact that the measure I_X is not σ-additive but rather additive, therefore many convergence and extension theorems can not be applied.

All the results in [MD] are measure theoretical results; now we would like to turn our attention to a more applied point of view. In particular, since many of the most important applications of stochastic analysis are obtained through the use of the Ito formula, in this article we lay the groundwork for establishing an Ito formula for locally additive processes.

The first question that arises when trying to establish an Ito formula for integration with respect to a process X is whether this process is integrable with respect to itself. This is one of the questions we are analyzing in this paper, for locally additive summable processes, by determining how large is the class of locally integrable processes.

For the sake of completeness In Section 2 we present the notations and definitions introduced in [MD]. In Section 3 we introduce the notions of local additive summability and local integrability with respect to a locally additive summable process, as well as the relationship between the two types of integrability, while in Section 4 we give three examples of locally integrable processes: elementary processes, σ-elementary processes and caglad processes. From here we deduce that if X is a locally additive summable process then X_- is integrable with respect to X, which will allow us, in future work, to determine an Ito formula for locally additive processes.

2. Notations and definitions

For the sake of completeness we introduce most of the definitions and notations used in this paper. For the remaining definitions and notations we might use, the reader is directed to [DM78] and [Din00].

2.1. **Additive Summable Processes.** We consider E, F, G Banach spaces with $E \subset L(F,G)$ continuously, that is, $|x(y)| \leq |x||y|$ for $x \in E$ and $y \in F$; for example, $E = L(\mathbb{R}, E)$.

Definition 1. If $m : \mathcal{R} \to E \subset L(F,G)$ is an additive measure defined on a ring $\mathcal{R}$ of subsets of a set S, for every set $A \subset S$ **the semivariation of** m on A relative to the embedding $E \subset L(F,G)$ (or relative to the pair (F,G)) is denoted by $\tilde{m}_{F,G}(A)$ and defined by the equality

$$\tilde{m}_{F,G}(A) = \sup |\sum_{i \in I} m(A_i)x_i|,$$

where the supremum is taken for all finite families $(A_i)_{i\in I}$ of disjoint sets from $\mathcal{R}$ contained in A and all families $(x_i)_{i\in I}$ of elements from F_1, the unit ball of F.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfies the usual conditions, and X be a cadlag, adapted process $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$, such that $X_t \in L^p_E$ for every $t \geq 0$ and $1 \leq p < \infty$.

Let $\mathcal{S}$ be the semiring of predictable rectangles and $I_X : \mathcal{S} \to L^p_E$ the stochastic measure defined by

$$I_X(\{0\} \times A) = 1_A X_0, \text{ for } A \in \mathcal{F}_0$$

and

$$I_X((s,t] \times A) = 1_A(X_t - X_s), \text{ for } A \in \mathcal{F}_s.$$

Note that I_X is finitely additive on $\mathcal{S}$ therefore it can be extended uniquely to a finitely additive measure on the ring $\mathcal{R}$ generated by $\mathcal{S}$.

Let $Z \subset (L^p_G)^*$ be a norming space for L^p_G (a subspace Z of the dual space B^* of a Banach space B is called a *norming space for* B, if for every $x \in B$ we have $|x| = \sup_{z\in Z_1} |\langle x, z\rangle|$, Z_1 being the unit ball of Z.) For each $z \in Z$ we define a measure $(I_X)_z : \mathcal{R} \to F^*$ by

$$\langle y, (I_X)_z(A)\rangle = \langle I_X(A)y, z\rangle = \int \langle I_X(A)(\omega)y, z(\omega)\rangle dP(\omega), \text{ for } A \in \mathcal{P} \text{ and } y \in F$$

where the bracket in the integral represents the duality between G and G^*.

Since $L^p_E \subset L(F, L^p_G)$, we can consider the semivariation of I_X relative to the pair (F, L^p_G). To simplify the notation, we shall write $(\tilde{I}_X)_{F,G}$ instead of $(\tilde{I}_X)_{F,L^p_G}$ and we shall call it the semivariation of I_X relative to (F,G):

Definition 2. Let $\mathcal{P}$ be the σ-algebra generated by $\mathcal{R}$. We say that X is p**-additive summable** relative to the pair (F,G) if I_X has an additive extension $I_X : \mathcal{P} \to L^p_E$ with finite semivariation relative to (F,G), and such that the measure $(I_X)_z$ is σ-additive for each $z \in (L^p_G)^*$.

If $p = 1$, we say, simply, that X is additive summable relative to (F,G).

Remark 3. A summable process was defined in a similar fashion, with the difference that the measure I_X to have a $\sigma-$additive extension to $\mathcal{P}$, hence the definition of additive summability is weaker.

Remark 4. The problems that might appear if (I_X) is not $\sigma-$additive are convergence problems (most of the convergence theorem are stated for $\sigma-$additive measures) and extension problems (the uniqueness of extensions of measures usually requires $\sigma-$additivity).

2.2. **The Stochastic Integral.** Let X be a p-additive summable process relative to (F, G).

Consider the additive measure $I_X : \mathcal{P} \to L^p_E \subset L(F, L^p_G)$ with bounded semivariation $\tilde{I}_{F,G}$ relative to (F, L^p_G) for which each measure $(I_X)_z$ is σ-additive and with finite variation $|(I_X)_z|$, for every $z \in (L^p_F)^*$.

Then we have

$$(\tilde{I}_X)_{F,G} = \sup\{|(I_X)_z| : \|z\| \leq 1, z \in (L^p_F)^*\},$$

(See Proposition 4.13 in [Din00].)

We denote by $\mathcal{F}_{F,G}(X)$ the space of predictable processes $H : \mathbb{R}_+ \times \Omega \to F$ such that

$$\tilde{I}_{F,G}(H) = \sup\{\int |H| d|(I_X)_z| : \|z\|_q \leq 1\} < \infty.$$

Definition 5. For any $H \in \mathcal{F}_{F,G}(X)$ We define the integral $\int H dI_X$ to be the mapping $z \mapsto \int H d(I_X)_z$.

Remark 6. If $H \in \mathcal{F}_{F,G}(X)$ the integral $\int H d(I_X)_z$ is defined and is a scalar for each $z \in Z$, hence the mapping $z \mapsto \int H d(I_X)_z$ is a continuous linear functional on $(L^p_G)^*$. Therefore, $\int H dI_X \in (L^p_G)^{**}$,

$$\langle \int H dI_X, z\rangle = \int H d(I_X)_z, \text{ for } z \in Z$$

and

$$|\int H dI_X| \leq \tilde{I}_{F,G}(H).$$

Remark 7. Let $H \in \mathcal{F}_{F,G}(X)$. Then, for every $t \geq 0$ we have $1_{[0,t]}H \in \mathcal{F}_{F,G}(X)$.

Definition 8. We denote by $\int_{[0,t]} H dI_X$ the integral $\int 1_{[0,t]} H dI_X \in (L^p_G)^{**}$. We define

$$\int_{[0,\infty]} H dI_X := \int_{[0,\infty)} H dI_X := \int H dI_X.$$

For each $H \in \mathcal{F}_{F,G}(X)$ we obtain a family $(\int_{[0,t]} H dI_X)_{t \in \mathbb{R}_+}$ of elements of $(L^p_G)^{**}$.

We restrict ourselves to processes H for which $\int_{[0,t]} H dI_X \in L^p_G$ for each $t \geq 0$. Since L^p_G is a set of equivalence classes, $\int_{[0,t]} H dI_X$ represents an equivalence class. We use the same notation for any random variable in its equivalence class. We are interested to see whether or not the process $(\int_{[0,t]} H dI_X)_{t \geq 0}$ is adapted and if it admits a cadlag modification.

It is not clear weather there is a cadlag modification of the previously defined process $(\int_{[0,t]} H dI_X)_t$. Therefore we use the following definition

Definition 9. We define by $L^1_{F,G}(X)$ the set of all processes $H \in \mathcal{F}_{F,G}(I_X)$ that satisfy the following two conditions:

a) $\int_{[0,t]} H dI_X \in L^p_G$ for every $t \in \mathbb{R}_+$;

b) The process $(\int_{[0,t]} H dI_X)_{t \geq 0}$ has a cadlag modification.

The processes $H \in L^1_{F,G}(X)$ are said to be **integrable with respect to** X.

If $H \in L^1_{F,G}(X)$, then any cadlag modification of the process $(\int_{[0,t]} HdI_X)_{t\geq 0}$ is called **the stochastic integral of H with respect to X** and is denoted by $H \cdot X$ or $\int HdX$:

$$(H \cdot X)_t(\omega) = (\int HdX)_t(\omega) = (\int_{[0,t]} HdI_X)(\omega), \text{ a.s.}$$

Therefore the stochastic integral is defined up to an evanescent process. For $t = \infty$ we have

$$(H \cdot X)_\infty = \int_{[0,\infty]} HdI_X = \int_{[0,\infty)} HdI_X = \int HdI_X.$$

Remark 10. In [MD] we showed that the stochastic integral $H \cdot X$ is a cadlag, adapted process.

3. Local Summability and Local Integrability

Definition 11. We say X is **locally p-additive summable** relative to (F, G) if there is an increasing sequence (T_n) of stopping times, with $T_n \uparrow \infty$, such that for each n, the stopped process X^{T_n} is p-additive summable relative to (F, G). The sequence (T_n) is called a determining sequence for the local additive summability of X.

Definition 12. A predictable process $H : \mathbb{R}_+ \times \Omega \to F$ is said to be **locally integrable with respect to** X, if there is an increasing sequence (T_n) of stopping times with $T_n \uparrow \infty$, such that, for each n, X^{T_n} is p-additive summable relative to (F, G) and $1_{[0,T_n]}H$ is integrable with respect to X^{T_n}. We say that (T_n) is a determining sequence for the local integrability of H with respect to X.

Theorem 13. *Let X be a p-additive summable process relative to (F, G) and $H \in \mathcal{F}_{F,G}(X)$. Then H is integrable with respect to X iff H is locally integrable with respect to X. Regardless of the type of integrability (i.e. local or not) the stochastic integral $H \cdot X$ is the same.*

Proof. The proof follows from Theorems 4 and 15 b) in [MD]. Indeed, if H is integrable with respect to X, and $T_n \uparrow \infty$ is a sequence of stopping times, then by Theorem 15 b) in [MD], we have $1_{[0,T_n]}H \in L^1_{F,G}(X)$ and $1_{[0,T_n]}H \in L^1_{F,G}(X^{T_n})$; therefore H is locally integrable with respect to X. Then

$$\lim_n (1_{[0,T_n]}H \cdot X^{T_n}) = \lim_n (H \cdot X)^{T_n} = H \cdot X,$$

hence the two stochastic integrals coincide.

On the other hand, if H is locally integrable with respect to X and (T_n) is a determining sequence of stopping times, then $1_{[0,T_n]}H \in L^1_{F,G}(X^{T_n})$ and by Theorem 15 b) in [MD] we have $1_{[0,T_n]}H \in L^1_{F,G}(X)$. If we show that the sequence $H^n = 1_{[0,t]}1_{[0,T_n]}H$ satisfies the hypothesis of Theorem 4 in [MD] then we conclude that $\int 1_{[0,t]}HdI_X \in L^p_G$ and the only statement to prove for the proof of our present theorem to be complete is that the process $(\int_{[0,t]} HdI_X)_{t\geq 0}$ is cadlag.

Let us verify first the assumption of Theorem 4 in [MD]. We observe that for each $t \geq 0$ we have $1_{[0,t]}1_{[0,T_n]}H \to 1_{[0,t]}H$, pointwise, $|1_{[0,t]}1_{[0,T_n]}H| \leq |H|$, for each n, and $\int_{[0,t]} 1_{[0,T_n]}HdI_{X^{T_n}} \in L^p_G$. Also by theorem 15 b) in [MD] $\int_{[0,t]} 1_{[0,T_n]}HdI_X = ((1_{[0,T_n]}H) \cdot X)_t = ((1_{[0,T_n]}H) \cdot X^{T_n})_t$. It remains to show that this last sequence converges pointwise. Indeed, for each $t \geq 0$ fixed, and $\omega \in \Omega$, we choose $N = N_\omega$ such that $t < T_N(\omega)$. Then, for $n \geq N$ we have

$$\left(\int_{[0,t]} 1_{[0,T_n]} H dI_X\right)(\omega) = ((1_{[0,T_n]}H) \cdot X)_t(\omega) = (1_{[0,T_n]}H \cdot X)_t^{T_N}(\omega)$$
$$= (1_{[0,T_N]}1_{[0,T_n]}H \cdot X)_t(\omega) = (1_{[0,T_N]}H \cdot X)_t(\omega) \tag{1}$$

where the equalities follow from Theorem 15 b) in [MD] and the fact that $t < T_N(\omega) \leq T_n(\omega)$. Hence the sequence is pointwise convergent and now we are able to apply Theorem 4 in [MD] to conclude that $\int 1_{[0,t]} H dI_X \in L^p_G$

$$\lim_n \left(\int_{[0,t]} 1_{[0,T_n]} H dI_X\right)(\omega) = \int_{[0,t]} H dI_X, \text{ pointwise.}$$

As we said above it remains to show that the process $(\int_{[0,t]} H dI_X)_{t\geq 0}$ is cadlag. Indeed, for each $\omega \in \Omega$ and $N = N_\omega$ as above, we have, by equality 1

$$(\int_{[0,t]} H dI_X)(\omega) = (1_{[0,T_N]}H \cdot X)_t(\omega),$$

hence, the process $(\int_{[0,t]} H dI_X)_{t\geq 0}$ is cadlag. Therefore $H \cdot X$ exists and

$$(H \cdot X)_t = \int_{[0,t]} H dI_X = \lim_n (1_{[0,T_n]}H \cdot X)_t.$$

$\square$

4. Examples of locally integrable processes

4.1. **Elementary and σ-elementary processes.** In this section we show that certain elementary and σ-elementary processes are integrable, but, in general, not all of them are integrable or locally integrable. Again, we remind the reader that for a process to be integrable, we need not only the integral $\int_{[0,t]} H dI_X$ to exist for each t, but also the process $(\int_{[0,t]} H dI_X)_t$ to be cadlag.

Theorem 14. *a) Let H be an F-valued, elementary process of the form*

$$H = H_0 1_{\{0\}} + \sum_{1\leq i\leq n} H_i 1_{(T_i, T_{i+1}]},$$

where $0 = T_0 \leq T_1 \leq \cdots \leq T_{n+1}$ are stopping times and for each $i = 0, 1, 2, \ldots, n$, H_i is an F-valued, $\mathcal{F}_{T_i}$-measurable, bounded random variable. Then $H \in L^1_{F,G}(X)$ and the stochastic integral $H \cdot X$ can be computed pathwise:

$$(H \cdot X) = H_0 X_0 + \sum_{1\leq i\leq n} H_i (X^{T_{i+1}} - X^{T_i}).$$

b) Let $0 = T_0 \leq T_1 \leq \cdots \leq T_{n+1}$ be predictable stopping times and for each $i = 0, 1, 2, \ldots, n$, H_i be an F-valued, $\mathcal{F}_{T_i -}$-measurable, bounded random variable. Then an F-valued, elementary process of the form

$$H = \sum_{0\leq i\leq n} H_i 1_{[T_i, T_{i+1})},$$

in general, is not in $L^1_{F,G}(X)$, unless the additive summable process X is continuous. In that case, the stochastic integral $H \cdot X$ can be computed pathwise:

$$(H \cdot X) = \sum_{0 \leq i \leq n} H_i(X^{T_{i+1}} - X^{T_i}).$$

Proof. For each $1 \leq i \leq n$, we have, by Proposition 8 in [MD],

$$X_t^{T_i} = X_{T_i \wedge t} \in L_E^p, \text{ hence } H_i X_t^{T_i} \in L_G^p.$$

Assume now that H_i are simple random variables. Then, by Proposition 9 in [MD], for any pair $(T_i^n)_n$, $(T_{i+1}^n)_n$ of sequences of simple stopping times, with $T_i^n \downarrow T_i, T_{i+1}^n \downarrow T_{i+1}$, such that $T_i^n \leq T_{i+1}^n$ for each n, we have

$$(2) \quad \langle \int H_i 1_{(T_i, T_{i+1}]}\, dI_X, z \rangle = \lim_n \langle H_i(X_{T_i^n} - X_{T_{i+1}^n}), z \rangle = \langle H_i(X_{T_i} - X_{T_{i+1}}), z \rangle,$$

for $z \in (L_G^p)^*$, where the bracket represents the duality between L_G^p and $(L_G^p)^*$. Since this is true for every $z \in (L_G^p)^*$ we deduce that, if H_i are simple random variables for every $0 \leq i \leq n$, then

$$\int_{(0,t]} H_i 1_{(T_i, T_{i+1}]} dI_X = H_i(X_{T_{i+1} \wedge t} - X_{T_i \wedge t}).$$

If H_i are bounded random variables, then there is a sequence H_i^n of simple random variables from $\mathcal{F}_{T_i}$, with $H_i^n \to H_i$ and $|H_i^n| \leq |H_i|$ for every n, and every i. Then $H_i^n 1_{(T_i \wedge t, T_{i+1} \wedge t]} \to H_i 1_{(T_i \wedge t, T_{i+1} \wedge t]}$ pointwise. Also, since

$$\int_{(0,t]} H_i^n 1_{(T_i, T_{i+1}]} dI_X = H_i^n(X_{T_{i+1} \wedge t} - X_{T_i \wedge t}) \in L_G^p$$

for every $n \geq 1$, and $H_i^n(X_{T_{i+1} \wedge t} - X_{T_i \wedge t}) \to H_i^n(X_{T_{i+1} \wedge t} - X_{T_i \wedge t})$ in G, pointwise on Ω we can apply Theorem 4 a) and b) in [MD] and deduce that $\int_{(0,t]} H_i 1_{(T_i, T_{i+1}]} dI_X \in L_G^p$ and

$$\int_{(0,t]} H_i^n 1_{(T_i, T_{i+1}]} dI_X \to \int_{(0,t]} H 1_{(T_i, T_{i+1}]} dI_X,$$

pointwise and in L_G^1. Hence

$$\int_{(0,t]} H_i 1_{(T_i, T_{i+1}]} dI_X = H_i(X_{T_{i+1} \wedge t} - X_{T_i \wedge t}).$$

Moreover, since X is cadlag, each process X^{T_i} is cadlag, hence $H_i 1_{(T_i, T_{i+1}]} \in L^1_{F,G}(X)$ and

$$(H_i 1_{(T_i, t_{i+1}]} \cdot X)_t = H_i(X_t^{T_{i+1}} - X_t^{T_i}).$$

We have to argue separately the case $i = 0$, but the proof uses the argument from above. Take, now, $T_i = T_{i+1} = 0$. Then

$$\int_{[0,t]} H_0 1_{\{0\}} dI_X = H_0 X_0,$$

hence $H_0 1_{\{0\}} \cdot X \in L^1_{F,G}(X)$ and

$$(H_0 1_{\{0\}} \cdot X)_t = H_0 X_0.$$

It follows that $H \in L^1_{F,G}(X)$ and

$$(H \cdot X)_t = H_0 X_0 + \sum_{1 \leq i \leq n} H_i(X_t^{T_{i+1}} - X_t^{T_i}).$$

b) Since T_i are predictable stopping times there are increasing sequences of stopping times $(T_i^n)_n$, with $T_i^n \uparrow T_i$. Then the equality 2 in assertion a) becomes

$$\langle \int H_i 1_{(T_i,T_{i+1}]}\, dI_X, z\rangle = \lim_n \langle H_i(X_{T_i^n} - X_{T_{i+1}^n}), z\rangle = \langle H_i(X_{T_i-} - X_{T_{i+1}-}), z\rangle,$$

for $z \in (L_G^p)^*$, and with the same argument as in assertion a) we can prove that

$$\int H_i 1_{(T_i,T_{i+1}]} dI_X = H_i(X_{T_{i+1}-} - X_{T_i-}) \in L_G^p$$

but this process is not cadlag, hence the integral $\int H_i 1_{(T_i,T_{i+1}]} dI_X$ can not be the stochastic integral. If the process X is continuous, then $X_{T_i-} = X_{T_i}$ and the process $H_i(X_{T_{i+1}-} - X_{T_i-})$ is cadlag, hence $H_i 1_{(T_i,T_{i+1}]} \in L^1_{F,G}(X)$, and, as above,

$$(H \cdot X)_t = \sum_{1\le i\le n} H_i(X_t^{T_{i+1}} - X_t^{T_i}).$$

□

If the process is σ-elementary rather than elementary then the process might not be integrable, but as we will see in the next theorem, it will be locally integrable, even if the random variables H_i are not bounded.

Theorem 15. *Assume X is locally p-additive summable relative to (F, G) and let H be a σ-elementary process of the form*

$$H = H_0 1_{\{0\}} + \sum_{1\le i<\infty} H_i 1_{(T_i,T_{i+1}]},$$

where $0 = T_0 \le T_1 \le T_2 \le \dots$ is a sequence of stopping times with $T_i \uparrow \infty$ and for $0 \le i < \infty$, H_i is $\mathcal{F}_{T_i}$-measurable. Then H is locally integrable with respect to X

Proof. The idea is to reduce this case to the case in assertion a) of the previous theorem. There are three main differences between the two case:
1) The process X is not p-additive summable but rather locally p-additive summable.
2) The random variables H_i are not necessarily bounded.
3) The sum in the formula of the process H is not finite.
All of the differences could be addressed in an simple manner.

For 1) we consider $S_n \uparrow \infty$ a sequence of stopping times, determining for the local p-additive summability of X. Then X^{S_n} is p-additive summable for each n. For 2) and 3) we observe that for each t and ω fixed the sum in the formula of H is a finite sum, and we consider , for each n, the stopping time $R_n = \inf\{t : |H_t| > n\}$. Since H is caglad, we have $R_n \uparrow \infty$ and $1_{[0,R_n]}|H| \le n$. Then for each i we have $1_{[0,R_n]}|H_i| \le n$, and $1_{[0,R_n\wedge T_n]}H$ is an elementary process.

In order to address now all problems at the same time we are looking at the sequence of stopping times $S_n \wedge R_n \wedge T_n$. Indeed, since $X^{S_n\wedge R_n\wedge T_n} = (X^{S_n})^{R_n\wedge T_n}$ and X^{S_n} is p-additive summable, by the results in Section 2.6 of [MD], the process $X^{S_n\wedge R_n\wedge T_n}$ is p-additive summable. Also, as above, $1_{[0,S_n\wedge R_n\wedge T_n]}H$ is an elementary process, hence by the previous theorem, it is integrable. It follows that H is locally integrable with respect to X, where the determining sequence is $S_n \wedge R_n \wedge T_n$ □

Remark 16. If the σ-elementary process is of the form

$$H = \sum_{1 \le i < \infty} H_i 1_{[T_i, T_{i+1})},$$

the process X would also need to be continuous in order for H to be locally integrable.

4.2. **Caglad processes.**

Theorem 17. *Let $H : \mathbb{R}_+ \times \Omega \to F$ be a caglad, adapted process and X be a locally p-additive summable process relative to (F, G). Then $H \in L^1_{F,G}(X)_{loc}$.*

Proof. The proof follows in three steps. In the first step we define a sequence of σ-elementary processes H^n such that $H^n \to H$ uniformly. In the second step we show that if a sequence of locally integrable processes H^n that are in $L^1_{F,G}(X)_{\mathrm{loc}}$, converges pointwise uniformly to a process H then $H \in L^1_{F,G}(X)_{\mathrm{loc}}$. The third step puts everything together to deduce the conclusion of the theorem.

STEP 1) Construct the sequence of H^n.

For fixed n, define the stopping times $T_0^n = 0$, and for each $k \ge 1$, define T_k^n by

$$T_k^n = \inf\{t > T_{k-1}^n : |H_t - (H_+)_{T_{k-1}^n}| > \frac{1}{n}\},$$

as long as $T_{k-1} < \infty$.

Observe that $(H_+)_{T_{k-1}}$ always exists since H is caglad, and that $T_{k+1}^n > T_k^n$ for all k. Indeed, since H is left continuous, for each $\omega \in \Omega$, there exists $\delta_\omega > 0$ such that for all $t \in (T_k^n(\omega), T_k^n(\omega) + \delta_\omega)$ we have $|H_t - (H_+)_{T_k^n(\omega)}| < \frac{1}{n}$, so $T_{k+1}^n(\omega) \ge T_k^n(\omega) + \delta(\omega) > T_k^n(\omega)$, thus $T_0^n, T_1^n, \cdots$ is a strictly increasing sequence, in k, for each n.

Now, for each n, define the σ-elementary process

$$H^n = H_0 1_{\{0\}} + \sum_{k=0}^{\infty} (H_+)_{T_k^n} 1_{(T_k^n, T_{k+1}^n]}.$$

There are two possibilities. Either $T_k^n \uparrow \infty$ as $k \to \infty$, or $T_k^n \uparrow a < \infty$ as $k \to \infty$. In the first case, for all $t \in [0, \infty)$ either $t = 0$, in which case $H_t^n = H_t$, or there is a k such that $t \in (T_k^n, T_{k+1}^n]$. Then, $|H_t - H_t^n| = |H_t - (H_+)_{T_k^n}| \le \frac{1}{n}$. Hence $\sup_{t \in [0,\infty)} |H_t - H_t^n| \le \frac{1}{n}$ and H^n converges uniformly to H.

The second case, $T_k^n \uparrow a < \infty$ as $k \to \infty$, is impossible, because it implies that $\lim_{t \to a^-} H_t \ne H_a$ contradicting the caglad assumption. Indeed, suppose that $\lim_{t \to a^-} H_t = H_a \in \mathbb{R}$. Take $\epsilon = \frac{1}{3n}$, then there exists $\delta > 0$ such that for all $t \in (a - \delta, a), |H_t - H_a| < \epsilon$. However, since $a = \lim_{t \to \infty} T_k^n$, there exists K such that $T_k^n > a - \delta$ for all $k \ge K$, and since T_k^n is an increasing sequence we have $T_k^n, T_{k+1}^n, \cdots \in (a - \delta, a)$. Thus, $|H_{T_k^n} - H_a| < \epsilon$ and $|H_{T_{k+1}^n} - H_a| < \epsilon$.

But

$$\begin{aligned}
\frac{1}{n} &\le |(H_+)_{T_k^n} - H_{T_{k+1}^n}| \le |(H_+)_{T_k^n} - H_a| + |H_{T_{k+1}^n} - H_a| \\
&= \lim_{t \to T_k^n +} |H_t - H_a| + |H_{T_{k+1}^n} - H_a| \le \frac{1}{3n} + \frac{1}{3n},
\end{aligned}$$

which is a contradiction. The inequality $\lim_{t \to T_k^n +} |H_t - H_a| \le \frac{1}{3n}$ takes place because, as stated above, for all $t \in (a - \delta, a), |H_t - H_a| < \epsilon$ and $T_k^n \in (a - \delta, a)$, hence for all $t \in (T_k^n, a)$ we have $|H_t - H_a| < \frac{1}{3n}$.

STEP 2) Show that if (H^n) is a sequence from $L^1_{F,G}(X)_{\mathrm{loc}}$ converging uniformly on $\mathbb{R}_+ \times \Omega$ to a process H, then H is locally integrable with respect to X.

Indeed, if N is such that $|H^n - H^N| \leq 1$ for $n \geq N$ and (T_k) is a determining sequence for the local integrability of $H^1, H^2, \cdots, H^N$ with respect to X then we have $1_{[0,T_k]}H^n \in L^1_{F,G}(X^{T_k})$ for every k and every $n \leq N$. Moreover, since $1_{[0,T_k]}H^N \in L^1_{F,G}(X^{T_k})$ for each k and $n \geq N$, we have

$$\begin{aligned}\tilde{I}_{F,G}(1_{[0,T_k]}H^n) &= \sup\{\int |1_{[0,T_k]}H^n| d|(I_X)_z| : \|z\|_q \leq 1\}\\ &\leq \sup\{\int |1_{[0,T_k]}H^n - 1_{[0,T_k]}H^N| + |1_{[0,T_k]}H^N| d|(I_X)_z| : \|z\|_q \leq 1\}\\ &\leq \sup\{\int 1 + |1_{[0,T_k]}H^N| d|(I_X)_z| : \|z\|_q \leq 1\} < \infty \qquad (3)\end{aligned}$$

where the last inequality is because the measures $(I_X)_z$ have finite variations and because $1_{[0,T_k]}H^N \in L^1_{F,G}(X^{T_k})$ for each k hence $1_{[0,T_k]}H^N \in \mathcal{F}_{F,G}(X^{T_k})$ By 3, we deduce that for each k, $1_{[0,T_k]}H^n \in \mathcal{F}_{F,G}(X^{T_k})$ for $n \geq N$. Since H^n is locally integrable with respect to X, using Theorem 15 b) in [MD] we deduce that $1_{[0,T_k]}H^n$ is locally integrable with respect to X^{T_k}, hence by Theorem 13, $1_{[0,T_k]}H^n$ is integrable with respect to X^{T_k}.

Since $1_{[0,T_k]}H^n \to 1_{[0,T_k]}H$ uniformly, as $n \to \infty$, by Theorem 19 in [MD], we deduce that for each k we have

$$1_{[0,T_k]}H \in L^1_{F,G}(X^{T_k}) \text{ and } 1_{[0,T_k]}H^n \to 1_{[0,T_k]}H,$$

in $L^1_{F,G}(X^{T_k})$, as $n \to \infty$.

STEP 3) Let H^n be the sequence of σ-elementary processes converging uniformly to H from STEP 1). Since H is caglad and $(\mathcal{F}_t)_{t\in\mathbb{R}}$ satisfies the usual conditions, we deduce from IV.17 in [DM78], for example, that $(H_+)_{T^n_k} \in \mathcal{F}_{T^n_k}$ and from Theorem 15 we conclude that $H^n \in L^1_{F,G}(X)_{\mathrm{loc}}$. Then by STEP 2) we have $H \in L^1_{F,G}(X)_{\mathrm{loc}}$ which concludes the proof. □

Corollary 18. *Let $H : \mathbb{R}_+ \times \Omega \to F$ be a cadlag, adapted process and X be a continuous, locally p-additive summable process relative to (F,G). Then $H \in L^1_{F,G}(X)_{loc}$.*

Proof. The proof is the same as the proof for the previous theorem, with the modifications in the formula of the stopping times T^n_k and the processes H^n from STEP 1). Namely, now T^n_k should be given by

$$T^n_k = \inf\{t > T^n_{k-1} : |H_t - H_{T^n_{k-1}}| > \frac{1}{n}\},$$

and H^n by

$$H^n = \sum_{k=0}^{\infty} H_{T^n_k} 1_{[T^n_k, T^n_{k+1})}.$$

The using Remark 16 instead of Theorem 15 in STEP 3) we get to the conclusion of the Corollary.

□

Corollary 19. *Let X be an p-additive summable process relative to (F,G). Then the integral $X_- \cdot X$ exists.*

Proof. The process X is p-additive summable, hence is a cadlag process. Therefore X_- is a caglad process and by Theorem 17, the integral $X_- \cdot X$ exists. $\square$

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