Summable processes which are not semimartingales*

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Abstract

The classical stochastic integral $\int HdX$ is defined for *real-valued semimartingales* X. For processes with values in a Banach space E, the stochastic integral $\int HdX$ is defined for *summable* processes X.

We prove that for certain Banach spaces E, there are E-valued summable processes which are not semimartingales. This shows that the stochastic integral with respect to summable process is more comprehensive than the stochastic integral with respect to semimartingales.

1 Introduction

The purpose of this paper is to solve the problem whether there are locally summable processes which are not semimartingales. A positive solution of this problem would mean that the stochastic integrable integral $\int HdX$ with respect to locally summable process is more comprehensive than the classical stochastic integral which is defined for real-valued semimartingales X, as presented, for example, by Dellacherie and Mayer, in [D-M], chapter viii. In this paper we solve the problem for locally summable processes with values in an infinite dimensional Banach space E, with $c_0 \not\subset E$ and with Radon-Nikodym Property (Theorem 18), by constructing an example of a predictable process

*AMS subject classification: Primary G0 H05, Secondary G0 G20.
Key words and phrases: summable processes, semimartingale, integrable variation, integrable semivariation, local martingale, dual projection.

with integrable semivariation – hence summable, but with infinite variation – therefore not a semimartingale (Theorem 15).

The problem of finding an example of a real-valued locally summable process which is not a semimartingale, is still open.

In [K], Theorem 12.3, Kussmaul proved that a *predictable*, real-valued, locally summable process (in the sense of Definition 1 below) is necessarily a semimartingale.

In the first part of the paper, for the convenience of the reader, we state the definitions and the theorems which will be used in the proofs of the theorems in the second part, devoted to prove the existence of a locally summable process which is not a semimartingale. For the proofs of the theorems in the first part we refer the reader to [D].

2 Preliminaries

In this paragraph, we present the notations that will be used throughout the paper and state the definitions and the theorems that will be needed for the proof of the main results in the following paragraph.

2.1 Notations

$(\Omega, \mathcal{F}, P)$ is a probability space; $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a filtration satisfying the usual conditions; $\mathcal{R}$ is the ring generated by the semiring of predictable rectangles of the form $\{0\} \times A$ with $A \in \mathcal{F}_0$ and $(s,t] \times A$, with $0 \leq s \leq t < \infty$ and $A \in \mathcal{F}_s$; $\mathcal{P}$ is the predictable σ-algebra generated by the ring $\mathcal{R}$.

We denote $\mathcal{M} = \mathcal{B}(\mathbb{R}_+) \times \mathcal{F}$. Processes measurable with respect to $\mathcal{M}$ will be called, simply, measurable processes.

E, F, G are Banach spaces with $E \subset L(F, G)$; for example, $E = L(\mathbb{R}, E)$, $E \subset L(E^*, \mathbb{R})$, $E = L(E, \mathbb{R})$ if E is a Hilbert space.

For any Banach space M we denote by $|\cdot|$ its norm, by M^* its dual and by M_1 its unit ball.

2.2 Summable processes

Let $X : \mathbb{R}_+ \times \Omega \to E$ be a cadlag, adapted process, with $X_t \in L^1_E$ for every $t \geq 0$. We define the additive measure $I_X : \mathcal{R} \to L^1_E \subset L(F, L^1_G)$, first for predictable rectangles, by $I_X(\{0\} \times A) = 1_A X_0$, for $A \in \mathcal{F}_0$ and

$I_X((s,t] \times A) = 1_A(X_t - X_s),$ for $A \in \mathcal{F}_s$, and then, extended by additivity to $\mathcal{R}$.

Definition 1. *We say that X is summable relative to the embedding $E \subset L(F,G)$ or relative to the pair (F,G), if I_X has a σ-additive extension $I_X : \mathcal{P} \to L^1_E \subset L(F, L^1_G)$ with finite semivariation relative to the pair (F, L^1_G). We say that X is locally summable relative to (F,G) if there is an increasing sequence (T_n) of sopping times with $T_n \uparrow \infty$, such that, for each n, the stopped process X^{T_n} is summable relative to (F,G).*

For a detailed account of integration with respect to measures with finite semivariation, see [D], §5.

If X is a locally summable process, one can define the stochastic integral, denoted by $H \cdot X$ or $\int HdX$, for certain predictable, F-valued processes H. For the construction of the stochastic integral the reader is referred to [D], chapter 2.

For the definition of the classical stochastic integral $\int HdX$ with respect to real-valued semimartingales the reader is referred to [D – M], chapter viii.

In this paper we shall not be concerned with the stochastic integral, but only with the summability.

Example of (locally) summable processes are: (locally) square integrable martingales in Hilbert spaces, processes with (locally) integrable variation and processes with (locally) integrable semivariation.

2.3 The variation and the semivariation of a function

Let $g : \mathbb{R} \to E \subset L(F,G)$ be a function.

Definition 2. *a) For any interval $I \subset \mathbb{R}$ we define the variation $var(g,I)$ of g on I and the semivariation $svar_{F,G}(g,I)$ of g on I, relative to the embedding $E \subset L(F,G)$, or relative to the pair (F,G), by the following equalities:*

$$var(g,I) = \sup \sum_{i=1}^{n} |g(t_{i+1}) - g(t_i)|$$

and

$$svar_{F,G}(g,I) = \sup |\sum_{i=1}^{n} [g(t_{i+1}) - g(t_i)]x_i|,$$

where the supremum is taken for all finite divisions $d : t_1 < t_2 < \cdots < t_{n+1}$ consisting of points from I and all finite families $(x_i)_{1 \le i \le n}$ of elements of F_1.

b) The variation function $|g|$ is defined by

$|g|(t) = var(g, (-\infty, t])$, *for* $t \in \mathbb{R}$,

$|g|(\infty) = var(g, \mathbb{R})$.

b') The semivariation function $\tilde{g}_{F,G}$ *is defined by*
$\tilde{g}_{F,G}(t) = svar_{F,G}(g, (-\infty, t])$*, for* $t \in \mathbb{R}$*,*
$\tilde{g}_{F,G}(\infty) = svar_{F,G}(g, \mathbb{R})$*.*

c) We say that g *has finite (resp. bounded) variation if* $|g|(t) < \infty$ *for* $t \in \mathbb{R}$ *(resp.* $|g|(\infty) < \infty$*).*

c') We say that g *has finite (resp. bounded) semivariation if* $\tilde{g}_{F,G}(t) < \infty$ *for* $t \in \mathbb{R}$ *(resp.* $\tilde{g}_{F,G}(\infty) < \infty$*).*

For a detailed account of the variation and the semivariation of a function see [D], §§18 and 20. We mention here the following properties which will be used in the paper.

Proposition 3. *We have* $\tilde{g}_{F,G} \leq |g|$ *and* $|g(t)| \leq \tilde{g}_{F,G}(t)$*, for* $t \in \mathbb{R}$*.*

Proposition 4. *If* $E \subset L(F, \mathbb{R})$*, then* $svar_{F,\mathbb{R}} g = var\, g$*.*

(See [D], Proposition 20.4).

Proposition 5. *Let* $Z \subset G^*$ *be a closed subspace, norming for* G*. For each* $z \in Z$*, define the function* $g_z : \mathbb{R} \to F^*$ *by* $\langle x, g_z(t) \rangle = \langle g(t)x, z \rangle$*, for* $t \in \mathbb{R}$ *and* $x \in F$*.*

a) For any interval $I \subset \mathbb{R}$ *we have* $svar_{F,G}(g, I) = \sup_{z \in Z_1} var(g_z, I)$*.*

b) For every interval $I \subset \mathbb{R}$ *we have* $svar_{F,G}(g, I) < \infty$ *iff* $var(g_z, I) < \infty$ *for every* $z \in Z$*.*

For the proof of a) see [D], Proposition 20.7; for b) see [D], Proposition 20.9.

The *jump* $\Delta g(t)$ of g at t is define by $\Delta g(t) = g(t_+) - g(t_-)$. If g is right continuous, then $\Delta g(t) = g(t) - g(t_-)$.

Proposition 6. *Assume* g *is right continuous.*

a) If g *has finite variation* $|g|$*, then* $\Delta |g|(t) = |\Delta g(t)|$*, for* $t \in \mathbb{R}$*.*

b) If $c_0 \not\subset E$ *and* g *has finite semivariation* $\tilde{g}_{\mathbb{R},E}$*, then* $\Delta \tilde{g}_{\mathbb{R},E}(t) = |\Delta g(t)|$*.*

c) We have $|\Delta g(t)| \leq \tilde{g}_{\mathbb{R},E}(t)$*, for* $t \in \mathbb{R}$*.*

For assertion a) see [D], Theorem 18.20; for assertion b), see [D], Theorem 20.14; assertion c) follows from $|g(t) - g(s)| \leq svar(g, (-\infty, t])$, by taking the limit as $s \uparrow t$.

The following theorem associates to g a Borel measure.

Theorem 7. *Assume g is right continuous and has bounded variation $|g|$ (resp. $c_0 \not\subset E$ and has bounded semivariation $\tilde{g}_{F,G}$). Then there is a unique σ-additive Borel measure $m_g : \mathcal{B}(\mathbb{R}) \to E$ with bounded variation $|m|$ (resp. with bounded semivariation $\tilde{m}_{F,G}$) such that $m_g((s.t]) = g(t) - g(s)$, for $s \leq t$ in $\mathbb{R}$.*

For the proof, see [D], Theorems 18.19 and 20.13.

Under the hypothesis of Theorem 7, we denote the space $L^1(m_g)$ of m_g-integrable functions $f : \mathbb{R} \to F$ by $L^1_F(g)$. For every function $f \in L^1_F(g)$ we define the Lebesgue-Stieltges integral $\int f dg$ by $\int f dg = \int f dm_g$.

2.4 Processes with finite variation or finite semivariation

Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$ be a process. We consider X automatically extended to $\mathbb{R} \times \Omega$ by $X_t(\omega) = 0$ for $t < 0$ and $\omega \in \Omega$.

Definition 8. *a) The variation process $|X|$ is defined by the equalities*

$$|X|_t(\omega) = var(X.(\omega), (-\infty, t]), \text{ for } t \in \mathbb{R} \text{ and } \omega \in \Omega,$$
$$|X|_\infty(\omega) = var(X.(\omega), \mathbb{R}), \text{ for } \omega \in \Omega.$$

a') The semivariation process $\tilde{X}_{F,G}$ is defined by

$$(\tilde{X}_{F,G})_t(\omega) = svar_{F,G}(X.(\omega),\ (-\infty, t]), \text{ for } t \in \mathbb{R} \text{ and } \omega \in \Omega,$$
$$(\tilde{X}_{F,G})_\infty(\omega) = svar_{F,G}(X.(\omega), \mathbb{R}) \text{ for } \omega \in \Omega,$$

b) We say X has finite variation $|X|$ (resp. finite semivariation $\tilde{X}_{F,G}$) if $|X|_t(\omega) < \infty$ (resp. $(\tilde{X}_{F,G})_t(\omega) < \infty$), for every $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

c) We say X has integrable variation (resp. integrable semivariation) if $E(|X|_\infty) < \infty$ (resp. $E((\tilde{X}_{F,G})_\infty) < \infty$.

d) We say X has locally integrable variation $|X|$ (resp. locally integrable semivariation $\tilde{X}_{F,G}$) if there is an increasing sequence (T_n) of stopping times with $T_n \uparrow \infty$, such that for each n, the stopped process X^{T_n} has integrable variation $|X^{T_n}|$ (resp. integrable semivariation $(X^{T_n})^{\sim}_{F,G}$).

The following theorem states the local summability of the processes with (locally) integrable variation or semivariation.

Theorem 9. *Assume X is a right continuous, adapted process.*
a) If X has (locally) integrable variation $|X|$, then X is (locally) summable relative to any embedding $E \subset L(F,G)$.

b) If $c_0 \not\subset E$ and if X has (locally) integrable semivariation $\tilde{X}_{\mathbb{R},E}$, then X is (locally) summable relative to the embedding $E = L(\mathbb{R}, E)$.

For a) see [D], Theorem 19.13; for b) see [D], Theorem 21.12.

The following theorem gives sufficient conditions for two processes to be indistinguishable. For the proof see [D], Corollary 21.10 b').

Theorem 10. *Assume $c_0 \not\subset E$ and let $A, B : \mathbb{R}_+ \times \Omega \to E$ be two predictable processes with integrable semivariation relative to $(\mathbb{R}, E)$. If for every stopping time T we have $E(A_\infty - A_T) = E(B_\infty - B_T)$, then A and B are indistinguishable.*

The next theorem gives examples of processes with locally integrable variation or semivariation. For the proof see [D], Theorems 22.15 and 22.16.

Theorem 11. *Assume X is right continuous and has finite variation $|X|$ (resp. finite semivariation $\tilde{X}_{F,G}$). If X is either predictable or a local martingale, then X has locally integrable variation $|X|$ (resp. locally integrable semivariation $\tilde{X}_{F,G}$).*

2.5 Dual projections

Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F,G)$ be a right continuous, measurable process.

Definition 12. *a) Assume that X has integrable variation $|X|$ (resp. $c_0 \not\subset E$ and X has integrable semivariation $\tilde{X}_{\mathbb{R},E}$).A right continuous, predictable process $Z : \mathbb{R}_+ \times \Omega \to E$ with integrable variation $|Z|$ (resp. with integrable semivariation $\tilde{Z}_{\mathbb{R},E}$) is called the predictable dual projection with integrable variation (resp. with integrable semivariation relative $(\mathbb{R}, E)$), if, for every real-valued, bounded, measurable process ϕ we have $E(\int \phi_s dZ_s) = E(\int {}^p\phi_s dX_s)$. where ${}^p\phi$ is the predictable projection of ϕ and the integrals are Lebesgue-Stieltjes integrals. We denote $Z = X^p$.*

b) Assume X has locally integrable variation $|X|$ (resp. $c_0 \not\subset E$ and X has locally integrable semivariation $\tilde{X}_{\mathbb{R},E}$). A right continuous predictable process $Z : \mathbb{R}_+ \times \Omega \to E$ with locally integrable variation $|Z|$ (resp. with locally integrable semivariation $\tilde{Z}_{\mathbb{R},E}$) is called the predictable dual projection of X , if there is an increasing sequence (T_n) of stopping times with $T_n \uparrow \infty$, *such that for each n, X^{T_n} and Z^{T_n} have integrable variation (resp. integrable semivariation relative $(\mathbb{R}, E)$) and Z^{T_n} is the predictable dual projection of X^{T_n}. We denote $Z = X^p$.*

A similar definition is stated for the optional dual projection.

The existence of the dual projection is stated by the following theorem. For the proof see [D], Theorems 22.8 and 22.13.

Theorem 13. *Assume that E has the Radon–Nikodym Property and that X has integrable (resp. locally integrable) variation $|X|$. Then X has a predictable dual projection with integrable (resp. locally integrable) variation.*

3 Existence of summable processes that are not semimartingales

In this paragraph we construct an example of a predictable process with integrable semivariation and infinite variation (Theorem 14), and prove that such a process is not a semimartingale (Theorem 17)

Theorem 14. *Let E be an infinite dimensional Banach space. There are E-valued predictable processes with finite, integrable semivariation relative to $(\mathbb{R}, E)$ having infinite variation.*

Proof. Using the Dvoretzky-Rogers theorem, we can find a sequence $(x_n)_{1\leq n\leq\infty}$ of elements of E, such that the series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent but not absolutely convergent.

Denote $s_0 = 0$, $s_1 = x_1, \cdots, s_n = \sum_{i=1}^{i=n} x_i, \cdots$.Then $x_n = s_n - s_{n-1}$, $\lim_{n\to\infty} s_n = \sum_{i=1}^{\infty} x_i$ and $\sum_{n=1}^{\infty} |s_n - s_{n-1}| = \infty$.

Consider the function $g : \mathbb{R}_+ \to E$ defined by $g(0) = \sum_{n=1}^{\infty} x_n$ and $g(t) = \sum_{1\leq n<\infty} 1_{[\frac{1}{n+1}, \frac{1}{n})}(t) s_n,$ for $t > 0$. We define the deterministic process $X : \mathbb{R}_+ \times \Omega \to E$ by $X = g1_\Omega$, i.e., $X_t(\omega) = g(t)$ for $t \geq 0$ and $\omega \in \Omega$. We shall prove that X is a right continuous predictable process with finite, integrable semivariation $\tilde{X}_{\mathbb{R},E}$ and with infinite variation.

The proof is divided into several steps.

a) From the definition of g we deduce that g is right continuous; therefore X is right continuous.

b) g has infinite variation on $\mathbb{R}_+$; hence X has infinite variation on $\mathbb{R}_+$. In fact, we have $g(t) = s_n$ if $\frac{1}{n+1} \le t < \frac{1}{n}$. We have then for each k,

$$\sum_{n=1}^{k} |g(\frac{1}{n}) - g(\frac{1}{n+1})| = \sum_{n=1}^{k} |s_{n-1} - s_n| = \sum_{n=1}^{k} |x_n|.$$

It follows that $var(g, \mathbb{R}_+) \ge \sup_k \sum_{n=1}^{k} |g(\frac{1}{n}) - g(\frac{1}{n+1})| = \sup_k \sum_{n=1}^{k} |x_n|$ $=\sum_{n=1}^{\infty} |x_n| = \infty$.

c) g has bounded semivariation $\tilde{g} = \tilde{g}_{\mathbb{R},E}$ on $\mathbb{R}_+$; In fact, let $x^* \in E^*$. The jumps of x^*g are at 0, equal to $x^*g(0)$ and at $\frac{1}{n}$, equal to $x^*s_{n-1} - x^*s_n$. Therefore
$var(x^*g, \mathbb{R}_+) = |x^*g(0)| + \sum_{n=1}^{\infty} |x^*s_{n-1} - x^*s_n|$

$\le \sum_{n=1}^{\infty} |x^*(x_n)| + \sum_{n=1}^{\infty} |x^*(s_{n-1} - s_n)| = 2\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$.
Then, by Proposition 5b) we have $svar_{\mathbb{R},E}(g, \mathbb{R}_+) < \infty$; hence the semivariation $\tilde{g}$ is bounded. From $\tilde{X}_t(\omega) = \tilde{g}(t)$ we deduce that X has bounded semivariation $\tilde{X}$.

d) X has integrable semivariation $\tilde{X} = \tilde{X}_{\mathbb{R},E}$. In fact,
$E(\tilde{X}_\infty) = \int \tilde{X}_\infty dP = \int svar_{\mathbb{R},E}(X, \mathbb{R}_+) dP = svar_{\mathbb{R},E}(g, \mathbb{R}_+) < \infty$.

e) X is adapted since it is deterministic.

f) X is predictable since the sets $\{0\} \times \Omega$ and $[\frac{1}{n+1}, \frac{1}{n}) \times \Omega$ are predictable. □

Proposition 15. *Let $X : \mathbb{R}_+ \times \Omega \to E$ be a process with finite variation. If X has locally integrable semivariation $\tilde{X}_{\mathbb{R},E}$, then X has locally integrable variation.*

Proof. Assume X has locally integrable semivariation $\tilde{X}$ relative to $(\mathbb{R}, E)$. Then there is an increasing sequence S_n of stopping times with $S_n \uparrow \infty$ such that $E(\tilde{X}_{S_n}) < \infty$ for each n. For each n define the stopping times T_n by $T_n = S_n \wedge \inf\{t |\, |X|_t \ge n\}$. It follows that $|X|_{T_n-} \le n$. Since X has finite variation, by Proposition 6 we have $\Delta|X_{T_n}| = |\Delta X_{T_n}| \le \tilde{X}_{T_n}$. From $\Delta|X|_{T_n} = |X|_{T_n} - |X|_{T_n-}$ we deduce that $|X|_{T_n} = |X|_{T_n-} + \Delta|X_{T_n}| \le n + \tilde{X}_{T_n}$; Therefore $E(|X|_{T_n}) \le n + E(\tilde{X}_{T_n}) < \infty$; hence X has locally integrable variation. □

Proposition 16. *Let E be a Banach space with the Radon-Nikodym property and with $c_0 \not\subset E$ (for example, E can be a reflexive space). Let $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$ be a right continuous, predictable process with*

finite semivariation $\tilde{X}_{\mathbb{R},E}$.
If X is a semimartingale, then X has locally integrable variation, outside an evanescent set.

Proof. We remark that since X is predictable and has finite semivariation $\tilde{X}_{\mathbb{R},E}$, by Theorem 11, X has locally integrable semivariation. $\tilde{X}_{\mathbb{R},E}$.

a) We assume first that X has integrable semivariation $\tilde{X}_{\mathbb{R},E}$ and that X is a semimartingale of the form $X = M + A$ where M is a right continuous, uniformly integrable martingale with integrable semivariation $\tilde{M}_{\mathbb{R},E}$ and A is a right continuous process with integrable variation $|A|$.

Let $z \in E^*$. Then $\langle X, z\rangle = \langle M, z\rangle + \langle A, z\rangle$. The processes $\langle X, z\rangle$, $\langle M, z\rangle$, $\langle A, z\rangle$ are adapted, right continuous and have integrable variation; $\langle X, z\rangle$ is predictable, and $\langle M, z\rangle$ is a uniformly integrable martingale. Consider the σ-additive stochastic measures with finite variation $\mu_{\langle X,z\rangle}, \mu_{\langle M,z\rangle}, \mu_{\langle A,z\rangle} : \mathcal{M} \to \mathbb{R}$, defined for any real-valued, bounded, measurable process ϕ by $\int \phi d\mu_{\langle X,z\rangle} = E(\int \phi d\langle X, z\rangle)$, $\int \phi d\mu_{\langle M,z\rangle} = E(\int \phi d\langle M, z\rangle)$, $\int \phi d\mu_{\langle A,z\rangle} = E(\int \phi d\langle A, z\rangle)$. (see [D – M], vi.64). Then $\mu_{\langle X,z\rangle} = \mu_{\langle M,z\rangle} + \mu_{\langle A,z\rangle}$. Since $\langle M, z\rangle$ is a uniformly integrable martingale with integrable variation, we have $\mu_{\langle M,z\rangle}(B) = 0$, for $B \in \mathcal{P}$. It follows that $\mu_{\langle X,z\rangle} = \mu_{\langle A,z\rangle}$ on $\mathcal{P}$. Let ϕ be a real-valued, bounded, predictable process and let ${}^p\phi$ be its predictable projection. Then $\int {}^p\phi d\mu_{\langle X,z\rangle} = E(\int {}^p\phi d\mu_{\langle A,z\rangle})$, i.e. $E(\int {}^p\phi d\langle X, z\rangle) = E(\int {}^p\phi d\langle A, z\rangle)$. Since $\langle X, z\rangle$ is predictable, we have $E(\int {}^p\phi d\langle X, z\rangle) = E(\int \phi d\langle X, z\rangle)$. It follows that $E(\int \phi d\langle X, z\rangle) = E(\int {}^p\phi d\langle A, z\rangle)$, or $\langle E(\int \phi dX), z\rangle = \langle E(\int {}^p\phi dA), z\rangle$, where the integral are Stieltjes integral with respect to functions with finite semivariation.

Since $z \in E^*$ was arbitrary, we deduce that $E(\int \phi dX) = E(\int {}^p\phi dA)$. It follows that X is the predictable dual projection with integrable semivariation $\tilde{X}_{\mathbb{R},E}$ of the process A with integrable semivariation $\tilde{A}_{\mathbb{R},E}$.

Since E has the Radon Nikodym Property and A has integrable variation, by Theorem 13, A has a predictable dual projection Y with integrable variation, satisfying, $E(\int \phi dY) = E(\int {}^p\phi dA)$, where the integrals are Stieltjes integrals with respect to functions with finite variation, which are also Stieltjes integrals with respect to the same functions, considered with finite semivariation relative to $(\mathbb{R}, E)$. It follows that $E(\int \phi dX) = E(\int \phi dY)$. If T is any stopping time , we take $\phi = 1_{(T,\infty)}$ and we obtain $E(X_\infty - X_T) = E(Y_\infty - Y_T)$. Since both X and Y are predictable processes with integrable semivariation relative to $(\mathbb{R}, E)$, by Theorem 10, X and Y are indistinguishable. It follows that X has integrable variation except on an evanescent set.

b) Assume now that X is right continuous, predictable, with finite semivariation $\tilde{X}_{\mathbb{R},E}$ and that $X = M + A$ where M is a right continuous, local martingale and A is a right continuous process with finite variation.

From $M = X - A$ we deduce that M has finite semivariation $\tilde{M}_{\mathbb{R},E}$. Since $c_0 \not\subset E$, by Theorem 11, the martingale M has locally integrable semivariation $\tilde{M}_{\mathbb{R},E}$. Then, from $A = X - M$, it follows that A has locally integrable semivariation $\tilde{A}_{\mathbb{R},E}$; therefore by Proposition 15, A has locally integrable variation.

Let T_n be an increasing sequence of stopping times with $T_n \uparrow \infty$ such that for each n, X^{T_n} and M^{T_n} have integrable semivariation relative to $(\mathbb{R}, E)$, M^{T_n} is a uniformly integrable martingale and A^{T_n} has integrable variation. For each n we have $X^{T_n} = M^{T_n} + A^{T_n}$. By the first part of the proof, for each n, X^{T_n} has locally integrable variation, except on an evanescent set of the form $\mathbb{R}_+ \times N_n$ with $P(N_n) = 0$. The union $N = \cup_n N_n$ is P-negligible, and outside the evanescent set $\mathbb{R}_+ \times N$, each process X^{T_n} has integrable variation. It follows that X has locally integrable variation outside $\mathbb{R}_+ \times N$. □

Combining Theorem 14 and 16 we deduce the following corollary:

Theorem 17. *Let E be an infinite dimensional Banach space with $c_0 \not\subset E$ and having Radon-Nikodym Property. Then there are E-valued summable processes which are not semimartingales.*

Proof. By Theorem 14 there is an $E-$valued, right continuous, predictable process X with finite, integrable semivariation relative to $(\mathbb{R}, E)$ and with infinite variation. By Theorem 9, X is summable relative to $(\mathbb{R}, E)$. If X is a semimartingale, then by Theorem 16, X has locally integrable variation; hence finite variation outside an evanescent set. It follows that X is not a semimartingale. □

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