Chapter 10

Analytic Geometry Topics

Exercise Set 10.1

1. Graph (f) is the graph of $x^2 = 8y$.

2. Graph (c) is the graph of $y^2 = -10x$.

3. Graph (b) is the graph of $(y-2)^2 = -3(x+4)$.

4. Graph (e) is the graph of $(x+1)^2 = 5(y-2)$.

5. Graph (d) is the graph of $13x^2 - 8y - 9 = 0$.

6. Graph (a) is the graph of $41x + 6y^2 = 12$.

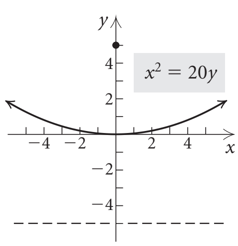
7. $x^2 = 20y$

$x^2 = 4 \cdot 5 \cdot y$ Writing $x^2 = 4py$

Vertex: $(0,0)$

Focus: $(0,5)$ $[(0,p)]$

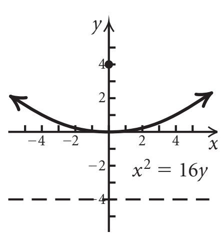
Directrix: $y = -5$ $(y = -p)$



8. $x^2 = 16y$

$x^2 = 4 \cdot 4 \cdot y$

$V: (0,0),\ F: (0,4),\ D: y = -4$



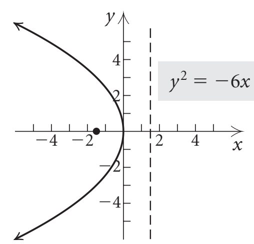
9. $y^2 = -6x$

$y^2 = 4\left(-\dfrac{3}{2}\right)x$ Writing $y^2 = 4px$

Vertex: (0,0)

Focus: $\left(-\dfrac{3}{2}, 0\right)$ $[(p,0)]$

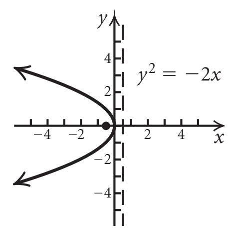
Directrix: $x = -\left(-\dfrac{3}{2}\right) = \dfrac{3}{2}$ $(x = -p)$



10. $y^2 = -2x$

$y^2 = 4\left(-\dfrac{1}{2}\right)x$

$V: (0,0), F: \left(-\dfrac{1}{2}, 0\right), D: x = \dfrac{1}{2}$



11. $x^2 - 4y = 0$

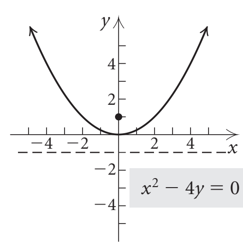
$x^2 = 4y$

$x^2 = 4 \cdot 1 \cdot y$ Writing $x^2 = 4py$

Vertex: $(0,0)$

Focus: $(0,1)$ $[(0,p)]$

Directrix: $y = -1$ $(y = -p)$

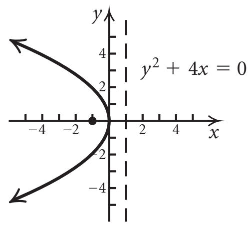


12. $y^2 + 4x = 0$

$y^2 = -4x$

$y^2 = 4(-1)x$

$V: (0,0), F: (-1,0), D: x = 1$

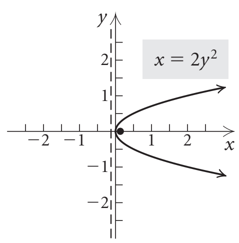


13.
$$\begin{aligned} x &= 2y^2 \\ y^2 &= \frac{1}{2}x \\ y^2 &= 4\cdot\frac{1}{8}\cdot x \quad \text{Writing } y^2 = 4px \end{aligned}$$

Vertex: $(0,0)$

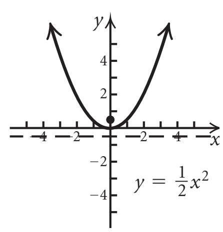
Focus: $\left(\frac{1}{8}, 0\right)$

Directrix: $x = -\frac{1}{8}$



14. $y = \frac{1}{2}x^2$

$V : (0,0),\ F : \left(0, \frac{1}{2}\right),\ D : y = -\frac{1}{2}$



15. Since the directrix, $x = -4$, is a vertical line, the equation is of the form $(y-k)^2 = 4p(x-h)$. The focus, $(4,0)$, is on the x-axis so the axis of symmetry is the x-axis and $p = 4$. The vertex, (h,k), is the point on the x-axis midway between the directrix and the focus. Thus, it is $(0,0)$. We have

$$\begin{aligned} (y-k)^2 &= 4p(x-h) \\ (y-0)^2 &= 4\cdot 4(x-0) \quad \text{Substituting} \\ y^2 &= 16x. \end{aligned}$$

16.
$$\begin{aligned} (x-h)^2 &= 4p(y-k) \\ (x-0)^2 &= 4\cdot\frac{1}{4}(y-0) \\ x^2 &= y \end{aligned}$$

17. Since the directrix, $y = \pi$, is a horizontal line, the equation is of the form $(x-h)^2 = 4p(y-k)$. The focus, $(0,-\pi)$, is on the y-axis so the axis of symmetry is the y-axis and $p = -\pi$. The vertex (h,k) is the point on the y-axis midway between the directrix and the focus. Thus, it is $(0,0)$. We have

$$\begin{aligned} (x-h)^2 &= 4p(y-k) \\ (x-0)^2 &= 4(-\pi)(y-0) \quad \text{Substituting} \\ x^2 &= -4\pi y \end{aligned}$$

18.
$$\begin{aligned} (y-k)^2 &= 4p(x-h) \\ (y-0)^2 &= 4(-\sqrt{2})(x-0) \\ y^2 &= -4\sqrt{2}x \end{aligned}$$

19. Since the directrix, $x = -4$, is a vertical line, the equation is of the form $(y-k)^2 = 4p(x-h)$. The focus, $(3,2)$, is on the horizontal line $y = 2$, so the axis of symmetry is $y = 2$. The vertex is the point on the line $y = 2$ that is midway between the directrix and the focus. That is, it is the midpoint of the segment from $(-4,2)$ to $(3,2)$: $\left(\frac{-4+3}{2}, \frac{2+2}{2}\right)$, or $\left(-\frac{1}{2}, 2\right)$. Then $h = -\frac{1}{2}$ and the directrix is $x = h - p$, so we have

$$\begin{aligned} x &= h - p \\ -4 &= -\frac{1}{2} - p \\ -\frac{7}{2} &= -p \\ \frac{7}{2} &= p. \end{aligned}$$

Now we find the equation of the parabola.

$$\begin{aligned} (y-k)^2 &= 4p(x-h) \\ (y-2)^2 &= 4\left(\frac{7}{2}\right)\left[x - \left(-\frac{1}{2}\right)\right] \\ (y-2)^2 &= 14\left(x + \frac{1}{2}\right) \end{aligned}$$

20. Since the directrix, $y = -3$, is a horizontal line, the equation is of the form $(x-h)^2 = 4p(y-k)$. The focus $(-2,3)$, is on the vertical line $x = -2$, so the axis of symmetry is $x = -2$. The vertex is the point on the line $x = -2$ that is midway between the directrix and the focus. That is, it is the midpoint of the segment from $(-2,3)$ to $(-2,-3)$: $\left(\frac{-2-2}{3}, \frac{3-3}{2}\right)$, or $(-2,0)$. Then $k = 0$ and the directrix is $y = k - p$, so we have

$$\begin{aligned} y &= k - p \\ -3 &= 0 - p \\ 3 &= p. \end{aligned}$$

Now we find the equation of the parabola.

$$\begin{aligned} (x-h)^2 &= 4p(y-k) \\ [x-(-2)]^2 &= 4\cdot 3(y-0) \\ (x+2)^2 &= 12y \end{aligned}$$

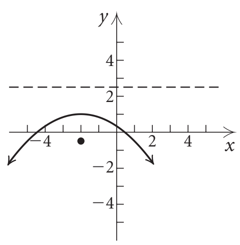
21.
$$\begin{aligned} (x+2)^2 &= -6(y-1) \\ [x-(-2)]^2 &= 4\left(-\frac{3}{2}\right)(y-1) \quad [(x-h)^2 = 4p(y-k)] \end{aligned}$$

Vertex: $(-2,1)$ $\qquad [(h,k)]$

Focus: $\left(-2, 1+\left(-\dfrac{3}{2}\right)\right)$, or $\left(-2, -\dfrac{1}{2}\right)$

$[(h, k+p)]$

Directrix: $y = 1 - \left(-\dfrac{3}{2}\right) = \dfrac{5}{2}$ $\quad (y = k - p)$

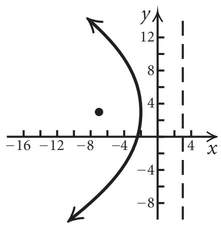


$(x+2)^2 = -6(y-1)$

22.
$$\begin{aligned}(y-3)^2 &= -20(x+2)\\ (y-3)^2 &= 4(-5)[x-(-2)]\end{aligned}$$
$V:\ (-2, 3)$

$F:\ (-2-5, 3)$, or $(-7, 3)$

$D:\ x = -2-(-5) = 3$



$(y-3)^2 = -20(x+2)$

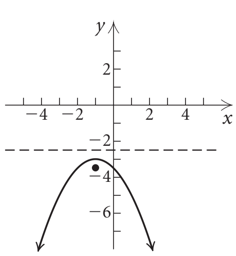
23.
$$\begin{aligned}x^2+2x+2y+7 &= 0\\ x^2+2x &= -2y-7\\ (x^2+2x+1) &= -2y-7+1 = -2y-6\\ (x+1)^2 &= -2(y+3)\\ [x-(-1)]^2 &= 4\left(-\frac{1}{2}\right)[y-(-3)]\end{aligned}$$
$[(x-h)^2 = 4p(y-k)]$

Vertex: $(-1, -3)$ $\quad [(h, k)]$

Focus: $\left(-1, -3+\left(-\dfrac{1}{2}\right)\right)$, or $\left(-1, -\dfrac{7}{2}\right)$

$[(h, k+p)]$

Directrix: $y = -3 - \left(-\dfrac{1}{2}\right) = -\dfrac{5}{2}$ $\quad (y = k - p)$

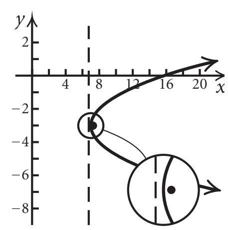


$x^2 + 2x + 2y + 7 = 0$

24.
$$\begin{aligned}y^2+6y-x+16 &= 0\\ y^2+6y+9 &= x-16+9\\ (y+3)^2 &= x-7\\ [y-(-3)]^2 &= 4\left(\frac{1}{4}\right)(x-7)\end{aligned}$$
$V:\ (7, -3)$

$F:\ \left(7+\dfrac{1}{4}, -3\right)$, or $\left(\dfrac{29}{4}, -3\right)$

$D:\ x = 7 - \dfrac{1}{4} = \dfrac{27}{4}$



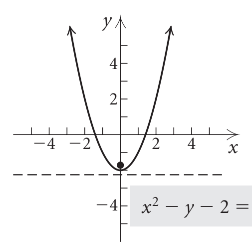
$y^2 + 6y - x + 16 = 0$

25.
$$\begin{aligned}x^2-y-2 &= 0\\ x^2 &= y+2\\ (x-0)^2 &= 4\cdot\frac{1}{4}\cdot[y-(-2)]\end{aligned}$$
$[(x-h)^2 = 4p(y-k)]$

Vertex: $(0, -2)$ $\quad [(h, k)]$

Focus: $\left(0, -2+\dfrac{1}{4}\right)$, or $\left(0, -\dfrac{7}{4}\right)$ $\quad [(h, k+p)]$

Directrix: $y = -2 - \dfrac{1}{4} = -\dfrac{9}{4}$ $\quad (y = k - p)$

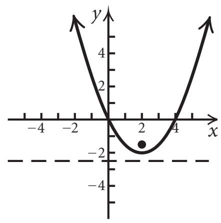


$x^2 - y - 2 = 0$

26.
$$\begin{aligned}x^2-4x-2y &= 0\\ x^2-4x+4 &= 2y+4\\ (x-2)^2 &= 2(y+2)\\ (x-2)^2 &= 4\left(\frac{1}{2}\right)[y-(-2)]\end{aligned}$$
$V:\ (2, -2)$

$F:\ \left(2, -2+\dfrac{1}{2}\right)$, or $\left(2, -\dfrac{3}{2}\right)$

$D:\ y = -2 - \dfrac{1}{2} = -\dfrac{5}{2}$



$x^2 - 4x - 2y = 0$

27.

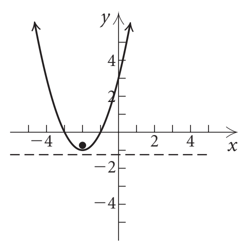
$$
\begin{aligned}
y &= x^2 + 4x + 3 \\
y - 3 &= x^2 + 4x \\
y - 3 + 4 &= x^2 + 4x + 4 \\
y + 1 &= (x+2)^2 \\
4 \cdot \frac{1}{4} \cdot [y - (-1)] &= [x - (-2)]^2
\end{aligned}
$$

$$[(x-h)^2 = 4p(y-k)]$$

Vertex: $(-2, -1)$ $\quad [(h, k)]$

Focus: $\left(-2, -1 + \frac{1}{4}\right)$, or $\left(-2, -\frac{3}{4}\right)$ $[(h, k+p)]$

Directrix: $y = -1 - \frac{1}{4} = -\frac{5}{4}$ $\quad (y = k - p)$



$y = x^2 + 4x + 3$

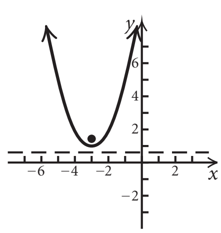
28.

$$
\begin{aligned}
y &= x^2 + 6x + 10 \\
y - 10 + 9 &= x^2 + 6x + 9 \\
y - 1 &= (x+3)^2 \\
4\left(\frac{1}{4}\right)(y - 1) &= [x - (-3)]^2
\end{aligned}
$$

$V:$ $(-3, 1)$

$F:$ $\left(-3, 1 + \frac{1}{4}\right)$, or $\left(-3, \frac{5}{4}\right)$

$D:$ $y = 1 - \frac{1}{4} = \frac{3}{4}$



$y = x^2 + 6x + 10$

29. $y^2 - y - x + 6 = 0$

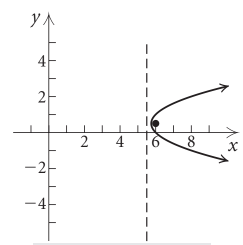
$$
\begin{aligned}
y^2 - y &= x - 6 \\
y^2 - y + \frac{1}{4} &= x - 6 + \frac{1}{4} \\
\left(y - \frac{1}{2}\right)^2 &= x - \frac{23}{4} \\
\left(y - \frac{1}{2}\right)^2 &= 4 \cdot \frac{1}{4}\left(x - \frac{23}{4}\right)
\end{aligned}
$$

$$[(y-k)^2 = 4p(x-h)]$$

Vertex: $\left(\frac{23}{4}, \frac{1}{2}\right)$ $\quad [(h, k)]$

Focus: $\left(\frac{23}{4} + \frac{1}{4}, \frac{1}{2}\right)$, or $\left(6, \frac{1}{2}\right)$ $\quad [(h+p, k)]$

Directrix: $x = \frac{23}{4} - \frac{1}{4} = \frac{22}{4}$ or $\frac{11}{2}$ $\quad (x = h - p)$



$y^2 - y - x + 6 = 0$

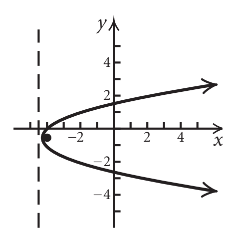
30. $y^2 + y - x - 4 = 0$

$$
\begin{aligned}
y^2 + y &= x + 4 \\
y^2 + y + \frac{1}{4} &= x + 4 + \frac{1}{4} \\
\left(y + \frac{1}{2}\right)^2 &= x + \frac{17}{4} \\
\left[y - \left(-\frac{1}{2}\right)\right]^2 &= 4 \cdot \frac{1}{4}\left[x - \left(-\frac{17}{4}\right)\right]
\end{aligned}
$$

$V:$ $\left(-\frac{17}{4}, -\frac{1}{2}\right)$

$F:$ $\left(-\frac{17}{4} + \frac{1}{4}, -\frac{1}{2}\right)$, or $\left(-4, -\frac{1}{2}\right)$

$D:$ $x = -\frac{17}{4} - \frac{1}{4} = -\frac{9}{2}$

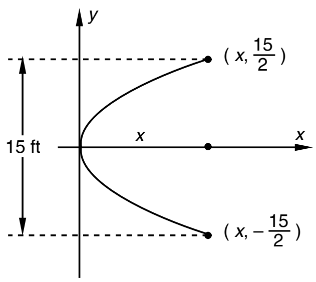


$y^2 + y - x - 4 = 0$

31. a) The vertex is $(0,0)$. The focus is $(4,0)$, so $p = 4$. The parabola has a horizontal axis of symmetry so the equation is of the form $y^2 = 4px$. We have

$$y^2 = 4px$$
$$y^2 = 4 \cdot 4 \cdot x$$
$$y^2 = 16x$$

b) We make a drawing.



The depth of the satellite dish at the vertex is x where $\left(x, \frac{15}{2}\right)$ is a point on the parabola.

$$y^2 = 16x$$
$$\left(\frac{15}{2}\right)^2 = 16x \qquad \text{Substituting } \frac{15}{2} \text{ for } y$$
$$\frac{225}{4} = 16x$$
$$\frac{225}{64} = x, \text{ or}$$
$$3\frac{33}{64} = x$$

The depth of the satellite dish at the vertex is $3\frac{33}{64}$ ft.

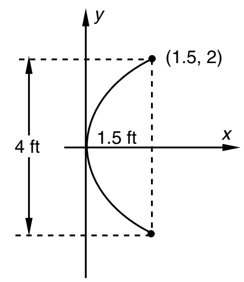
32. a) The parabola is of the form $y^2 = 4px$. A point on the parabola is $\left(1, \frac{6}{2}\right)$, or $(1,3)$.

$$y^2 = 4px$$
$$3^2 = 4 \cdot p \cdot 1$$
$$\frac{9}{4} = p$$

Then the equation of the parabola is $y^2 = 4 \cdot \frac{9}{4}x$, or $y^2 = 9x$.

b) The focus is at $(p,0)$, or $\left(\frac{9}{4}, 0\right)$, so the bulb should be placed $\frac{9}{4}$ in., or $2\frac{1}{4}$ in., from the vertex.

33. We position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry.

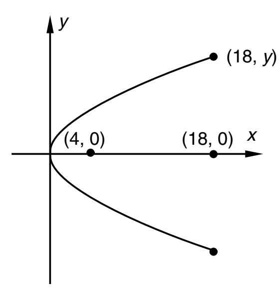


The parabola is of the form $y^2 = 4px$ and a point on the parabola is $\left(1.5, \frac{4}{2}\right)$, or $(1.5, 2)$.

$$y^2 = 4px$$
$$2^2 = 4 \cdot p \cdot (1.5) \quad \text{Substituting}$$
$$4 = 6p$$
$$\frac{4}{6} = p, \text{ or}$$
$$\frac{2}{3} = p$$

Since the focus is at $(p,0)$, or $\left(\frac{2}{3}, 0\right)$, the focus is $\frac{2}{3}$ ft, or 8 in., from the vertex.

34.



$$y^2 = 4px$$
$$y^2 = 4 \cdot 4 \cdot 18$$
$$y^2 = 288$$
$$y = 12\sqrt{2}$$

The width at the opening is $2 \cdot 12\sqrt{2}$, or $24\sqrt{2}$ in. ≈ 34 in.

35. When we let $y = 0$ and solve for x, the only equation for which $x = \frac{2}{3}$ is (h), so only equation (h) has x-intercept $\left(\frac{2}{3}, 0\right)$.

36. Equations (a) - (f) are in the form $y = mx + b$ and $b = 7$ in equation (d). When we solve equations (g) and (h) for y we get $y = 2x - \frac{7}{4}$ and $y = -\frac{1}{2}x + \frac{1}{3}$, respectively. Neither has $b = 7$, so only equation (d) has y-intercept $(0,7)$.

37. Note that equation (g) is equivalent to $y = 2x - \frac{7}{4}$ and equation (h) is equivalent to $y = -\frac{1}{2}x + \frac{1}{3}$. When we look at the equations in the form $y = mx+b$, we see that $m > 0$ for (a), (b), (f), and (g) so these equations have positive slope, or slant up front left to right.

38. The equation for which $|m|$ is smallest is (b), so it has the least steep slant.

39. When we look at the equations in the form $y = mx+b$ (See Exercise 37.), only (b) has $m = \dfrac{1}{3}$ so only (b) has slope $\dfrac{1}{3}$.

40. When we substitute 3 for x in each equation, we see that $y = 7$ only in equation (f) so only (f) contains the point $(3, 7)$.

41. Parallel lines have the same slope and different y-intercepts. When we look at the equations in the form $y = mx + b$ (See Exercise 37.), we see that (a) and (g) represent parallel lines.

42. The pairs of equations for which the product of the slopes is -1 are (a) and (h), (g) and (h), and (b) and (c).

43. A parabola with a vertical axis of symmetry has an equation of the type $(x-h)^2 = 4p(y-k)$.

Solve for p substituting $(-1, 2)$ for (h, k) and $(-3, 1)$ for (x, y).

$$\begin{aligned}[-3-(-1)]^2 &= 4p(1-2)\\ 4 &= -4p\\ -1 &= p\end{aligned}$$

The equation of the parabola is

$$[x-(-1)]^2 = 4(-1)(y-2), \text{ or}$$
$$(x+1)^2 = -4(y-2).$$

44. A parabola with a horizontal axis of symmetry has an equation of the type $(y-k)^2 = 4p(x-h)$.

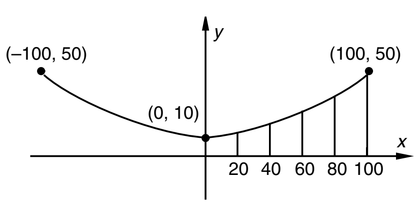
Find p by substituting $(-2, 1)$ for (h, k) and $(-3, 5)$ for (x, y).

$$\begin{aligned}(5-1)^2 &= 4p[-3-(-2)]\\ 16 &= 4p(-1)\\ 16 &= -4p\\ -4 &= p\end{aligned}$$

The equation of the parabola is

$$(y-1)^2 = 4(-4)[x-(-2)], \text{ or}$$
$$(y-1)^2 = -16(x+2).$$

45. Position a coordinate system as shown below with the y-axis on the parabola's axis of symmetry.



The equation of the parabola is of the form $(x-h)^2 = 4p(y-k)$. Substitute 100 for x, 50 for y, 0 for h, and 10 for k and solve for p.

$$\begin{aligned}(x-h)^2 &= 4p(y-k)\\ (100-0)^2 &= 4p(50-10)\\ 10,000 &= 160p\\ \frac{250}{4} &= p\end{aligned}$$

Then the equation is

$$x^2 = 4\left(\frac{250}{4}\right)(y-10), \text{ or}$$
$$x^2 = 250(y-10).$$

To find the lengths of the vertical cables, find y when $x = 0$, 20, 40, 60, 80, and 100.

When $x = 0$: $$\begin{aligned}0^2 &= 250(y-10)\\ 0 &= y-10\\ 10 &= y\end{aligned}$$

When $x = 20$: $$\begin{aligned}20^2 &= 250(y-10)\\ 400 &= 250(y-10)\\ 1.6 &= y-10\\ 11.6 &= y\end{aligned}$$

When $x = 40$: $$\begin{aligned}40^2 &= 250(y-10)\\ 1600 &= 250(y-10)\\ 6.4 &= y-10\\ 16.4 &= y\end{aligned}$$

When $x = 60$: $$\begin{aligned}60^2 &= 250(y-10)\\ 3600 &= 250(y-10)\\ 14.4 &= y-10\\ 24.4 &= y\end{aligned}$$

When $x = 80$: $$\begin{aligned}80^2 &= 250(y-10)\\ 6400 &= 250(y-10)\\ 25.6 &= y-10\\ 35.6 &= y\end{aligned}$$

When $x = 100$, we know from the given information that $y = 50$.

The lengths of the vertical cables are 10 ft, 11.6 ft, 16.4 ft, 24.4 ft, 35.6 ft, and 50 ft.

Exercise Set 10.2

1. Graph (b) is the graph of $x^2 + y^2 = 5$.

2. Graph (f) is the graph of $y^2 = 20 - x^2$.

3. Graph (d) is the graph of $x^2 + y^2 - 6x + 2y = 6$.

4. Graph (c) is the graph of $x^2 + y^2 + 10x - 12y = 3$.

5. Graph (a) is the graph of $x^2 + y^2 - 5x + 3y = 0$.

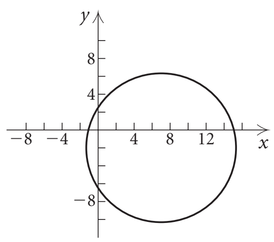
6. Graph (e) is the graph of $x^2 + 4x - 2 = 6y - y^2 - 6$.

7. Complete the square twice.

$$\begin{aligned}x^2+y^2-14x+4y &= 11\\ x^2-14x+y^2+4y &= 11\\ x^2-14x+49+y^2+4y+4 &= 11+49+4\\ (x-7)^2+(y+2)^2 &= 64\\ (x-7)^2+[y-(-2)]^2 &= 8^2\end{aligned}$$

Center: $(7, -2)$

Radius: 8

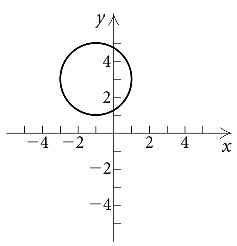


$x^2 + y^2 - 14x + 4y = 11$

8.
$$\begin{aligned} x^2 + y^2 + 2x - 6y &= -6 \\ x^2 + 2x + 1 + y^2 - 6y + 9 &= -6 + 1 + 9 \\ (x+1)^2 + (y-3)^2 &= 4 \\ [x-(-1)]^2 + (y-3)^2 &= 2^2 \end{aligned}$$

Center: $(-1, 3)$

Radius: 2

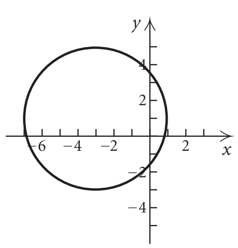


$x^2 + y^2 + 2x - 6y = -6$

9. Complete the square twice.
$$\begin{aligned} x^2 + y^2 + 6x - 2y &= 6 \\ x^2 + 6x + y^2 - 2y &= 6 \\ x^2 + 6x + 9 + y^2 - 2y + 1 &= 6 + 9 + 1 \\ (x+3)^2 + (y-1)^2 &= 16 \\ [x-(-3)]^2 + (y-1)^2 &= 4^2 \end{aligned}$$

Center: $(-3, 1)$

Radius: 4

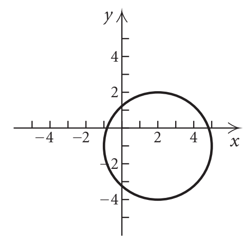


$x^2 + y^2 + 6x - 2y = 6$

10.
$$\begin{aligned} x^2 + y^2 - 4x + 2y &= 4 \\ x^2 - 4x + 4 + y^2 + 2y + 1 &= 4 + 4 + 1 \\ (x-2)^2 + (y+1)^2 &= 9 \\ (x-2)^2 + [y-(-1)]^2 &= 3^2 \end{aligned}$$

Center: $(2, -1)$

Radius: 3

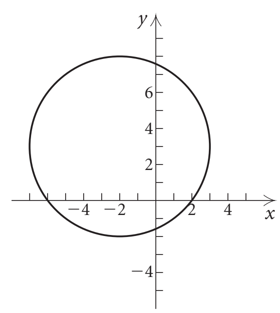


$x^2 + y^2 - 4x + 2y = 4$

11. Complete the square twice.
$$\begin{aligned} x^2 + y^2 + 4x - 6y - 12 &= 0 \\ x^2 + 4x + y^2 - 6y &= 12 \\ x^2 + 4x + 4 + y^2 - 6y + 9 &= 12 + 4 + 9 \\ (x+2)^2 + (y-3)^2 &= 25 \\ [x-(-2)]^2 + (y-3)^2 &= 5^2 \end{aligned}$$

Center: $(-2, 3)$

Radius: 5

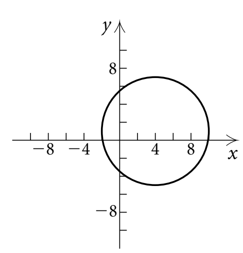


$x^2 + y^2 + 4x - 6y - 12 = 0$

12.
$$\begin{aligned} x^2 + y^2 - 8x - 2y - 19 &= 0 \\ x^2 - 8x + y^2 - 2y &= 19 \\ x^2 - 8x + 16 + y^2 - 2y + 1 &= 19 + 16 + 1 \\ (x-4)^2 + (y-1)^2 &= 36 \\ (x-4)^2 + (y-1)^2 &= 6^2 \end{aligned}$$

Center: $(4, 1)$

Radius: 6



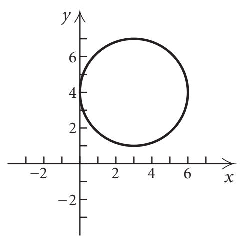
$x^2 + y^2 - 8x - 2y - 19 = 0$

13. Complete the square twice.

$$x^2+y^2-6x-8y+16=0$$
$$x^2-6x+y^2-8y=-16$$
$$x^2-6x+9+y^2-8y+16=-16+9+16$$
$$(x-3)^2+(y-4)^2=9$$
$$(x-3)^2+(y-4)^2=3^2$$

Center: $(3,4)$

Radius: 3

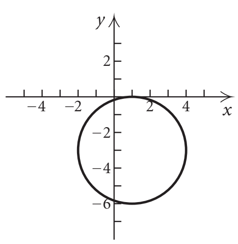


$x^2+y^2-6x-8y+16=0$

14.
$$x^2+y^2-2x+6y+1=0$$
$$x^2-2x+y^2+6y=-1$$
$$x^2-2x+1+y^2+6y+9=-1+1+9$$
$$(x-1)^2+(y+3)^2=9$$
$$(x-1)^2+[y-(-3)]^2=3^2$$

Center: $(1,-3)$

Radius: 3



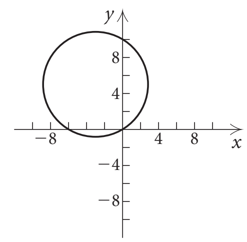
$x^2+y^2-2x+6y+1=0$

15. Complete the square twice.

$$x^2+y^2+6x-10y=0$$
$$x^2+6x+y^2-10y=0$$
$$x^2+6x+9+y^2-10y+25=0+9+25$$
$$(x+3)^2+(y-5)^2=34$$
$$[x-(-3)]^2+(y-5)^2=(\sqrt{34})^2$$

Center: $(-3,5)$

Radius: $\sqrt{34}$

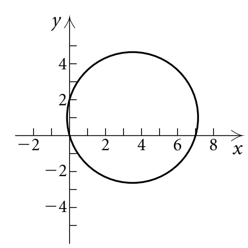


$x^2+y^2+6x-10y=0$

16.
$$x^2+y^2-7x-2y=0$$
$$x^2-7x+\frac{49}{4}+y^2-2y+1=\frac{49}{4}+1$$
$$\left(x-\frac{7}{2}\right)^2+(y-1)^2=\frac{53}{4}$$
$$\left(x-\frac{7}{2}\right)^2+(y-1)^2=\left(\frac{\sqrt{53}}{2}\right)^2$$

Center: $\left(\frac{7}{2},1\right)$

Radius: $\frac{\sqrt{53}}{2}$



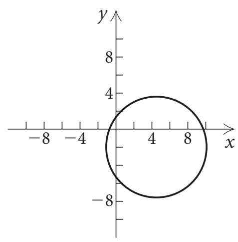
$x^2+y^2-7x-2y=0$

17. Complete the square twice.

$$x^2+y^2-9x=7-4y$$
$$x^2-9x+y^2+4y=7$$
$$x^2-9x+\frac{81}{4}+y^2+4y+4=7+\frac{81}{4}+4$$
$$\left(x-\frac{9}{2}\right)^2+(y+2)^2=\frac{125}{4}$$
$$\left(x-\frac{9}{2}\right)^2+[y-(-2)]^2=\left(\frac{5\sqrt{5}}{2}\right)^2$$

Center: $\left(\frac{9}{2},-2\right)$

Radius: $\frac{5\sqrt{5}}{2}$

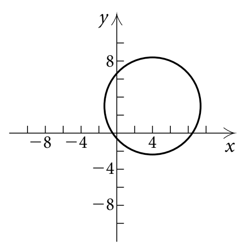


$x^2 + y^2 - 9x = 7 - 4y$

18.
$$\begin{aligned} y^2 - 6y - 1 &= 8x - x^2 + 3 \\ x^2 - 8x + y^2 - 6y &= 4 \\ x^2 - 8x + 16 + y^2 - 6y + 9 &= 4 + 16 + 9 \\ (x-4)^2 + (y-3)^2 &= 29 \\ (x-4)^2 + (y-3)^2 &= (\sqrt{29})^2 \end{aligned}$$

Center: $(4, 3)$

Radius: $\sqrt{29}$



$y^2 - 6y - 1 = 8x - x^2 + 3$

19. Graph (c) is the graph of $16x^2 + 4y^2 = 64$.

20. Graph (a) is the graph of $4x^2 + 5y^2 = 20$.

21. Graph (d) is the graph of $x^2 + 9y^2 - 6x + 90y = -225$.

22. Graph (b) is the graph of $9x^2 + 4y^2 + 18x - 16y = 11$.

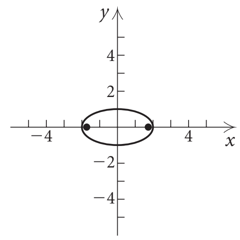
23. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

$\dfrac{x^2}{2^2} + \dfrac{y^2}{1^2} = 1$ Standard form

$a = 2$, $b = 1$

The major axis is horizontal, so the vertices are $(-2, 0)$ and $(2, 0)$. Since we know that $c^2 = a^2 - b^2$, we have $c^2 = 4 - 1 = 3$, so $c = \sqrt{3}$ and the foci are $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$.

To graph the ellipse, plot the vertices. Note also that since $b = 1$, the y-intercepts are $(0, -1)$ and $(0, 1)$. Plot these points as well and connect the four plotted points with a smooth curve.

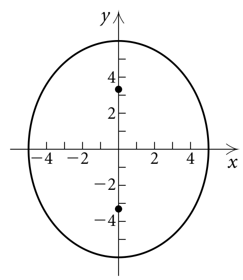


$\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

24. $\dfrac{x^2}{25} + \dfrac{y^2}{36} = 1$, or $\dfrac{x^2}{5^2} + \dfrac{y^2}{6^2} = 1$

$a = 6$, $b = 5$

The major axis is vertical, so the vertices are $(0, -6)$ and $(0, 6)$. Since $c^2 = a^2 - b^2$, we have $c^2 = 36 - 25 = 11$, so $c = \sqrt{11}$ and the foci are $(0, -\sqrt{11})$ and $(0, \sqrt{11})$.



$\dfrac{x^2}{25} + \dfrac{y^2}{36} = 1$

25. $16x^2 + 9y^2 = 144$

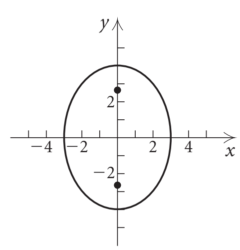
$\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ Dividing by 144

$\dfrac{x^2}{3^2} + \dfrac{y^2}{4^2} = 1$ Standard form

$a = 4$, $b = 3$

The major axis is vertical, so the vertices are $(0, -4)$ and $(0, 4)$. Since $c^2 = a^2 - b^2$, we have $c^2 = 16 - 9 = 7$, so $c = \sqrt{7}$ and the foci are $(0, -\sqrt{7})$ and $(0, \sqrt{7})$.

To graph the ellipse, plot the vertices. Note also that since $b = 3$, the x-intercepts are $(-3, 0)$ and $(3, 0)$. Plot these points as well and connect the four plotted points with a smooth curve.



$16x^2 + 9y^2 = 144$

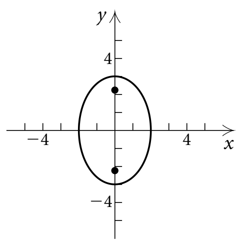
26. $9x^2 + 4y^2 = 36$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$$

$a = 3,\ b = 2$

The major axis is vertical, so the vertices are $(0, -3)$ and $(0, 3)$. Since $c^2 = a^2 - b^2$, we have $c^2 = 9 - 4 = 5$, so $c = \sqrt{5}$ and the foci are $(0, -\sqrt{5})$ and $(0, \sqrt{5})$.



$9x^2 + 4y^2 = 36$

27. $2x^2 + 3y^2 = 6$

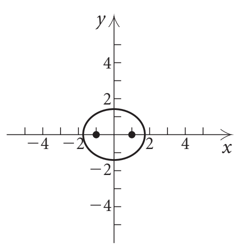
$$\frac{x^2}{3} + \frac{y^2}{2} = 1$$

$$\frac{x^2}{(\sqrt{3})^2} + \frac{y^2}{(\sqrt{2})^2} = 1$$

$a = \sqrt{3},\ b = \sqrt{2}$

The major axis is horizontal, so the vertices are $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$. Since $c^2 = a^2 - b^2$, we have $c^2 = 3 - 2 = 1$, so $c = 1$ and the foci are $(-1, 0)$ and $(1, 0)$.

To graph the ellipse, plot the vertices. Note also that since $b = \sqrt{2}$, the y-intercepts are $(0, -\sqrt{2})$ and $(0, \sqrt{2})$. Plot these points as well and connect the four plotted points with a smooth curve.



$2x^2 + 3y^2 = 6$

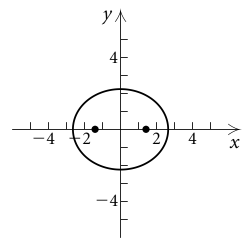
28. $5x^2 + 7y^2 = 35$

$$\frac{x^2}{7} + \frac{y^2}{5} = 1$$

$$\frac{x^2}{(\sqrt{7})^2} + \frac{y^2}{(\sqrt{5})^2} = 1$$

$a = \sqrt{7},\ b = \sqrt{5}$

The major axis is horizontal, so the vertices are $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$. Since $c^2 = a^2 - b^2$, we have $c^2 = 7 - 5 = 2$, so $c = \sqrt{2}$ and the foci are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$.



$5x^2 + 7y^2 = 35$

29. $4x^2 + 9y^2 = 1$

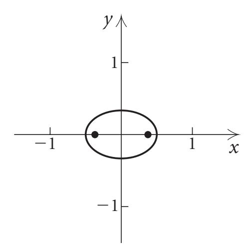
$$\frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{9}} = 1$$

$$\frac{x^2}{\left(\frac{1}{2}\right)^2} + \frac{y^2}{\left(\frac{1}{3}\right)^2} = 1$$

$a = \frac{1}{2},\ b = \frac{1}{3}$

The major axis is horizontal, so the vertices are $\left(-\frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, 0\right)$. Since $c^2 = a^2 - b^2$, we have $c^2 = \frac{1}{4} - \frac{1}{9} = \frac{5}{36}$, so $c = \frac{\sqrt{5}}{6}$ and the foci are $\left(-\frac{\sqrt{5}}{6}, 0\right)$ and $\left(\frac{\sqrt{5}}{6}, 0\right)$.

To graph the ellipse, plot the vertices. Note also that since $b = \frac{1}{3}$, the y-intercepts are $\left(0, -\frac{1}{3}\right)$ and $\left(0, \frac{1}{3}\right)$. Plot these points as well and connect the four plotted points with a smooth curve.



$4x^2 + 9y^2 = 1$

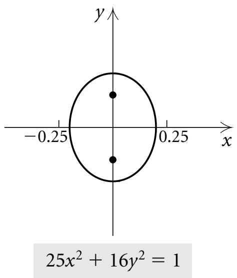
30. $25x^2 + 16y^2 = 1$

$$\frac{x^2}{\frac{1}{25}} + \frac{y^2}{\frac{1}{16}} = 1$$

$$\frac{x^2}{\left(\frac{1}{5}\right)^2} + \frac{y^2}{\left(\frac{1}{4}\right)^2} = 1$$

$a = \frac{1}{4},\ b = \frac{1}{5}$

The major axis is vertical, so the vertices are $\left(0, -\frac{1}{4}\right)$ and $\left(0, \frac{1}{4}\right)$. Since $c^2 = a^2 - b^2$, we have $c^2 = \frac{1}{16} - \frac{1}{25} = \frac{9}{400}$, so $c = \frac{3}{20}$ and the foci are $\left(0, -\frac{3}{20}\right)$ and $\left(0, \frac{3}{20}\right)$.



31. The vertices are on the x-axis, so the major axis is horizontal. We have $a = 7$ and $c = 3$, so we can find b^2:

$$c^2 = a^2 - b^2$$
$$3^2 = 7^2 - b^2$$
$$b^2 = 49 - 9 = 40$$

Write the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{x^2}{49} + \frac{y^2}{40} = 1$$

32. The major axis is vertical; $a = 6$ and $c = 4$.

$$c^2 = a^2 - b^2$$
$$16 = 36 - b^2$$
$$b^2 = 20$$

The equation is $\dfrac{x^2}{20} + \dfrac{y^2}{36} = 1$.

33. The vertices, $(0, -8)$ and $(0, 8)$, are on the y-axis, so the major axis is vertical and $a = 8$. Since the vertices are equidistant from the origin, the center of the ellipse is at the origin. The length of the minor axis is 10, so $b = 10/2$, or 5.

Write the equation:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$
$$\frac{x^2}{5^2} + \frac{y^2}{8^2} = 1$$
$$\frac{x^2}{25} + \frac{y^2}{64} = 1$$

34. The vertices, $(-5, 0)$ and $(5, 0)$ are on the x-axis, so the major axis is horizontal and $a = 5$. Since the vertices are equidistant from the origin, the center of the ellipse is at the origin. The length of the minor axis is 6, so $b = 6/2$, or 3. The equation is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$.

35. The foci, $(-2, 0)$ and $(2, 0)$ are on the x-axis, so the major axis is horizontal and $c = 2$. Since the foci are equidistant from the origin, the center of the ellipse is at the origin. The length of the major axis is 6, so $a = 6/2$, or 3. Now we find b^2:

$$c^2 = a^2 - b^2$$
$$2^2 = 3^2 - b^2$$
$$4 = 9 - b^2$$
$$b^2 = 5$$

Write the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

36. The foci, $(0, -3)$ and $(0, 3)$, are on the y-axis, so the major axis is vertical. The foci are equidistant from the origin, so the center of the ellipse is at the origin. The length of the major axis is 10, so $a = 10/2$, or 5. Find b^2:

$$c^2 = a^2 - b^2$$
$$9 = 25 - b^2$$
$$b^2 = 16$$

The equation is $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$.

37. $\dfrac{(x-1)^2}{9} + \dfrac{(y-2)^2}{4} = 1$

$\dfrac{(x-1)^2}{3^2} + \dfrac{(y-2)^2}{2^2} = 1$ Standard form

The center is $(1, 2)$. Note that $a = 3$ and $b = 2$. The major axis is horizontal so the vertices are 3 units left and right of the center:

$(1 - 3, 2)$ and $(1 + 3, 2)$, or $(-2, 2)$ and $(4, 2)$.

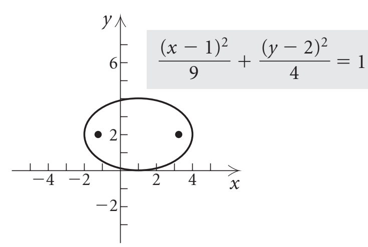
We know that $c^2 = a^2 - b^2$, so $c^2 = 9 - 4 = 5$ and $c = \sqrt{5}$. Then the foci are $\sqrt{5}$ units left and right of the center:

$$(1 - \sqrt{5}, 2) \text{ and } (1 + \sqrt{5}, 2).$$

To graph the ellipse, plot the vertices. Since $b = 2$, two other points on the graph are 2 units below and above the center:

$(1, 2 - 2)$ and $(1, 2 + 2)$ or $(1, 0)$ and $(1, 4)$

Plot these points also and connect the four plotted points with a smooth curve.



38. $\dfrac{(x-1)^2}{1} + \dfrac{(y-2)^2}{4} = 1$

$\dfrac{(x-1)^2}{1^2} + \dfrac{(y-2)^2}{2^2} = 1$ Standard form

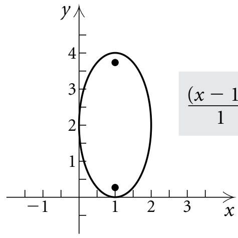
The center is $(1, 2)$. Note that $a = 2$ and $b = 1$. The major axis is vertical so the vertices are 2 units below and above the center:

$(1, 2 - 2)$ and $(1, 2 + 2)$, or $(1, 0)$ and $(1, 4)$.

We know that $c^2 = a^2 - b^2$, so $c^2 = 4 - 1 = 3$ and $c = \sqrt{3}$. Then the foci are $\sqrt{3}$ units below and above the center:

$$(1, 2 - \sqrt{3}) \text{ and } (1, 2 + \sqrt{3}).$$

Since $b=1$, two points on the graph other than the vertices are $(1-1,2)$ and $(1+1,2)$ or $(0,2)$ and $(2,2)$.



$\frac{(x-1)^2}{1}+\frac{(y-2)^2}{4}=1$

39.
$$\frac{(x+3)^2}{25}+\frac{(y-5)^2}{36}=1$$
$$\frac{[x-(-3)]^2}{5^2}+\frac{(y-5)^2}{6^2}=1 \quad \text{Standard form}$$

The center is $(-3,5)$. Note that $a=6$ and $b=5$. The major axis is vertical so the vertices are 6 units below and above the center:

$(-3,5-6)$ and $(-3,5+6)$, or $(-3,-1)$ and $(-3,11)$.

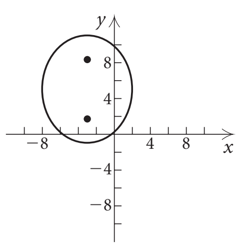
We know that $c^2=a^2-b^2$, so $c^2=36-25=11$ and $c=\sqrt{11}$. Then the foci are $\sqrt{11}$ units below and above the vertex:

$$(-3,5-\sqrt{11}) \text{ and } (-3,5+\sqrt{11}).$$

To graph the ellipse, plot the vertices. Since $b=5$, two other points on the graph are 5 units left and right of the center:

$(-3-5,5)$ and $(-3+5,5)$, or $(-8,5)$ and $(2,5)$

Plot these points also and connect the four plotted points with a smooth curve.



$\frac{(x+3)^2}{25}+\frac{(y-5)^2}{36}=1$

40.
$$\frac{(x-2)^2}{16}+\frac{(y+3)^2}{25}=1$$
$$\frac{(x-2)^2}{4^2}+\frac{[y-(-3)]^2}{5^2}=1 \quad \text{Standard form}$$

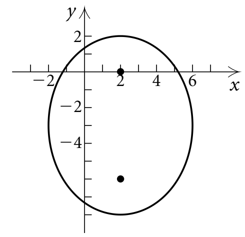
The center is $(2,-3)$. Note that $a=5$ and $b=4$. The major axis is vertical so the vertices are 5 units below and above the center:

$(2,-3-5)$ and $(2,-3+5)$, or $(2,-8)$ and $(2,2)$.

We know that $c^2=a^2-b^2$, so $c^2=25-16=9$ and $c=3$. Then the foci are 3 units below and above the center:

$(2,-3-3)$ and $(2,-3+3)$, or $(2,-6)$ and $(2,0)$.

Since $b=4$, two points on the graph other than the vertices are $(2-4,-3)$ and $(2+4,-3)$, or $(-2,-3)$ and $(6,-3)$.



$\frac{(x-2)^2}{16}+\frac{(y+3)^2}{25}=1$

41.
$$3(x+2)^2+4(y-1)^2=192$$
$$\frac{(x+2)^2}{64}+\frac{(y-1)^2}{48}=1 \quad \text{Dividing by 192}$$
$$\frac{[x-(-2)]^2}{8^2}+\frac{(y-1)^2}{(\sqrt{48})^2}=1 \quad \text{Standard form}$$

The center is $(-2,1)$. Note that $a=8$ and $b=\sqrt{48}$, or $4\sqrt{3}$. The major axis is horizontal so the vertices are 8 units left and right of the center:

$(-2-8,1)$ and $(-2+8,1)$, or $(-10,1)$ and $(6,1)$.

We know that $c^2=a^2-b^2$, so $c^2=64-48=16$ and $c=4$. Then the foci are 4 units left and right of the center:

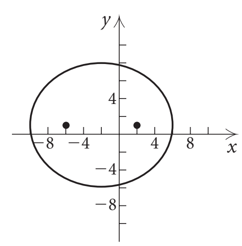
$(-2-4,1)$ and $(-2+4,1)$ or $(-6,1)$ and $(2,1)$.

To graph the ellipse, plot the vertices. Since $b=4\sqrt{3}\approx 6.928$, two other points on the graph are about 6.928 units below and above the center:

$(-2,1-6.928)$ and $(-2,1+6.928)$, or

$(-2,-5.928)$ and $(-2,7.928)$.

Plot these points also and connect the four plotted points with a smooth curve.



$3(x+2)^2+4(y-1)^2=192$

42.
$$4(x-5)^2+3(y-4)^2=48$$
$$\frac{(x-5)^2}{12}+\frac{(y-4)^2}{16}=1$$
$$\frac{(x-5)^2}{(\sqrt{12})^2}+\frac{(y-4)^2}{4^2}=1 \quad \text{Standard form}$$

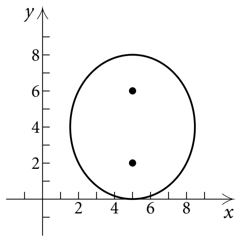
The center is $(5,4)$. Note that $a=16$ and $b=\sqrt{12}$, or $2\sqrt{3}$. The major axis is vertical so the vertices are 4 units below and above the center:

$(5,4-4)$ and $(5,4+4)$, or $(5,0)$ and $(5,8)$.

We know that $c^2=a^2-b^2$, so $c^2=16-12=4$ and $c=2$. Then the foci are 2 units below and above the center:

$(5,4-2)$ and $(5,4+2)$, or $(5,2)$ and $(5,6)$.

Since $b = 2\sqrt{3} \approx 3.464$, two points on the graph other than the vertices are about $(5 - 3.464, 4)$ and $(5 + 3.464, 4)$, or $(1.536, 4)$ and $(8.464, 4)$.



$4(x-5)^2 + 3(y-4)^2 = 48$

43. Begin by completing the square twice.

$$\begin{aligned}
4x^2 + 9y^2 - 16x + 18y - 11 &= 0 \\
4x^2 - 16x + 9y^2 + 18y &= 11 \\
4(x^2 - 4x) + 9(y^2 + 2y) &= 11 \\
4(x^2 - 4x + 4) + 9(y^2 + 2y + 1) &= 11 + 4 \cdot 4 + 9 \cdot 1 \\
4(x-2)^2 + 9(y+1)^2 &= 36 \\
\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} &= 1 \\
\frac{(x-2)^2}{3^2} + \frac{[y-(-1)]^2}{2^2} &= 1
\end{aligned}$$

The center is $(2, -1)$. Note that $a = 3$ and $b = 2$. The major axis is horizontal so the vertices are 3 units left and right of the center:

$(2 - 3, -1)$ and $(2 + 3, -1)$, or $(-1, -1)$ and $(5, -1)$.

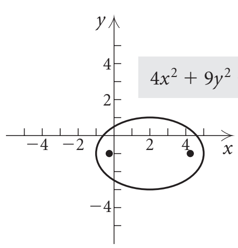
We know that $c^2 = a^2 - b^2$, so $c^2 = 9 - 4 = 5$ and $c = \sqrt{5}$. Then the foci are $\sqrt{5}$ units left and right of the center:

$(2 - \sqrt{5}, -1)$ and $(2 + \sqrt{5}, -1)$.

To graph the ellipse, plot the vertices. Since $b = 2$, two other points on the graph are 2 units below and above the center:

$(2, -1-2)$ and $(2, -1+2)$, or $(2, -3)$ and $(2, 1)$.

Plot these points also and connect the four plotted points with a smooth curve.



$4x^2 + 9y^2 - 16x + 18y - 11 = 0$

44. Begin by completing the square twice.

$$\begin{aligned}
x^2 + 2y^2 - 10x + 8y + 29 &= 0 \\
x^2 - 10x + 2(y^2 + 4y) &= -29 \\
x^2 - 10x + 25 + 2(y^2 + 4y + 4) &= -29 + 25 + 2 \cdot 4 \\
(x-5)^2 + 2(y+2)^2 &= 4 \\
\frac{(x-5)^2}{4} + \frac{(y+2)^2}{2} &= 1 \\
\frac{(x-5)^2}{2^2} + \frac{[y-(-2)]^2}{(\sqrt{2})^2} &= 1
\end{aligned}$$

The center is $(5, -2)$. Note that $a = 2$ and $b = \sqrt{2}$. The major axis is horizontal so the vertices are 2 units left and right of the center:

$(5-2, -2)$ and $(5+2, -2)$, or $(3, -2)$ and $(7, -2)$.

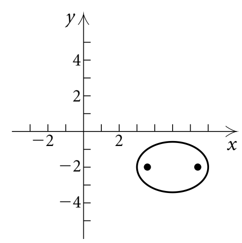
We know that $c^2 = a^2 - b^2$, so $c^2 = 4 - 2 = 2$ and $c = \sqrt{2}$. Then the foci are $\sqrt{2}$ units left and right of the center:

$(5 - \sqrt{2}, -2)$ and $(5 + \sqrt{2}, -2)$.

Since $b = \sqrt{2} \approx 1.414$, two points on the graph other than the vertices are about

$(5, -2 - 1.414)$ and $(5, -2 + 1.414)$, or

$(5, -3.414)$ and $(5, -0.586)$.



$x^2 + 2y^2 - 10x + 8y + 29 = 0$

45. Begin by completing the square twice.

$$\begin{aligned}
4x^2 + y^2 - 8x - 2y + 1 &= 0 \\
4x^2 - 8x + y^2 - 2y &= -1 \\
4(x^2 - 2x) + y^2 - 2y &= -1 \\
4(x^2 - 2x + 1) + y^2 - 2y + 1 &= -1 + 4 \cdot 1 + 1 \\
4(x-1)^2 + (y-1)^2 &= 4 \\
\frac{(x-1)^2}{1} + \frac{(y-1)^2}{4} &= 1 \\
\frac{(x-1)^2}{1^2} + \frac{(y-1)^2}{2^2} &= 1
\end{aligned}$$

The center is $(1, 1)$. Note that $a = 2$ and $b = 1$. The major axis is vertical so the vertices are 2 units below and above the center:

$(1, 1-2)$ and $(1, 1+2)$, or $(1, -1)$ and $(1, 3)$.

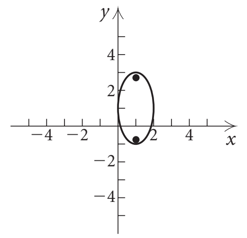
We know that $c^2 = a^2 - b^2$, so $c^2 = 4 - 1 = 3$ and $c = \sqrt{3}$. Then the foci are $\sqrt{3}$ units below and above the center:

$(1, 1 - \sqrt{3})$ and $(1, 1 + \sqrt{3})$.

To graph the ellipse, plot the vertices. Since $b = 1$, two other points on the graph are 1 unit left and right of the center:

$(1-1, 1)$ and $(1+1, 1)$ or $(0, 1)$ and $(2, 1)$.

Plot these points also and connect the four plotted points with a smooth curve.



$4x^2 + y^2 - 8x - 2y + 1 = 0$

46. Begin by completing the square twice.

$$9x^2 + 4y^2 + 54x - 8y + 49 = 0$$
$$9(x^2 + 6x) + 4(y^2 - 2y) = -49$$
$$9(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -49 + 9\cdot 9 + 4\cdot 1$$
$$9(x + 3)^2 + 4(y - 1)^2 = 36$$
$$\frac{(x-3)^2}{4} + \frac{(y-1)^2}{9} = 1$$
$$\frac{[x-(-3)]^2}{2^2} + \frac{(y-1)^2}{3^2} = 1$$

The center is $(-3, 1)$. Note that $a = 3$ and $b = 2$. The major axis is vertical so the vertices are 3 units below and above the center:

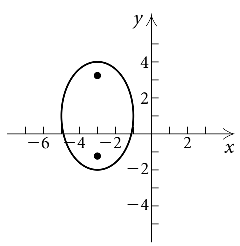
$(-3, 1-3)$ and $(-3, 1+3)$, or $(-3, -2)$ and $(-3, 4)$.

We know that $c^2 = a^2 - b^2$, so $c^2 = 9 - 4 = 5$ and $c = \sqrt{5}$. Then the foci are $\sqrt{5}$ units below and above the center:

$(-3, 1 - \sqrt{5})$ and $(-3, 1 + \sqrt{5})$.

Since $b = 2$, two points on the graph other than the vertices are

$(-3 - 2, 1)$ and $(-3 + 2, 1)$, or $(-5, 1)$ and $(-1, 1)$.



$9x^2 + 4y^2 + 54x - 8y + 49 = 0$

47. The ellipse in Example 4 is flatter than the one in Example 2, so the ellipse in Example 2 has the smaller eccentricity.

We compute the eccentricities: In Example 2, $c = 3$ and $a = 5$, so $e = c/a = 3/5 = 0.6$. In Example 4, $c = 2\sqrt{3}$ and $a = 4$, so $e = c/a = 2\sqrt{3}/4 \approx 0.866$. These computations confirm that the ellipse in Example 2 has the smaller eccentricity.

48. Ellipse (b) is flatter than ellipse (a), so ellipse (a) has the smaller eccentricity.

49. Since the vertices, $(0, -4)$ and $(0, 4)$ are on the y-axis and are equidistant from the origin, we know that the major axis of the ellipse is vertical, its center is at the origin, and $a = 4$. Use the information that $e = 1/4$ to find c:

$$e = \frac{c}{a}$$
$$\frac{1}{4} = \frac{c}{4} \quad \text{Substituting}$$
$$c = 1$$

Now $c^2 = a^2 - b^2$, so we can find b^2:

$$1^2 = 4^2 - b^2$$
$$1 = 16 - b^2$$
$$b^2 = 15$$

Write the equation of the ellipse:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$
$$\frac{x^2}{15} + \frac{y^2}{16} = 1$$

50. Since the vertices, $(-3, 0)$ and $(3, 0)$, are on the x-axis and are equidistant from the origin, we know that the major axis of the ellipse is horizontal, its center is at the origin, and $a = 3$.

Find c:

$$\frac{c}{a} = \frac{7}{10}$$
$$\frac{c}{3} = \frac{7}{10}$$
$$c = \frac{21}{10}$$

Now find b^2:

$$c^2 = a^2 - b^2$$
$$\left(\frac{21}{10}\right)^2 = 3^2 - b^2$$
$$b^2 = \frac{459}{100}$$

The equation of the ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{459/100} = 1$.

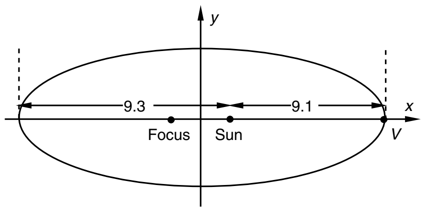
51. From the figure in the text we see that the center of the ellipse is $(0, 0)$, the major axis is horizontal, the vertices are $(-50, 0)$ and $(50, 0)$, and one y-intercept is $(0, 12)$. Then $a = 50$ and $b = 12$. The equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{x^2}{50^2} + \frac{y}{12^2} = 1$$
$$\frac{x^2}{2500} + \frac{y^2}{144} = 1.$$

52. Find the equation of the ellipse with center $(0, 0)$, $a = 1048/2 = 524$, $b = 898/2 = 449$, and a horizontal major axis:

$$\frac{x^2}{524^2} + \frac{y^2}{449^2} = 1$$
$$\frac{x^2}{274{,}576} + \frac{y^2}{201{,}601} = 1$$

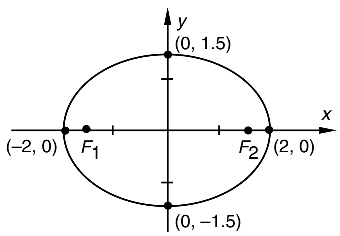
53. Position a coordinate system as shown below where 1 unit $= 10^7$ mi.



The length of the major axis is $9.3 + 9.1$, or 18.4. Then the distance from the center of the ellipse (the origin) to V is $18.4/2$, or 9.2. Since the distance from the sun to V is 9.1, the distance from the sun to the center is $9.2 - 9.1$, or 0.1. Then the distance from the sun to the other focus is twice this distance:

$$\begin{aligned}2(0.1 \times 10^7 \text{ mi}) &= 0.2 \times 10^7 \text{ mi}\\ &= 2 \times 10^6 \text{ mi}\end{aligned}$$

54. Position a coordinate system as shown.



a) We have an ellipse with $a = 2$ and $b = 1.5$. The foci are at $(\pm c, 0)$ where $c^2 = a^2 - b^2$, so $c^2 = 4 - 2.25 = 1.75$ and $c = \sqrt{1.75}$. Then the string should be attached $2 - \sqrt{1.75}$ ft, or about 0.7 ft from the ends of the board.

b) The string should be the length of the major axis, 4 ft.

55. midpoint

56. zero

57. y-intercept

58. two different real-number solutions

59. remainder

60. ellipse

61. parabola

62. circle

63. The center of the ellipse is the midpoint of the segment connecting the vertices:

$$\left(\frac{3+3}{2}, \frac{-4+6}{2}\right), \text{ or } (3, 1).$$

Now a is the distance from the origin to a vertex. We use the vertex $(3, 6)$.

$$a = \sqrt{(3-3)^2 + (6-1)^2} = 5$$

Also b is one-half the length of the minor axis.

$$b = \frac{\sqrt{(5-1)^2 + (1-1)^2}}{2} = \frac{4}{2} = 2$$

The vertices lie on the vertical line $x = 3$, so the major axis is vertical. We write the equation of the ellipse.

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-3)^2}{4} + \frac{(y-1)^2}{25} = 1$$

64. Center: $\left(\dfrac{-1-1}{2}, \dfrac{-1+5}{2}\right)$, or $(-1, 2)$

$$a = \sqrt{[1-(-1)]^2 + (-1-2)^2} = 3$$

$$b = \frac{\sqrt{(-3-1)^2 + (2-2)^2}}{2} = \frac{4}{2} = 2$$

The vertices are on the line $x = -1$, so the major axis is vertical. The equation is

$$\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1.$$

65. The center is the midpoint of the segment connecting the vertices:

$$\left(\frac{-3+3}{2}, \frac{0+0}{2}\right), \text{ or } (0, 0).$$

Then $a = 3$ and since the vertices are on the x-axis, the major axis is horizontal. The equation is of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Substitute 3 for a, 2 for x, and $\dfrac{22}{3}$ for y and solve for b^2.

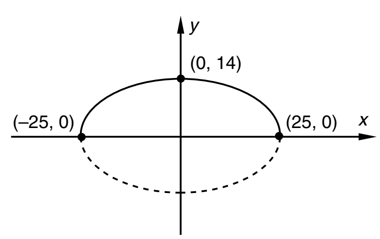
$$\begin{aligned}\frac{4}{9} + \frac{\frac{484}{9}}{b^2} &= 1\\ \frac{4}{9} + \frac{484}{9b^2} &= 1\\ 4b^2 + 484 &= 9b^2\\ 484 &= 5b^2\\ \frac{484}{5} &= b^2\end{aligned}$$

Then the equation is $\dfrac{x^2}{9} + \dfrac{y^2}{484/5} = 1$.

66. $a = 4/2 = 2$; $b = 1/2$

The equation is $\dfrac{(x+2)^2}{1/4} + \dfrac{(y-3)^2}{4} = 1$.

67. Position a coordinate system as shown.



The equation of the ellipse is

$$\frac{x^2}{25^2}+\frac{y^2}{14^2}=1$$
$$\frac{x^2}{625}+\frac{y^2}{196}=1.$$

A point 6 ft from the riverbank corresponds to $(25-6,0)$, or $(19,0)$ or to $(-25+6,0)$, or $(-19,0)$. Substitute either 19 or -19 for x and solve for y, the clearance.

$$\frac{19^2}{625}+\frac{y^2}{196}=1$$
$$\frac{y^2}{196}=1-\frac{361}{625}$$
$$y^2=196\left(1-\frac{361}{625}\right)$$
$$y\approx 9.1$$

The clearance 6 ft from the riverbank is about 9.1 ft.

Exercise Set 10.3

1. Graph (b) is the graph of $\dfrac{x^2}{25}-\dfrac{y^2}{9}=1$.

2. Graph (e) is the graph of $\dfrac{y^2}{4}-\dfrac{x^2}{36}=1$.

3. Graph (c) is the graph of $\dfrac{(y-1)^2}{16}-\dfrac{(x+3)^2}{1}=1$.

4. Graph (f) is the graph of $\dfrac{(x+4)^2}{100}-\dfrac{(y-2)^2}{81}=1$.

5. Graph (a) is the graph of $25x^2-16y^2=400$.

6. Graph (d) is the graph of $y^2-x^2=9$.

7. The vertices are equidistant from the origin and are on the y-axis, so the center is at the origin and the transverse axis is vertical. Since $c^2=a^2+b^2$, we have $5^2=3^2+b^2$ so $b^2=16$.

The equation is of the form $\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=1$, so we have $\dfrac{y^2}{9}-\dfrac{x^2}{16}=1$.

8. The vertices are equidistant from the origin and are on the x-axis, so the center is at the origin and the transverse axis is horizontal. Since $c^2=a^2+b^2$, we have $2^2=1^2+b^2$ so $b^2=3$. The equation is $\dfrac{x^2}{1}-\dfrac{y^2}{3}=1$.

9. The asymptotes pass through the origin, so the center is the origin. The given vertex is on the x-axis, so the transverse axis is horizontal. Since $\dfrac{b}{a}x=\dfrac{3}{2}x$ and $a=2$, we have $b=3$. The equation is of the form $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, so we have $\dfrac{x^2}{2^2}-\dfrac{y^2}{3^2}=1$, or $\dfrac{x^2}{4}-\dfrac{y^2}{9}=1$.

10. The asymptotes pass through the origin, so the center is the origin. The given vertex is on the y-axis, so the transverse axis is vertical. We use the equation of an asymptote to find b.

$$\frac{a}{b}x=\frac{5}{4}x$$
$$\frac{3}{b}x=\frac{5}{4}x \quad \text{Substituting 3 for } a$$
$$\frac{12}{5}=b$$

The equation is $\dfrac{y^2}{9}-\dfrac{x^2}{144/25}=1$.

11. $\dfrac{x^2}{4}-\dfrac{y^2}{4}=1$

$\dfrac{x^2}{2^2}-\dfrac{y^2}{2^2}=1$ Standard form

The center is $(0,0)$; $a=2$ and $b=2$. The transverse axis is horizontal so the vertices are $(-2,0)$ and $(2,0)$. Since $c^2=a^2+b^2$, we have $c^2=4+4=8$ and $c=\sqrt{8}$, or $2\sqrt{2}$. Then the foci are $(-2\sqrt{2},0)$ and $(2\sqrt{2},0)$.

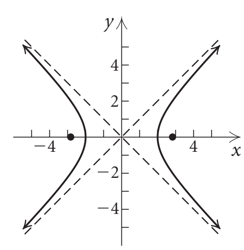
Find the asymptotes:

$y=\dfrac{b}{a}x$ and $y=-\dfrac{b}{a}x$

$y=\dfrac{2}{2}x$ and $y=-\dfrac{2}{2}x$

$y=x$ and $y=-x$

To draw the graph sketch the asymptotes, plot the vertices, and draw the branches of the hyperbola outward from the vertices toward the asymptotes.



$\dfrac{x^2}{4}-\dfrac{y^2}{4}=1$

12. $\dfrac{x^2}{1}-\dfrac{y^2}{9}=1$

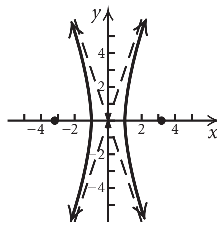
$\dfrac{x^2}{1^2}-\dfrac{y^2}{3^2}=1$ Standard form

The center is $(0,0)$; $a=1$ and $b=3$. The transverse axis is horizontal so the vertices are $(-1,0)$ and $(1,0)$. Since $c^2=a^2+b^2$, we have $c^2=1+9=10$ and $c=\sqrt{10}$. Then the foci are $(-\sqrt{10},0)$ and $(\sqrt{10},0)$. Find the asymptotes:

$y=\dfrac{b}{a}x$ and $y=-\dfrac{b}{a}x$

$y=\dfrac{3}{1}x$ and $y=-\dfrac{3}{1}x$

$y=3x$ and $y=-3x$



$\frac{x^2}{1} - \frac{y^2}{9} = 1$

13. $\frac{(x-2)^2}{9} - \frac{(y+5)^2}{1} = 1$

$\frac{(x-2)^2}{3^2} - \frac{[y-(-5)]^2}{1^2} = 1$ Standard form

The center is $(2,-5)$; $a = 3$ and $b = 1$. The transverse axis is horizontal, so the vertices are 3 units left and right of the center:

$(2-3,-5)$ and $(2+3,-5)$, or $(-1,-5)$ and $(5,-5)$.

Since $c^2 = a^2 + b^2$, we have $c^2 = 9 + 1 = 10$ and $c = \sqrt{10}$. Then the foci are $\sqrt{10}$ units left and right of the center:

$(2-\sqrt{10},-5)$ and $(2+\sqrt{10},-5)$.

Find the asymptotes:

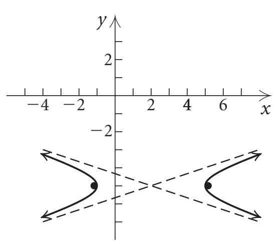
$$y-k=\frac{b}{a}(x-h) \text{ and } y-k=-\frac{b}{a}(x-h)$$

$$y-(-5)=\frac{1}{3}(x-2) \text{ and } y-(-5)=-\frac{1}{3}(x-2)$$

$$y+5=\frac{1}{3}(x-2) \text{ and } y+5=-\frac{1}{3}(x-2), \text{ or}$$

$$y=\frac{1}{3}x-\frac{17}{3} \text{ and } y=-\frac{1}{3}x-\frac{13}{3}$$

Sketch the asymptotes, plot the vertices, and draw the graph.



$\frac{(x-2)^2}{9} - \frac{(y+5)^2}{1} = 1$

14. $\frac{(x-5)^2}{16} - \frac{(y+2)^2}{9} = 1$

$\frac{(x-5)^2}{4^2} - \frac{[y-(-2)]^2}{3^2} = 1$ Standard form

The center is $(5,-2)$; $a = 4$ and $b = 3$. The transverse axis is horizontal, so the vertices are 4 units left and right of the center:

$(5-4,-2)$ and $(5+4,-2)$, or $(1,-2)$ and $(9,-2)$.

Since $c^2 = a^2 + b^2$, we have $c^2 = 16 + 9 = 25$ and $c = 5$. Then the foci are 5 units left and right of the center:

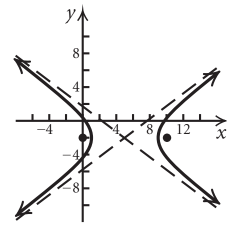
$(5-5,-2)$ and $(5+5,-2)$, or $(0,-2)$ and $(10,-2)$.

Find the asymptotes:

$$y-k=\frac{b}{a}(x-h) \text{ and } y-k=-\frac{b}{a}(x-h)$$

$$y+2=\frac{3}{4}(x-5) \text{ and } y+2=-\frac{3}{4}(x-5), \text{ or}$$

$$y=\frac{3}{4}x-\frac{23}{4} \text{ and } y=-\frac{3}{4}x+\frac{7}{4}$$



$\frac{(x-5)^2}{16} - \frac{(y+2)^2}{9} = 1$

15. $\frac{(y+3)^2}{4} - \frac{(x+1)^2}{16} = 1$

$\frac{[y-(-3)]^2}{2^2} - \frac{[x-(-1)]^2}{4^2} = 1$ Standard form

The center is $(-1,-3)$; $a = 2$ and $b = 4$. The transverse axis is vertical, so the vertices are 2 units below and above the center:

$(-1,-3-2)$ and $(1,-3+2)$, or $(-1,-5)$ and $(-1,-1)$.

Since $c^2 = a^2 + b^2$, we have $c^2 = 4 + 16 = 20$ and $c = \sqrt{20}$, or $2\sqrt{5}$. Then the foci are $2\sqrt{5}$ units below and above of the center:

$(-1,-3-2\sqrt{5})$ and $(-1,-3+2\sqrt{5})$.

Find the asymptotes:

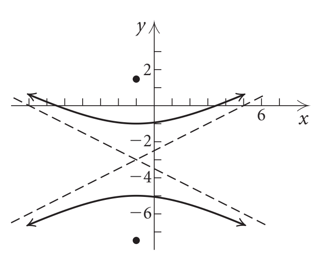
$$y-k=\frac{a}{b}(x-h) \text{ and } y-k=-\frac{a}{b}(x-h)$$

$$y-(-3)=\frac{2}{4}(x-(-1)) \text{ and } y-(-3)=-\frac{2}{4}(x-(-1))$$

$$y+3=\frac{1}{2}(x+1) \text{ and } y+3=-\frac{1}{2}(x+1), \text{ or}$$

$$y=\frac{1}{2}x-\frac{5}{2} \text{ and } y=-\frac{1}{2}x-\frac{7}{2}$$

Sketch the asymptotes, plot the vertices, and draw the graph.



$\frac{(y+3)^2}{4} - \frac{(x+1)^2}{16} = 1$

16. $\dfrac{(y+4)^2}{25} - \dfrac{(x+2)^2}{16} = 1$

$\dfrac{[y-(-4)]^2}{5^2} - \dfrac{[x-(-2)]^2}{4^2} = 1$ Standard form

The center is $(-2,-4)$; $a = 5$ and $b = 4$. The transverse axis is vertical, so the vertices are 5 units below and above the center:

$(-2,-4-5)$ and $(-2,-4+5)$, or $(-2,-9)$ and $(-2,1)$.

Since $c^2 = a^2+b^2$, we have $c^2 = 25+16 = 41$ and $c = \sqrt{41}$. Then the foci are $\sqrt{41}$ units below and above the center:

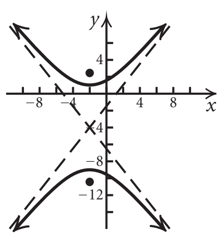
$(-2,-4-\sqrt{41})$ and $(-2,-4+\sqrt{41})$.

Find the asymptotes:

$y-k = \dfrac{a}{b}(x-h)$ and $y-k = -\dfrac{a}{b}(x-h)$

$y+4 = \dfrac{5}{4}(x+2)$ and $y+4 = -\dfrac{5}{4}(x+2)$, or

$y = \dfrac{5}{4}x - \dfrac{3}{2}$ and $y = -\dfrac{5}{4}x - \dfrac{13}{2}$



$\dfrac{(y+4)^2}{25} - \dfrac{(x+2)^2}{16} = 1$

17. $x^2 - 4y^2 = 4$

$\dfrac{x^2}{4} - \dfrac{y^2}{1} = 1$

$\dfrac{x^2}{2^2} - \dfrac{y^2}{1^2} = 1$ Standard form

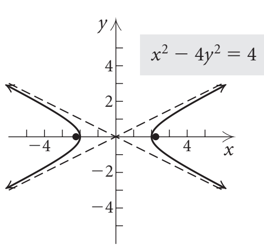
The center is $(0,0)$; $a = 2$ and $b = 1$. The transverse axis is horizontal, so the vertices are $(-2,0)$ and $(2,0)$. Since $c^2 = a^2+b^2$, we have $c^2 = 4+1 = 5$ and $c = \sqrt{5}$. Then the foci are $(-\sqrt{5},0)$ and $(\sqrt{5},0)$.

Find the asymptotes:

$y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$

$y = \dfrac{1}{2}x$ and $y = -\dfrac{1}{2}x$

Sketch the asymptotes, plot the vertices, and draw the graph.



18. $4x^2 - y^2 = 16$

$\dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$

$\dfrac{x^2}{2^2} - \dfrac{y^2}{4^2} = 1$ Standard form

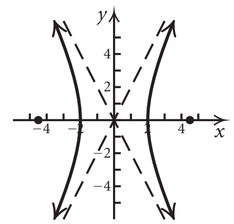
The center is $(0,0)$; $a = 2$ and $b = 4$. The transverse axis is horizontal, so the vertices are $(-2,0)$ and $(2,0)$. Since $c^2 = a^2+b^2$, we have $c^2 = 4+16 = 20$ and $c = \sqrt{20}$, or $2\sqrt{5}$. Then the foci are $(-2\sqrt{5},0)$ and $(2\sqrt{5},0)$.

Find the asymptotes:

$y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$

$y = \dfrac{4}{2}x$ and $y = -\dfrac{4}{2}x$

$y = 2x$ and $y = -2x$



$4x^2 - y^2 = 16$

19. $9y^2 - x^2 = 81$

$\dfrac{y^2}{9} - \dfrac{x^2}{81} = 1$

$\dfrac{y^2}{3^2} - \dfrac{x^2}{9^2} = 1$ Standard form

The center is $(0,0)$; $a = 3$ and $b = 9$. The transverse axis is vertical, so the vertices are $(0,-3)$ and $(0,3)$. Since $c^2 = a^2+b^2$, we have $c^2 = 9+81 = 90$ and $c = \sqrt{90}$, or $3\sqrt{10}$. Then the foci are $(0,-3\sqrt{10})$ and $(0,3\sqrt{10})$.

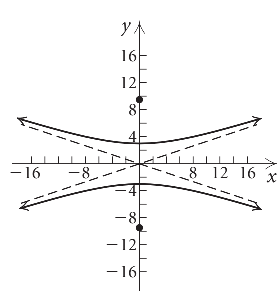
Find the asymptotes:

$y = \dfrac{a}{b}x$ and $y = -\dfrac{a}{b}x$

$y = \dfrac{3}{9}x$ and $y = -\dfrac{3}{9}x$

$y = \dfrac{1}{3}x$ and $y = -\dfrac{1}{3}x$

Sketch the asymptotes, plot the vertices, and draw the graph.



$9y^2 - x^2 = 81$

20. $y^2 - 4x^2 = 4$

$$\frac{y^2}{4} - \frac{x^2}{1} = 1$$

$$\frac{y^2}{2^2} - \frac{x^2}{1^2} = 1 \quad \text{Standard form}$$

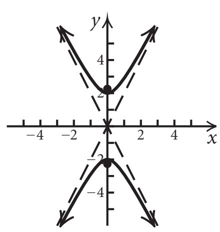
The center is $(0, 0)$; $a = 2$ and $b = 1$. The transverse axis is vertical, so the vertices are $(0, -2)$ and $(0, 2)$. Since $c^2 = a^2 + b^2$, we have $c^2 = 4 + 1 = 5$ and $c = \sqrt{5}$. Then the foci are $(0, -\sqrt{5})$ and $(0, \sqrt{5})$.

Find the asymptotes:

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x$$

$$y = \frac{2}{1}x \quad \text{and} \quad y = -\frac{2}{1}x$$

$$y = 2x \quad \text{and} \quad y = -2x$$



$y^2 - 4x^2 = 4$

21. $x^2 - y^2 = 2$

$$\frac{x^2}{2} - \frac{y^2}{2} = 1$$

$$\frac{x^2}{(\sqrt{2})^2} - \frac{y^2}{(\sqrt{2})^2} = 1 \quad \text{Standard form}$$

The center is $(0, 0)$; $a = \sqrt{2}$ and $b = \sqrt{2}$. The transverse axis is horizontal, so the vertices are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$. Since $c^2 = a^2 + b^2$, we have $c^2 = 2 + 2 = 4$ and $c = 2$. Then the foci are $(-2, 0)$ and $(2, 0)$.

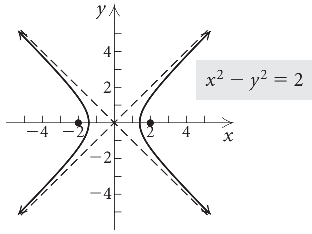
Find the asymptotes:

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

$$y = \frac{\sqrt{2}}{\sqrt{2}}x \quad \text{and} \quad y = -\frac{\sqrt{2}}{\sqrt{2}}x$$

$$y = x \quad \text{and} \quad y = -x$$

Sketch the asymptotes, plot the vertices, and draw the graph.



22. $x^2 - y^2 = 3$

$$\frac{x^2}{3} - \frac{y^2}{3} = 1$$

$$\frac{x^2}{(\sqrt{3})^2} - \frac{y^2}{(\sqrt{3})^2} = 1 \quad \text{Standard form}$$

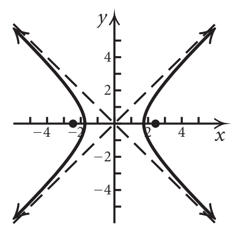
The center is $(0, 0)$; $a = \sqrt{3}$ and $b = \sqrt{3}$. The transverse axis is horizontal, so the vertices are $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$. Since $c^2 = a^2 + b^2$, we have $c^2 = 3 + 3 = 6$ so $c = \sqrt{6}$. Then the foci are $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$.

Find the asymptotes:

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

$$y = \frac{\sqrt{3}}{\sqrt{3}}x \quad \text{and} \quad y = -\frac{\sqrt{3}}{\sqrt{3}}x$$

$$y = x \quad \text{and} \quad y = -x$$



$x^2 - y^2 = 3$

23. $y^2 - x^2 = \frac{1}{4}$

$$\frac{y^2}{1/4} - \frac{x^2}{1/4} = 1$$

$$\frac{y^2}{(1/2)^2} - \frac{x^2}{(1/2)^2} = 1 \quad \text{Standard form}$$

The center is $(0, 0)$; $a = \frac{1}{2}$ and $b = \frac{1}{2}$. The transverse axis is vertical, so the vertices are $\left(0, -\frac{1}{2}\right)$ and $\left(0, \frac{1}{2}\right)$. Since $c^2 = a^2 + b^2$, we have $c^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and $c = \sqrt{\frac{1}{2}}$, or $\frac{\sqrt{2}}{2}$. Then the foci are $\left(0, -\frac{\sqrt{2}}{2}\right)$ and $\left(0, \frac{\sqrt{2}}{2}\right)$.

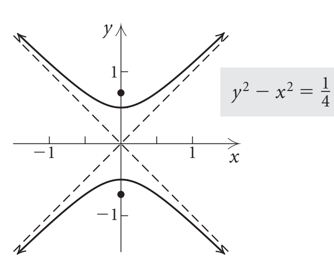
Find the asymptotes:

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x$$

$$y = \frac{1/2}{1/2}x \quad \text{and} \quad y = -\frac{1/2}{1/2}x$$

$$y = x \quad \text{and} \quad y = -x$$

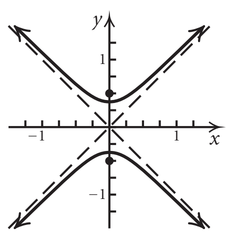
Sketch the asymptotes, plot the vertices, and draw the graph.



24.
$$y^2 - x^2 = \frac{1}{9}$$
$$\frac{y^2}{1/9} - \frac{x^2}{1/9} = 1$$
$$\frac{y^2}{(1/3)^2} - \frac{x^2}{(1/3)^2} = 1 \quad \text{Standard form}$$

The center is $(0, 0)$; $a = \dfrac{1}{3}$ and $b = \dfrac{1}{3}$. The transverse axis is vertical, so the vertices are $\left(0, -\dfrac{1}{3}\right)$ and $\left(0, \dfrac{1}{3}\right)$. Since $c^2 = a^2 + b^2$, we have $c^2 = \dfrac{1}{9} + \dfrac{1}{9} = \dfrac{2}{9}$ and $c = \dfrac{\sqrt{2}}{3}$. Then the foci are $\left(0, -\dfrac{\sqrt{2}}{3}\right)$ and $\left(0, \dfrac{\sqrt{2}}{3}\right)$.

Find the asymptotes:
$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x$$
$$y = \frac{1/3}{1/3}x \quad \text{and} \quad y = -\frac{1/3}{1/3}x$$
$$y = x \quad \text{and} \quad y = -x$$



$y^2 - x^2 = \frac{1}{9}$

25. Begin by completing the square twice.
$$x^2 - y^2 - 2x - 4y - 4 = 0$$
$$(x^2 - 2x) - (y^2 + 4y) = 4$$
$$(x^2 - 2x + 1) - (y^2 + 4y + 4) = 4 + 1 - 1 \cdot 4$$
$$(x-1)^2 - (y+2)^2 = 1$$
$$\frac{(x-1)^2}{1^2} - \frac{[y-(-2)]^2}{1^2} = 1 \quad \text{Standard form}$$

The center is $(1, -2)$; $a = 1$ and $b = 1$. The transverse axis is horizontal, so the vertices are 1 unit left and right of the center:

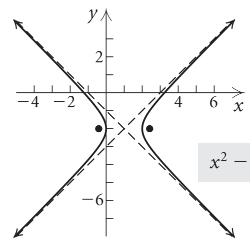
$(1 - 1, -2)$ and $(1 + 1, -2)$ or $(0, -2)$ and $(2, -2)$

Since $c^2 = a^2 + b^2$, we have $c^2 = 1 + 1 = 2$ and $c = \sqrt{2}$. Then the foci are $\sqrt{2}$ units left and right of the center:

$(1 - \sqrt{2}, -2)$ and $(1 + \sqrt{2}, -2)$.

Find the asymptotes:
$$y - k = \frac{b}{a}(x - h) \quad \text{and} \quad y - k = -\frac{b}{a}(x - h)$$
$$y - (-2) = \frac{1}{1}(x - 1) \quad \text{and} \quad y - (-2) = -\frac{1}{1}(x - 1)$$
$$y + 2 = x - 1 \quad \text{and} \quad y + 2 = -(x - 1), \text{ or}$$
$$y = x - 3 \quad \text{and} \quad y = -x - 1$$

Sketch the asymptotes, plot the vertices, and draw the graph.



$x^2 - y^2 - 2x - 4y - 4 = 0$

26. Begin by completing the square twice.
$$4x^2 - y^2 + 8x - 4y - 4 = 0$$
$$4(x^2 + 2x) - (y^2 + 4y) = 4$$
$$4(x^2 + 2x + 1) - (y^2 + 4y + 4) = 4 + 4 \cdot 1 - 1 \cdot 4$$
$$4(x+1)^2 - (y+2)^2 = 4$$
$$\frac{(x+1)^2}{1} - \frac{(y+2)^2}{4} = 1$$
$$\frac{[x-(-1)]^2}{1^2} - \frac{[y-(-2)]^2}{2^2} = 1 \quad \text{Standard form}$$

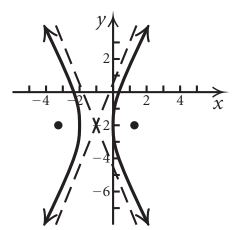
The center is $(-1, -2)$; $a = 1$ and $b = 2$. The transverse axis is horizontal, so the vertices are 1 unit left and right of the center:

$(-1 - 1, -2)$ and $(-1 + 1, -2)$ or $(-2, -2)$ and $(0, -2)$

Since $c^2 = a^2 + b^2$, we have $c^2 = 1 + 4 = 5$ and $c = \sqrt{5}$. Then the foci are $\sqrt{5}$ units left and right of the center:

$(-1 - \sqrt{5}, -2)$ and $(1 + \sqrt{5}, -2)$.

Find the asymptotes:
$$y - k = \frac{b}{a}(x - h) \quad \text{and} \quad y - k = -\frac{b}{a}(x - h)$$
$$y + 2 = \frac{2}{1}(x + 1) \quad \text{and} \quad y + 2 = -\frac{2}{1}(x + 1)$$
$$y + 2 = 2(x + 1) \quad \text{and} \quad y + 2 = -2(x + 1), \text{ or}$$
$$y = 2x \quad \text{and} \quad y = -2x - 4$$



$4x^2 - y^2 + 8x - 4y - 4 = 0$

27. Begin by completing the square twice.

$$36x^2 - y^2 - 24x + 6y - 41 = 0$$
$$(36x^2 - 24x) - (y^2 - 6y) = 41$$
$$36\left(x^2 - \frac{2}{3}x\right) - (y^2 - 6y) = 41$$
$$36\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) - (y^2 - 6y + 9) = 41 + 36 \cdot \frac{1}{9} - 1 \cdot 9$$
$$36\left(x - \frac{1}{3}\right)^2 - (y-3)^2 = 36$$
$$\frac{\left(x - \frac{1}{3}\right)^2}{1} - \frac{(y-3)^2}{36} = 1$$
$$\frac{\left(x - \frac{1}{3}\right)^2}{1^2} - \frac{(y-3)^2}{6^2} = 1 \quad \text{Standard form}$$

The center is $\left(\frac{1}{3}, 3\right)$; $a = 1$ and $b = 6$. The transverse axis is horizontal, so the vertices are 1 unit left and right of the center:

$\left(\frac{1}{3} - 1, 3\right)$ and $\left(\frac{1}{3} + 1, 3\right)$ or $\left(-\frac{2}{3}, 3\right)$ and $\left(\frac{4}{3}, 3\right)$.

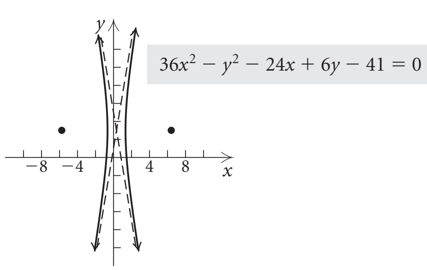
Since $c^2 = a^2 + b^2$, we have $c^2 = 1 + 36 = 37$ and $c = \sqrt{37}$. Then the foci are $\sqrt{37}$ units left and right of the center:

$$\left(\frac{1}{3} - \sqrt{37}, 3\right) \text{ and } \left(\frac{1}{3} + \sqrt{37}, 3\right).$$

Find the asymptotes:

$$y - k = \frac{b}{a}(x - h) \quad \text{and} \quad y - k = -\frac{b}{a}(x - h)$$
$$y - 3 = \frac{6}{1}\left(x - \frac{1}{3}\right) \quad \text{and} \quad y - 3 = -\frac{6}{1}\left(x - \frac{1}{3}\right)$$
$$y - 3 = 6\left(x - \frac{1}{3}\right) \quad \text{and} \quad y - 3 = -6\left(x - \frac{1}{3}\right), \text{ or}$$
$$y = 6x + 1 \quad \text{and} \quad y = -6x + 5$$

Sketch the asymptotes, plot the vertices, and draw the graph.



28. Begin by completing the square twice.

$$9x^2 - 4y^2 + 54x + 8y + 41 = 0$$
$$9(x^2 + 6x) - 4(y^2 - 2y) = -41$$
$$9(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = -41 + 9 \cdot 9 - 4 \cdot 1$$
$$9(x+3)^2 - 4(y-1)^2 = 36$$
$$\frac{(x+3)^2}{4} - \frac{(y-1)^2}{9} = 1$$
$$\frac{[x-(-3)]^2}{2^2} - \frac{(y-1)^2}{3^2} = 1 \quad \text{Standard form}$$

The center is $(-3, 1)$; $a = 2$ and $b = 3$. The transverse axis is horizontal, so the vertices are 2 units left and right of the center:

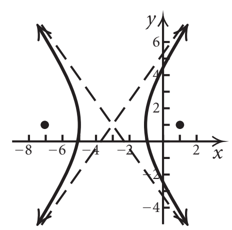
$(-3-2, 1)$ and $(-3+2, 1)$, or $(-5, 1)$ and $(-1, 1)$.

Since $c^2 = a^2 + b^2$, we have $c^2 = 4 + 9 = 13$ and $c = \sqrt{13}$. Then the foci are $\sqrt{13}$ units left and right of the center:

$$(-3 - \sqrt{13}, 1) \text{ and } (-3 + \sqrt{13}, 1).$$

Find the asymptotes:

$$y - k = \frac{b}{a}(x - h) \text{ and } y - k = -\frac{b}{a}(x - h)$$
$$y - 1 = \frac{3}{2}(x + 3) \text{ and } y - 1 = -\frac{3}{2}(x + 3), \text{ or}$$
$$y = \frac{3}{2}x + \frac{11}{2} \quad \text{and} \quad y = -\frac{3}{2}x - \frac{7}{2}$$



$9x^2 - 4y^2 + 54x + 8y + 41 = 0$

29. Begin by completing the square twice.

$$9y^2 - 4x^2 - 18y + 24x - 63 = 0$$
$$9(y^2 - 2y) - 4(x^2 - 6x) = 63$$
$$9(y^2 - 2y + 1) - 4(x^2 - 6x + 9) = 63 + 9 \cdot 1 - 4 \cdot 9$$
$$9(y-1)^2 - 4(x-3)^2 = 36$$
$$\frac{(y-1)^2}{4} - \frac{(x-3)^2}{9} = 1$$
$$\frac{(y-1)^2}{2^2} - \frac{(x-3)^2}{3^2} = 1 \quad \text{Standard form}$$

The center is $(3, 1)$; $a = 2$ and $b = 3$. The transverse axis is vertical, so the vertices are 2 units below and above the center:

$(3, 1-2)$ and $(3, 1+2)$, or $(3, -1)$ and $(3, 3)$.

Since $c^2 = a^2 + b^2$, we have $c^2 = 4 + 9 = 13$ and $c = \sqrt{13}$. Then the foci are $\sqrt{13}$ units below and above the center:

$$(3, 1 - \sqrt{13}) \text{ and } (3, 1 + \sqrt{13}).$$

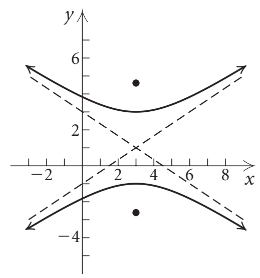
Find the asymptotes:

$$y-k=\frac{a}{b}(x-h) \text{ and } y-k=-\frac{a}{b}(x-h)$$

$$y-1=\frac{2}{3}(x-3) \text{ and } y-1=-\frac{2}{3}(x-3), \text{ or}$$

$$y=\frac{2}{3}x-1 \quad \text{and} \quad y=-\frac{2}{3}x+3$$

Sketch the asymptotes, plot the vertices, and draw the graph.



$9y^2 - 4x^2 - 18y + 24x - 63 = 0$

30. Begin by completing the square twice.

$$x^2-25y^2+6x-50y=41$$
$$x^2+6x+9-25(y^2+2y+1)=41+9-25\cdot 1$$
$$(x+3)^2-25(y+1)^2=25$$
$$\frac{(x+3)^2}{25}-\frac{(y+1)^2}{1}=1$$
$$\frac{[x-(-3)]^2}{5^2}-\frac{[y-(-1)]^2}{1^2}=1$$

The center is $(-3,-1)$; $a=5$ and $b=1$. The transverse axis is horizontal, so the vertices are 5 units left and right of the center:

$(-3-5,-1)$ and $(-3+5,-1)$, or $(-8,-1)$ and $(2,-1)$.

Since $c^2=a^2+b^2$, we have $c^2=25+1=26$ and $c=\sqrt{26}$. Then the foci are $\sqrt{26}$ units left and right of the center:

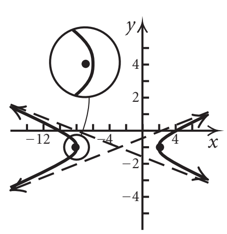
$(-3-\sqrt{26},-1)$ and $(-3+\sqrt{26},-1)$.

Find the asymptotes:

$$y-k=\frac{b}{a}(x-h) \text{ and } y-k=-\frac{b}{a}(x-h)$$

$$y+1=\frac{1}{5}(x+3) \text{ and } y+1=-\frac{1}{5}(x+3), \text{ or}$$

$$y=\frac{1}{5}x-\frac{2}{5} \quad \text{and} \quad y=-\frac{1}{5}x-\frac{8}{5}$$



$x^2 - 25y^2 + 6x - 50y = 41$

31. Begin by completing the square twice.

$$x^2-y^2-2x-4y=4$$
$$(x^2-2x+1)-(y^2+4y+4)=4+1-4$$
$$(x-1)^2-(y+2)^2=1$$
$$\frac{(x-1)^2}{1^2}-\frac{[y-(-2)]^2}{1^2}=1 \quad \text{Standard form}$$

The center is $(1,-2)$; $a=1$ and $b=1$. The transverse axis is horizontal, so the vertices are 1 unit left and right of the center:

$(1-1,-2)$ and $(1+1,-2)$, or $(0,-2)$ and $(2,-2)$.

Since $c^2=a^2+b^2$, we have $c^2=1+1=2$ and $c=\sqrt{2}$. Then the foci are $\sqrt{2}$ units left and right of the center:

$(1-\sqrt{2},-2)$ and $(1+\sqrt{2},-2)$.

Find the asymptotes:

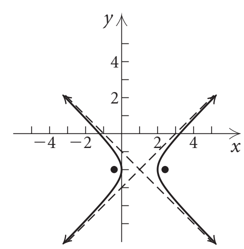
$$y-k=\frac{b}{a}(x-h) \text{ and } \quad y-k=-\frac{b}{a}(x-h)$$

$$y-(-2)=\frac{1}{1}(x-1) \text{ and } y-(-2)=-\frac{1}{1}(x-1)$$

$$y+2=x-1 \quad \text{and} \quad y+2=-(x-1), \text{ or}$$

$$y=x-3 \quad \text{and} \quad y=-x-1$$

Sketch the asymptotes, plot the vertices, and draw the graph.



$x^2 - y^2 - 2x - 4y = 4$

32. Begin by completing the square twice.

$$9y^2-4x^2-54y-8x+41=0$$
$$9(y^2-6y+9)-4(x^2+2x+1)=-41+9\cdot 9-4\cdot 1$$
$$9(y-3)^2-4(x+1)^2=36$$
$$\frac{(y-3)^2}{4}-\frac{(x+1)^2}{9}=1$$
$$\frac{(y-3)^2}{2^2}-\frac{[x-(-1)]^2}{3^2}=1 \quad \text{Standard form}$$

The center is $(-1,3)$; $a=2$ and $b=3$. The transverse axis is vertical, so the vertices are 2 units below and above the center:

$(-1,3-2)$ and $(-1,3+2)$, or $(-1,1)$ and $(-1,5)$.

Since $c^2=a^2+b^2$, we have $c^2=4+9$ and $c=\sqrt{13}$. Then the foci are $\sqrt{13}$ units below and above the center:

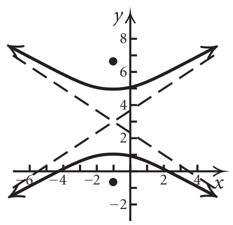
$(-1,3-\sqrt{13})$ and $(-1,3+\sqrt{13})$.

Find the asymptotes:

$$y-k=\frac{a}{b}(x-h) \text{ and } y-k=-\frac{a}{b}(x-h)$$

$$y-3=\frac{2}{3}(x+1) \text{ and } y-3=-\frac{2}{3}(x+1), \text{ or}$$

$$y=\frac{2}{3}x+\frac{11}{3} \quad \text{and} \quad y=-\frac{2}{3}x+\frac{7}{3}$$



$9y^2 - 4x^2 - 54y - 8x + 41 = 0$

33. Begin by completing the square twice.

$$y^2-x^2-6x-8y-29=0$$
$$(y^2-8y+16)-(x^2+6x+9)=29+16-9$$
$$(y-4)^2-(x+3)^2=36$$
$$\frac{(y-4)^2}{36}-\frac{(x+3)^2}{36}=1$$
$$\frac{(y-4)^2}{6^2}-\frac{[x-(-3)]^2}{6^2}=1 \quad \text{Standard form}$$

The center is $(-3,4)$; $a=6$ and $b=6$. The transverse axis is vertical, so the vertices are 6 units below and above the center:

$(-3,4-6)$ and $(-3,4+6)$, or $(-3,-2)$ and $(-3,10)$.

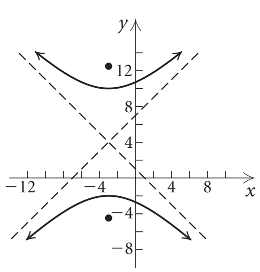
Since $c^2=a^2+b^2$, we have $c^2=36+36=72$ and $c=\sqrt{72}$, or $6\sqrt{2}$. Then the foci are $6\sqrt{2}$ units below and above the center:

$(-3,4-6\sqrt{2})$ and $(-3,4+6\sqrt{2})$.

Find the asymptotes:

$$y-k=\frac{a}{b}(x-h) \quad \text{and} \quad y-k=-\frac{a}{b}(x-h)$$
$$y-4=\frac{6}{6}(x-(-3)) \quad \text{and} \quad y-4=-\frac{6}{6}(x-(-3))$$
$$y-4=x+3 \quad \text{and} \quad y-4=-(x+3), \text{ or}$$
$$y=x+7 \quad \text{and} \quad y=-x+1$$

Sketch the asymptotes, plot the vertices, and draw the graph.



$y^2 - x^2 - 6x - 8y - 29 = 0$

34. Begin by completing the square twice.

$$x^2-y^2=8x-2y-13$$
$$x^2-8x-y^2+2y=-13$$
$$x^2-8x+16-(y^2-2y+1)=-13+16-1$$
$$(x-4)^2-(y-1)^2=2$$
$$\frac{(x-4)^2}{(\sqrt{2})^2}-\frac{(y-1)^2}{(\sqrt{2})^2}=1 \quad \text{Standard form}$$

The center is $(4,1)$; $a=\sqrt{2}$ and $b=\sqrt{2}$. The transverse axis is horizontal, so the vertices are $\sqrt{2}$ units left and right of the center:

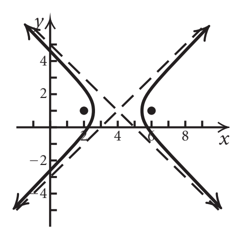
$(4-\sqrt{2},1)$ and $(4+\sqrt{2},1)$.

Since $c^2=a^2+b^2$, we have $c^2=2+2=4$ and $c=2$. Then the foci are 2 units left and right of the center:

$(4-2,1)$ and $(4+2,1)$, or $(2,1)$ and $(6,1)$.

Find the asymptotes:

$$y-k=\frac{b}{a}(x-h) \quad \text{and } y-k=-\frac{b}{a}(x-h)$$
$$y-1=\frac{\sqrt{2}}{\sqrt{2}}(x-4) \text{ and } y-1=-\frac{\sqrt{2}}{\sqrt{2}}(x-4)$$
$$y-1=x-4 \quad \text{and } y-1=-(x-4), \text{ or}$$
$$y=x-3 \quad \text{and} \quad y=-x+5$$



$x^2 - y^2 = 8x - 2y - 13$

35. The hyperbola in Example 3 is wider than the one in Example 2, so the hyperbola in Example 3 has the larger eccentricity.

Compute the eccentricities: In Example 2, $c=5$ and $a=4$, so $e=5/4$, or 1.25. In Example 3, $c=\sqrt{5}$ and $a=1$, so $e=\sqrt{5}/1\approx 2.24$. These computations confirm that the hyperbola in Example 3 has the larger eccentricity.

36. Hyperbola (b) is wider so it has the larger eccentricity.

37. The center is the midpoint of the segment connecting the vertices:

$$\left(\frac{3-3}{2},\frac{7+7}{2}\right), \text{ or } (0,7).$$

The vertices are on the horizontal line $y=7$, so the transverse axis is horizontal. Since the vertices are 3 units left and right of the center, $a=3$.

Find c:

$$e=\frac{c}{a}=\frac{5}{3}$$
$$\frac{c}{3}=\frac{5}{3} \quad \text{Substituting 3 for } a$$
$$c=5$$

Now find b^2:

$$c^2 = a^2 + b^2$$
$$5^2 = 3^2 + b^2$$
$$16 = b^2$$

Write the equation:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
$$\frac{x^2}{9} - \frac{(y-7)^2}{16} = 1$$

38. The center is the midpoint of the segment connecting the vertices:

$$\left(\frac{-1-1}{2}, \frac{3+7}{2}\right), \text{ or } (-1, 5).$$

The vertices are on the vertical line $x = -1$, so the transverse axis is vertical. Since the vertices are 2 units below and above the center, $a = 2$.

Find c:

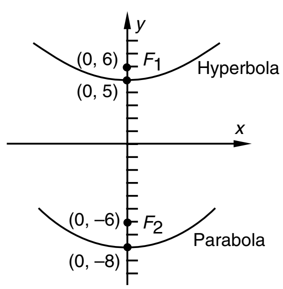
$$e = \frac{c}{a} = 4$$
$$\frac{c}{2} = 4$$
$$c = 8$$

Now find b^2:

$$c^2 = a^2 + b^2$$
$$64 = 4 + b^2$$
$$60 = b^2$$

The equation is $\dfrac{(y-5)^2}{4} - \dfrac{(x+1)^2}{60} = 1$.

39.



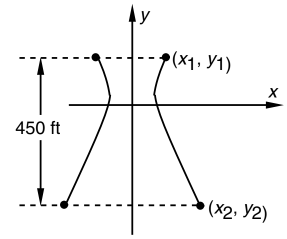
One focus is 6 units above the center of the hyperbola, so $c = 6$. One vertex is 5 units above the center, so $a = 5$. Find b^2:

$$c^2 = a^2 + b^2$$
$$6^2 = 5^2 + b^2$$
$$11 = b^2$$

Write the equation:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$
$$\frac{y^2}{25} - \frac{x^2}{11} = 1$$

40.



$y_1 = \dfrac{1}{2}y_2$, so $y_2 + \dfrac{1}{2}y_2 = 450$. Then $y_2 = 300$ and $y_1 = \dfrac{1}{2} \cdot 300 = 150$.

Find x_1:

$$\frac{x_1^2}{90^2} - \frac{150^2}{130^2} = 1$$
$$x_1^2 = 90^2\left(1 + \frac{150^2}{130^2}\right)$$
$$x_1 \approx 137.4$$

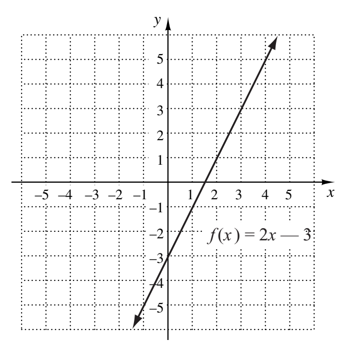
Then the diameter of the top of the tower is $2x_1 \approx 2(137.4) \approx 275$ ft.

Find x_2:

$$\frac{x_2^2}{90^2} - \frac{300^2}{130^2} = 1$$
$$x_2^2 = 90^2\left(1 + \frac{300^2}{130^2}\right)$$
$$x_2 \approx 226.4$$

Then the diameter of the bottom of the tower is $2x_2 \approx 2(226.4) \approx 453$ ft.

41. a) The graph of $f(x) = 2x - 3$ is shown below.



Since there is no horizontal line that crosses the graph more than once, the function is one-to-one.

b) Replace $f(x)$ with y: $y = 2x - 3$

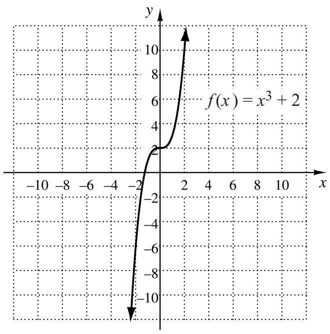
Interchange x and y: $x = 2y - 3$

Solve for y: $x + 3 = 2y$

$$\frac{x+3}{2} = y$$

Replace y with $f^{-1}(x)$: $f^{-1}(x) = \dfrac{x+3}{2}$

42. a) The graph of $f(x) = x^3 + 2$ is shown below. It passes the horizontal-line test, so it is one-to-one.



b) Replace $f(x)$ with y: $y = x^3 + 2$

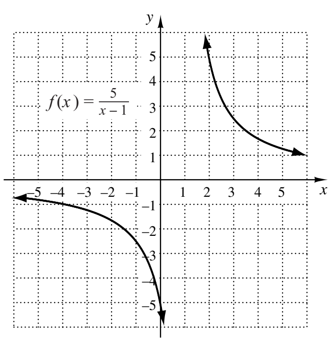
Interchange x and y: $x = y^3 + 2$

Solve for y: $x - 2 = y^3$

$$\sqrt[3]{x-2} = y$$

Replace y with $f^{-1}(x)$: $f^{-1}(x) = \sqrt[3]{x-2}$

43. a) The graph of $f(x) = \dfrac{5}{x-1}$ is shown below.



Since there is no horizontal line that crosses the graph more than once, the function is one-to-one.

b) Replace $f(x)$ with y: $y = \dfrac{5}{x-1}$

Interchange x and y: $x = \dfrac{5}{y-1}$

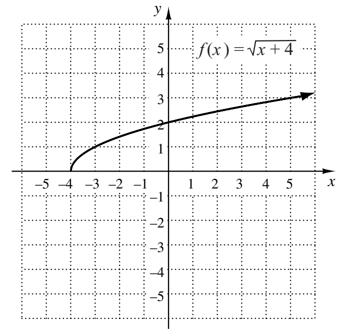
Solve for y: $x(y-1) = 5$

$$y - 1 = \frac{5}{x}$$

$$y = \frac{5}{x} + 1$$

Replace y with $f^{-1}(x)$: $f^{-1} = \dfrac{5}{x} + 1$, or $\dfrac{5+x}{x}$

44. a) The graph of $f(x) = \sqrt{x+4}$ is shown below. It passes the horizontal-line test, so it is one-to-one.



b) Replace $f(x)$ with y: $y = \sqrt{x+4}$

Interchange x and y: $x = \sqrt{y+4}$

Solve for y: $x^2 = y + 4$

$$x^2 - 4 = y$$

Replace y with $f^{-1}(x)$: $f^{-1}(x) = x^2 - 4$, $x \geq 0$

45.
$$\begin{array}{rl} x + y = 5, & (1) \\ x - y = 7 & (2) \\ \hline 2x \quad = 12 & \text{Adding} \\ x = 6 & \end{array}$$

Back-substitute in either equation (1) or (2) and solve for y. We use equation (1).

$$6 + y = 5$$
$$y = -1$$

The solution is $(6, -1)$.

46.
$$\begin{array}{rl} 3x - 2y = 5, & (1) \\ 5x + 2y = 3 & (2) \\ \hline 8x \quad = 8 & \text{Adding} \\ x = 1 & \end{array}$$

Back-substitute and solve for y.

$$5 \cdot 1 + 2y = 3 \quad \text{Using equation (2)}$$
$$2y = -2$$
$$y = -1$$

The solution is $(1, -1)$.

47.
$$\begin{array}{rl} 2x - 3y = 7, & (1) \\ 3x + 5y = 1 & (2) \end{array}$$

Multiply equation (1) by 5 and equation (2) by 3 and add to eliminate y.

$$\begin{array}{r} 10x - 15y = 35 \\ 9x + 15y = 3 \\ \hline 19x \quad = 38 \\ x = 2 \end{array}$$

Back-substitute and solve for y.

$$3 \cdot 2 + 5y = 1 \quad \text{Using equation (2)}$$
$$5y = -5$$
$$y = -1$$

The solution is $(2, -1)$.

48. $3x + 2y = -1$ (1)

$2x + 3y = 6$ (2)

Multiply equation (1) by 3 and equation (2) by -2 and add.

$$\begin{array}{rl} 9x + 6y &= -3 \\ -4x - 6y &= -12 \\ \hline 5x \qquad &= -15 \\ x &= -3 \end{array}$$

Back-substitute and solve for y.

$$\begin{aligned} 3(-3) + 2y &= -1 \quad \text{Using equation (1)} \\ 2y &= 8 \\ y &= 4 \end{aligned}$$

The solution is $(-3, 4)$.

49. The center is the midpoint of the segment connecting $(3, -8)$ and $(3, -2)$:

$$\left(\frac{3+3}{2}, \frac{-8-2}{2}\right), \text{ or } (3, -5).$$

The vertices are on the vertical line $x = 3$ and are 3 units above and below the center so the transverse axis is vertical and $a = 3$. Use the equation of an asymptote to find b:

$$\begin{aligned} y - k &= \frac{a}{b}(x - h) \\ y + 5 &= \frac{3}{b}(x - 3) \\ y &= \frac{3}{b}x - \frac{9}{b} - 5 \end{aligned}$$

This equation corresponds to the asymptote $y = 3x - 14$, so $\frac{3}{b} = 3$ and $b = 1$.

Write the equation of the hyperbola:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y+5)^2}{9} - \frac{(x-3)^2}{1} = 1$$

50. The center is the midpoint of the segment connecting the vertices:

$$\left(\frac{-9-5}{2}, \frac{4+4}{2}\right), \text{ or } (-7, 4).$$

The vertices are on the horizontal line $y = 4$ and are 2 units left and right of the center, so the transverse axis is horizontal and $a = 2$. Use the equation of an asymptote to find b:

$$\begin{aligned} y - k &= \frac{b}{a}(x - h) \\ y - 4 &= \frac{b}{2}(x + 7) \\ y &= \frac{b}{2}x + \frac{7}{2}b + 4 \end{aligned}$$

This equation corresponds to the asymptote $y = 3x + 25$, so $\frac{b}{2} = 3$ and $b = 6$.

Write the equation of the hyperbola:

$$\frac{(x+7)^2}{4} - \frac{(y-4)^2}{36} = 1$$

51. S and T are the foci of the hyperbola, so $c = 300/2 = 150$.

200 microseconds $\cdot \dfrac{0.186 \text{ mi}}{1 \text{ microsecond}} = 37.2$ mi, the difference of the ships' distances from the foci. That is, $2a = 37.2$, so $a = 18.6$.

Find b^2:

$$\begin{aligned} c^2 &= a^2 + b^2 \\ 150^2 &= 18.6^2 + b^2 \\ 22{,}154.04 &= b^2 \end{aligned}$$

Then the equation of the hyperbola is

$$\frac{x^2}{18.6^2} - \frac{y^2}{22{,}154.04} = 1, \text{ or } \frac{x^2}{345.96} - \frac{y^2}{22{,}154.04} = 1.$$

Exercise Set 10.4

1. The correct graph is (e).

2. The correct graph is (a).

3. The correct graph is (c).

4. The correct graph is (f).

5. The correct graph is (b).

6. The correct graph is (d).

7. $x^2 + y^2 = 25,$ (1)

$y - x = 1$ (2)

First solve equation (2) for y.

$y = x + 1$ (3)

Then substitute $x + 1$ for y in equation (1) and solve for x.

$$\begin{aligned} x^2 + y^2 &= 25 \\ x^2 + (x+1)^2 &= 25 \\ x^2 + x^2 + 2x + 1 &= 25 \\ 2x^2 + 2x - 24 &= 0 \\ x^2 + x - 12 &= 0 \quad \text{Multiplying by } \frac{1}{2} \\ (x+4)(x-3) &= 0 \quad \text{Factoring} \end{aligned}$$

$x + 4 = 0$ or $x - 3 = 0$ Principle of zero products

$x = -4$ or $x = 3$

Now substitute these numbers into equation (3) and solve for y.

$y = -4 + 1 = -3$

$y = 3 + 1 = 4$

The pairs $(-4, -3)$ and $(3, 4)$ check, so they are the solutions.

8. $x^2 + y^2 = 100,$
$y - x = 2$

$y = x + 2$

$$\begin{aligned} x^2 + (x+2)^2 &= 100 \\ x^2 + x^2 + 4x + 4 &= 100 \\ 2x^2 + 4x - 96 &= 0 \\ x^2 + 2x - 48 &= 0 \\ (x+8)(x-6) &= 0 \end{aligned}$$

$x = -8$ or $x = 6$

$y = -8 + 2 = -6$

$y = 6 + 2 = 8$

The pairs $(-8, -6)$ and $(6, 8)$ check.

9. $4x^2 + 9y^2 = 36, \quad (1)$
$3y + 2x = 6 \quad (2)$

First solve equation (2) for y.

$3y = -2x + 6$

$y = -\frac{2}{3}x + 2 \qquad (3)$

Then substitute $-\frac{2}{3}x + 2$ for y in equation (1) and solve for x.

$$\begin{aligned} 4x^2 + 9y^2 &= 36 \\ 4x^2 + 9\Big(-\frac{2}{3}x + 2\Big)^2 &= 36 \\ 4x^2 + 9\Big(\frac{4}{9}x^2 - \frac{8}{3}x + 4\Big) &= 36 \\ 4x^2 + 4x^2 - 24x + 36 &= 36 \\ 8x^2 - 24x &= 0 \\ x^2 - 3x &= 0 \\ x(x-3) &= 0 \end{aligned}$$

$x = 0$ or $x = 3$

Now substitute these numbers in equation (3) and solve for y.

$y = -\frac{2}{3} \cdot 0 + 2 = 2$

$y = -\frac{2}{3} \cdot 3 + 2 = 0$

The pairs $(0, 2)$ and $(3, 0)$ check, so they are the solutions.

10. $9x^2 + 4y^2 = 36,$
$3x + 2y = 6$

$$\begin{aligned} x &= 2 - \frac{2}{3}y \\ 9\Big(2 - \frac{2}{3}y\Big)^2 + 4y^2 &= 36 \\ 9\Big(4 - \frac{8}{3}y + \frac{4}{9}y^2\Big) + 4y^2 &= 36 \\ 36 - 24y + 4y^2 + 4y^2 &= 36 \\ 8y^2 - 24y &= 0 \\ y^2 - 3y &= 0 \\ y(y-3) &= 0 \end{aligned}$$

$y = 0$ or $y = 3$

$x = 2 - \frac{2}{3}(0) = 2$

$x = 2 - \frac{2}{3}(3) = 0$

The pairs $(2, 0)$ and $(0, 3)$ check.

11. $x^2 + y^2 = 25, \quad (1)$
$y^2 = x + 5 \quad (2)$

We substitute $x + 5$ for y^2 in equation (1) and solve for x.

$$\begin{aligned} x^2 + y^2 &= 25 \\ x^2 + (x+5) &= 25 \\ x^2 + x - 20 &= 0 \\ (x+5)(x-4) &= 0 \end{aligned}$$

$x + 5 = 0 \quad$ or $\quad x - 4 = 0$

$x = -5 \quad$ or $\quad x = 4$

We substitute these numbers for x in either equation (1) or equation (2) and solve for y. Here we use equation (2).

$y^2 = -5 + 5 = 0$ and $y = 0$.

$y^2 = 4 + 5 = 9$ and $y = \pm 3$.

The pairs $(-5, 0)$, $(4, 3)$ and $(4, -3)$ check. They are the solutions.

12. $y = x^2,$
$x = y^2$

$x = (x^2)^2$

$x = x^4$

$0 = x^4 - x$

$0 = x(x^3 - 1)$

$0 = x(x-1)(x^2 + x + 1)$

$x = 0 \;$ or $\; x = 1 \;$ or $\; x = \dfrac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2}$

$x = 0 \;$ or $\; x = 1 \;$ or $\; x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$y = 0^2 = 0$

$y = 1^2 = 1$

$y = \Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\Big)^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$y = \Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\Big)^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

The pairs $(0, 0)$, $(1, 1)$, $\Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\Big)$, and $\Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i\Big)$ check.

13. $x^2 + y^2 = 9,\quad (1)$
$x^2 - y^2 = 9\quad (2)$

Here we use the elimination method.

$$\begin{array}{rll} x^2 + y^2 = & 9 & (1) \\ x^2 - y^2 = & 9 & (2) \\ \hline 2x^2 \quad = & 18 & \text{Adding} \\ x^2 = & 9 & \\ x = & \pm 3 & \end{array}$$

If $x = 3,\ x^2 = 9$, and if $x = -3,\ x^2 = 9$, so substituting 3 or -3 in equation (1) gives us

$$x^2 + y^2 = 9$$
$$9 + y^2 = 9$$
$$y^2 = 0$$
$$y = 0.$$

The pairs $(3, 0)$ and $(-3, 0)$ check. They are the solutions.

14. $y^2 - 4x^2 = 4\quad (1)$
$4x^2 + y^2 = 4\quad (2)$

$$\begin{array}{rll} -4x^2 + \ y^2 = 4 & & (1) \\ 4x^2 + \ y^2 = 4 & & (2) \\ \hline 2y^2 = 8 & & \text{Adding} \\ y^2 = 4 & & \\ y = \pm 2 & & \end{array}$$

Substitute for y in equation (2).

$$4x^2 + 4 = 4$$
$$4x^2 = 0$$
$$x = 0$$

The pairs $(0, 2)$ and $(0, -2)$ check.

15. $y^2 - x^2 = 9\quad (1)$
$2x - 3 = y\quad (2)$

Substitute $2x - 3$ for y in equation (1) and solve for x.

$$y^2 - x^2 = 9$$
$$(2x - 3)^2 - x^2 = 9$$
$$4x^2 - 12x + 9 - x^2 = 9$$
$$3x^2 - 12x = 0$$
$$x^2 - 4x = 0$$
$$x(x - 4) = 0$$

$x = 0$ or $x = 4$

Now substitute these numbers into equation (2) and solve for y.

If $x = 0$, $y = 2 \cdot 0 - 3 = -3$.

If $x = 4$, $y = 2 \cdot 4 - 3 = 5$.

The pairs $(0, -3)$ and $(4, 5)$ check. They are the solutions.

16. $x + y = -6,$
$xy = -7$

$$y = -x - 6$$
$$x(-x - 6) = -7$$
$$-x^2 - 6x = -7$$
$$0 = x^2 + 6x - 7$$
$$0 = (x + 7)(x - 1)$$

$x = -7$ or $x = 1$

$y = -(-7) - 6 = 1$

$y = -1 - 6 = -7$

The pairs $(-7, 1)$ and $(1, -7)$ check.

17. $y^2 = x + 3,\quad (1)$
$2y = x + 4\quad (2)$

First solve equation (2) for x.

$2y - 4 = x\quad (3)$

Then substitute $2y - 4$ for x in equation (1) and solve for y.

$$y^2 = x + 3$$
$$y^2 = (2y - 4) + 3$$
$$y^2 = 2y - 1$$
$$y^2 - 2y + 1 = 0$$
$$(y - 1)(y - 1) = 0$$

$y - 1 = 0$ or $y - 1 = 0$
$y = 1$ or $y = 1$

Now substitute 1 for y in equation (3) and solve for x.

$$2 \cdot 1 - 4 = x$$
$$-2 = x$$

The pair $(-2, 1)$ checks. It is the solution.

18. $y = x^2,$
$3x = y + 2$

$y = 3x - 2$

$$3x - 2 = x^2$$
$$0 = x^2 - 3x + 2$$
$$0 = (x - 2)(x - 1)$$

$x = 2$ or $x = 1$

$y = 3 \cdot 2 - 2 = 4$

$y = 3 \cdot 1 - 2 = 1$

The pairs $(2, 4)$ and $(1, 1)$ check.

19. $x^2+y^2=25,\quad (1)$
$xy=12\quad (2)$

First we solve equation (2) for y.

$$xy=12$$
$$y=\frac{12}{x}$$

Then we substitute $\frac{12}{x}$ for y in equation (1) and solve for x.

$$\begin{aligned} x^2+y^2&=25\\ x^2+\left(\frac{12}{x}\right)^2&=25\\ x^2+\frac{144}{x^2}&=25\\ x^4+144&=25x^2\quad\text{Multiplying by } x^2\\ x^4-25x^2+144&=0\\ u^2-25u+144&=0\quad\text{Letting } u=x^2\\ (u-9)(u-16)&=0 \end{aligned}$$

$u=9$ or $u=16$

We now substitute x^2 for u and solve for x.

$x^2=9$ or $x^2=16$

$x=\pm 3$ or $x=\pm 4$

Since $y=12/x$, if $x=3$, $y=4$; if $x=-3$, $y=-4$; if $x=4$, $y=3$; and if $x=-4$, $y=-3$. The pairs $(3,4)$, $(-3,-4)$, $(4,3)$, and $(-4,-3)$ check. They are the solutions.

20.
$$\begin{aligned} x^2-y^2&=16,\quad (1)\\ x+y^2&=4\quad (2)\\ \hline x^2+x&=20\quad\text{Adding} \end{aligned}$$

$$x^2+x-20=0$$
$$(x+5)(x-4)=0$$

$x=-5$ or $x=4$

$y^2=4-x$ Solving equation (2) for y^2

$y^2=4-(-5)=9$ and $y=\pm 3$

$y^2=4-4=0$ and $y=0$

The pairs $(-5,3)$, $(-5,-3)$, and $(4,0)$ check.

21. $x^2+y^2=4,\quad (1)$
$16x^2+9y^2=144\quad (2)$

$$\begin{aligned} -9x^2-9y^2&=-36\quad\text{Multiplying (1) by } -9\\ 16x^2+9y^2&=144\\ \hline 7x^2&=108\quad\text{Adding} \end{aligned}$$

$$x^2=\frac{108}{7}$$
$$x=\pm\sqrt{\frac{108}{7}}=\pm 6\sqrt{\frac{3}{7}}$$
$$x=\pm\frac{6\sqrt{21}}{7}\quad\text{Rationalizing the denominator}$$

Substituting $\frac{6\sqrt{21}}{7}$ or $-\frac{6\sqrt{21}}{7}$ for x in equation (1) gives us

$$\begin{aligned} \frac{36\cdot 21}{49}+y^2&=4\\ y^2&=4-\frac{108}{7}\\ y^2&=-\frac{80}{7}\\ y&=\pm\sqrt{-\frac{80}{7}}=\pm 4i\sqrt{\frac{5}{7}}\\ y&=\pm\frac{4i\sqrt{35}}{7}.\quad\text{Rationalizing the denominator} \end{aligned}$$

The pairs $\left(\frac{6\sqrt{21}}{7},\frac{4i\sqrt{35}}{7}\right)$, $\left(\frac{6\sqrt{21}}{7},-\frac{4i\sqrt{35}}{7}\right)$, $\left(-\frac{6\sqrt{21}}{7},\frac{4i\sqrt{35}}{7}\right)$, and $\left(-\frac{6\sqrt{21}}{7},-\frac{4i\sqrt{35}}{7}\right)$ check. They are the solutions.

22. $x^2+y^2=25,\quad (1)$
$25x^2+16y^2=400\quad (2)$

$$\begin{aligned} -16x^2-16y^2&=-400\quad\text{Multiplying (1) by } -16\\ 25x^2+16y^2&=400\\ \hline 9x^2&=0\quad\text{Adding}\\ x&=0 \end{aligned}$$

$0^2+y^2=25$ Substituting in (1)

$y=\pm 5$

The pairs $(0,5)$ and $(0,-5)$ check.

23. $x^2+4y^2=25,\quad (1)$
$x+2y=7\quad (2)$

First solve equation (2) for x.

$x=-2y+7\quad (3)$

Then substitute $-2y+7$ for x in equation (1) and solve for y.

$$\begin{aligned} x^2+4y^2&=25\\ (-2y+7)^2+4y^2&=25\\ 4y^2-28y+49+4y^2&=25\\ 8y^2-28y+24&=0\\ 2y^2-7y+6&=0\\ (2y-3)(y-2)&=0 \end{aligned}$$

$y=\frac{3}{2}$ or $y=2$

Now substitute these numbers in equation (3) and solve for x.

$x=-2\cdot\frac{3}{2}+7=4$

$x=-2\cdot 2+7=3$

The pairs $\left(4,\frac{3}{2}\right)$ and $(3,2)$ check, so they are the solutions.

24. $y^2 - x^2 = 16,$

$2x - y = 1$

$y = 2x - 1$

$$(2x-1)^2 - x^2 = 16$$
$$4x^2 - 4x + 1 - x^2 = 16$$
$$3x^2 - 4x - 15 = 0$$
$$(3x+5)(x-3) = 0$$
$$x = -\frac{5}{3} \text{ or } x = 3$$
$$y = 2\left(-\frac{5}{3}\right) - 1 = -\frac{13}{3}$$
$$y = 2(3) - 1 = 5$$

The pairs $\left(-\frac{5}{3}, -\frac{13}{3}\right)$ and $(3,5)$ check.

25. $x^2 - xy + 3y^2 = 27,$ (1)

$x - y = 2$ (2)

First solve equation (2) for y.

$x - 2 = y$ (3)

Then substitute $x - 2$ for y in equation (1) and solve for x.

$$x^2 - xy + 3y^2 = 27$$
$$x^2 - x(x-2) + 3(x-2)^2 = 27$$
$$x^2 - x^2 + 2x + 3x^2 - 12x + 12 = 27$$
$$3x^2 - 10x - 15 = 0$$
$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(-15)}}{2 \cdot 3}$$
$$x = \frac{10 \pm \sqrt{100+180}}{6} = \frac{10 \pm \sqrt{280}}{6}$$
$$x = \frac{10 \pm 2\sqrt{70}}{6} = \frac{5 \pm \sqrt{70}}{3}$$

Now substitute these numbers in equation (3) and solve for y.

$$y = \frac{5+\sqrt{70}}{3} - 2 = \frac{-1+\sqrt{70}}{3}$$
$$y = \frac{5-\sqrt{70}}{3} - 2 = \frac{-1-\sqrt{70}}{3}$$

The pairs $\left(\frac{5+\sqrt{70}}{3}, \frac{-1+\sqrt{70}}{3}\right)$ and $\left(\frac{5-\sqrt{70}}{3}, \frac{-1-\sqrt{70}}{3}\right)$ check, so they are the solutions.

26. $2y^2 + xy + x^2 = 7,$

$x - 2y = 5$

$x = 2y + 5$

$$2y^2 + (2y+5)y + (2y+5)^2 = 7$$
$$2y^2 + 2y^2 + 5y + 4y^2 + 20y + 25 = 7$$
$$8y^2 + 25y + 18 = 0$$
$$(8y+9)(y+2) = 0$$
$$y = -\frac{9}{8} \text{ or } y = -2$$
$$x = 2\left(-\frac{9}{8}\right) + 5 = \frac{11}{4}$$
$$x = 2(-2) + 5 = 1$$

The pairs $\left(\frac{11}{4}, -\frac{9}{8}\right)$ and $(1,-2)$ check.

27. $x^2 + y^2 = 16,$ or $x^2 + y^2 = 16,$ (1)

$y^2 - 2x^2 = 10$ $\quad$ $-2x^2 + y^2 = 10$ (2)

Here we use the elimination method.

$$\begin{array}{rl} 2x^2 + 2y^2 = 32 & \text{Multiplying (1) by 2} \\ -2x^2 + y^2 = 10 & \\ \hline 3y^2 = 42 & \text{Adding} \\ y^2 = 14 & \\ y = \pm\sqrt{14} & \end{array}$$

Substituting $\sqrt{14}$ or $-\sqrt{14}$ for y in equation (1) gives us

$$x^2 + 14 = 16$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

The pairs $(-\sqrt{2}, -\sqrt{14})$, $(-\sqrt{2}, \sqrt{14})$, $(\sqrt{2}, -\sqrt{14})$, and $(\sqrt{2}, \sqrt{14})$ check. They are the solutions.

28.
$$\begin{array}{rll} x^2 + y^2 = & 14, & (1) \\ x^2 - y^2 = & 4 & (2) \\ \hline 2x^2 \quad = & 18 & \text{Adding} \\ x^2 = & 9 & \\ x = & \pm 3 & \end{array}$$

$9 + y^2 = 14$ Substituting in equation (1)

$$y^2 = 5$$
$$y = \pm\sqrt{5}$$

The pairs $(-3, -\sqrt{5})$, $(-3, \sqrt{5})$, $(3, -\sqrt{5})$, and $(3, \sqrt{5})$ check.

29. $x^2 + y^2 = 5,$ (1)

$xy = 2$ (2)

First we solve equation (2) for y.

$$xy = 2$$
$$y = \frac{2}{x}$$

Then we substitute $\frac{2}{x}$ for y in equation (1) and solve for x.

$$x^2 + y^2 = 5$$
$$x^2 + \left(\frac{2}{x}\right)^2 = 5$$
$$x^2 + \frac{4}{x^2} = 5$$
$$x^4 + 4 = 5x^2 \quad \text{Multiplying by } x^2$$
$$x^4 - 5x^2 + 4 = 0$$
$$u^2 - 5u + 4 = 0 \quad \text{Letting } u = x^2$$
$$(u-4)(u-1) = 0$$
$$u = 4 \text{ or } u = 1$$

We now substitute x^2 for u and solve for x.

$x^2 = 4 \quad \text{or} \quad x^2 = 1$

$x = \pm 2 \qquad x = \pm 1$

Since $y = 2/x$, if $x = 2$, $y = 1$; if $x = -2$, $y = -1$; if $x = 1$, $y = 2$; and if $x = -1$, $y = -2$. The pairs $(2, 1)$, $(-2, -1)$, $(1, 2)$, and $(-1, -2)$ check. They are the solutions.

30. $x^2 + y^2 = 20$,

$xy = 8$

$$y = \frac{8}{x}$$

$$x^2 + \left(\frac{8}{x}\right)^2 = 20$$

$$x^2 + \frac{64}{x^2} = 20$$

$$x^4 + 64 = 20x^2$$

$$x^4 - 20x^2 + 64 = 0$$

$$u^2 - 20u + 64 = 0 \qquad \text{Letting } u = x^2$$

$$(u - 16)(u - 4) = 0$$

$u = 16 \quad \text{or} \quad u = 4$

$x^2 = 16 \quad \text{or} \quad x^2 = 4$

$x = \pm 4 \quad \text{or} \quad x = \pm 2$

$y = 8/x$, so if $x = 4$, $y = 2$; if $x = -4$, $y = -2$; if $x = 2$, $y = 4$; if $x = -2$, $y = -4$. The pairs $(4, 2)$, $(-4, -2)$, $(2, 4)$, and $(-2, -4)$ check.

31. $3x + y = 7 \quad (1)$

$4x^2 + 5y = 56 \quad (2)$

First solve equation (1) for y.

$3x + y = 7$

$y = 7 - 3x \quad (3)$

Next substitute $7 - 3x$ for y in equation (2) and solve for x.

$$4x^2 + 5y = 56$$

$$4x^2 + 5(7 - 3x) = 56$$

$$4x^2 + 35 - 15x = 56$$

$$4x^2 - 15x - 21 = 0$$

Using the quadratic formula, we find that

$x = \dfrac{15 - \sqrt{561}}{8}$ or $x = \dfrac{15 + \sqrt{561}}{8}$.

Now substitute these numbers into equation (3) and solve for y.

If $x = \dfrac{15 - \sqrt{561}}{8}$, $y = 7 - 3\left(\dfrac{15 - \sqrt{561}}{8}\right)$, or $\dfrac{11 + 3\sqrt{561}}{8}$.

If $x = \dfrac{15 + \sqrt{561}}{8}$, $y = 7 - 3\left(\dfrac{15 + \sqrt{561}}{8}\right)$, or $\dfrac{11 - 3\sqrt{561}}{8}$.

The pairs $\left(\dfrac{15 - \sqrt{561}}{8}, \dfrac{11 + 3\sqrt{561}}{8}\right)$ and $\left(\dfrac{15 + \sqrt{561}}{8}, \dfrac{11 - 3\sqrt{561}}{8}\right)$ check and are the solutions.

32. $2y^2 + xy = 5$,

$4y + x = 7$

$x = -4y + 7$

$$2y^2 + (-4y + 7)y = 5$$

$$2y^2 - 4y^2 + 7y = 5$$

$$0 = 2y^2 - 7y + 5$$

$$0 = (2y - 5)(y - 1)$$

$y = \dfrac{5}{2}$ or $y = 1$

$x = -4\left(\dfrac{5}{2}\right) + 7 = -3$

$x = -4(1) + 7 = 3$

The pairs $\left(-3, \dfrac{5}{2}\right)$ and $(3, 1)$ check.

33. $a + b = 7, \quad (1)$

$ab = 4 \qquad (2)$

First solve equation (1) for a.

$a = -b + 7 \quad (3)$

Then substitute $-b + 7$ for a in equation (2) and solve for b.

$$(-b + 7)b = 4$$

$$-b^2 + 7b = 4$$

$$0 = b^2 - 7b + 4$$

$$b = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1}$$

$$b = \frac{7 \pm \sqrt{33}}{2}$$

Now substitute these numbers in equation (3) and solve for a.

$$a = -\left(\frac{7 + \sqrt{33}}{2}\right) + 7 = \frac{7 - \sqrt{33}}{2}$$

$$a = -\left(\frac{7 - \sqrt{33}}{2}\right) + 7 = \frac{7 + \sqrt{33}}{2}$$

The pairs $\left(\dfrac{7 - \sqrt{33}}{2}, \dfrac{7 + \sqrt{33}}{2}\right)$ and $\left(\dfrac{7 + \sqrt{33}}{2}, \dfrac{7 - \sqrt{33}}{2}\right)$ check, so they are the solutions.

34. $p + q = -4$,

$pq = -5$

$p = -q - 4$

$$(-q - 4)q = -5$$

$$-q^2 - 4q = -5$$

$$0 = q^2 + 4q - 5$$

$$0 = (q + 5)(q - 1)$$

$q = -5$ or $q = 1$

$p = -(-5) - 4 = 1$

$p = -1 - 4 = -5$

The pairs $(1, -5)$ and $(-5, 1)$ check.

35. $x^2+y^2=13,\quad (1)$

$xy=6 \quad (2)$

First we solve Equation (2) for y.

$$xy=6$$
$$y=\frac{6}{x}$$

Then we substitute $\frac{6}{x}$ for y in equation (1) and solve for x.

$$x^2+y^2=13$$
$$x^2+\left(\frac{6}{x}\right)^2=13$$
$$x^2+\frac{36}{x^2}=13$$
$$x^4+36=13x^2 \quad \text{Multiplying by } x^2$$
$$x^4-13x^2+36=0$$
$$u^2-13u+36=0 \quad \text{Letting } u=x^2$$
$$(u-9)(u-4)=0$$
$$u=9 \quad \text{or} \quad u=4$$

We now substitute x^2 for u and solve for x.

$$x^2=9 \quad \text{or} \quad x^2=4$$
$$x=\pm 3 \quad \text{or} \quad x=\pm 2$$

Since $y=6/x$, if $x=3$, $y=2$; if $x=-3$, $y=-2$; if $x=2$, $y=3$; and if $x=-2$, $y=-3$. The pairs $(3,2)$, $(-3,-2)$, $(2,3)$, and $(-2,-3)$ check. They are the solutions.

36. $x^2+4y^2=20,$

$xy=4$

$$y=\frac{4}{x}$$
$$x^2+4\left(\frac{4}{x}\right)^2=20$$
$$x^2+\frac{64}{x^2}=20$$
$$x^4+64=20x^2$$
$$x^4-20x^2+64=0$$
$$u^2-20u+64=0 \quad \text{Letting } u=x^2$$
$$(u-16)(u-4)=0$$
$$u=16 \quad \text{or} \quad u=4$$
$$x^2=16 \quad \text{or} \quad x^2=4$$
$$x=\pm 4 \quad \text{or} \quad x=\pm 2$$

$y=4/x$, so if $x=4$, $y=1$; if $x=-4$, $y=-1$; if $x=2$, $y=2$; and if $x=-2$, $y=-2$. The pairs $(4,1)$, $(-4,-1)$, $(2,2)$, and $(-2,-2)$ check.

37. $x^2+y^2+6y+5=0 \quad (1)$

$x^2+y^2-2x-8=0 \quad (2)$

Using the elimination method, multiply equation (2) by -1 and add the result to equation (1).

$$\begin{array}{rl} x^2+y^2+6y+5=0 & (1) \\ -x^2-y^2+2x+8=0 & (2) \\ \hline 2x+6y+13=0 & (3) \end{array}$$

Solve equation (3) for x.

$$2x+6y+13=0$$
$$2x=-6y-13$$
$$x=\frac{-6y-13}{2}$$

Substitute $\frac{-6y-13}{2}$ for x in equation (1) and solve for y.

$$x^2+y^2+6y+5=0$$
$$\left(\frac{-6y-13}{2}\right)^2+y^2+6y+5=0$$
$$\frac{36y^2+156y+169}{4}+y^2+6y+5=0$$
$$36y^2+156y+169+4y^2+24y+20=0$$
$$40y^2+180y+189=0$$

Using the quadratic formula, we find that $y=\frac{-45\pm 3\sqrt{15}}{20}$. Substitute $\frac{-45\pm 3\sqrt{15}}{20}$ for y in $x=\frac{-6y-13}{2}$ and solve for x.

If $y=\frac{-45+3\sqrt{15}}{20}$, then

$$x=\frac{-6\left(\frac{-45+3\sqrt{15}}{20}\right)-13}{2}=\frac{5-9\sqrt{15}}{20}.$$

If $y=\frac{-45-3\sqrt{15}}{20}$, then

$$x=\frac{-6\left(\frac{-45-3\sqrt{15}}{20}\right)-13}{2}=\frac{5+9\sqrt{15}}{20}.$$

The pairs $\left(\frac{5+9\sqrt{15}}{20},\frac{-45-3\sqrt{15}}{20}\right)$ and $\left(\frac{5-9\sqrt{15}}{20},\frac{-45+3\sqrt{15}}{20}\right)$ check and are the solutions.

38. $2xy+3y^2=7, \quad (1)$

$3xy-2y^2=4 \quad (2)$

$$\begin{array}{rl} 6xy+9y^2=21 & \text{Multiplying (1) by 3} \\ -6xy+4y^2=-8 & \text{Multiplying (2) by } -2 \\ \hline 13y^2=13 & \\ y^2=1 & \\ y=\pm 1 & \end{array}$$

Substitute for y in equation (1) and solve for x.

When $y=1$: $\quad 2\cdot x\cdot 1+3\cdot 1^2=7$

$$2x=4$$
$$x=2$$

When $y=-1$: $\quad 2\cdot x\cdot(-1)+3(-1)^2=7$

$$-2x=4$$
$$x=-2$$

The pairs $(2,1)$ and $(-2,-1)$ check.

39. $2a + b = 1,$ (1)

$b = 4 - a^2$ (2)

Equation (2) is already solved for b. Substitute $4 - a^2$ for b in equation (1) and solve for a.

$$2a + 4 - a^2 = 1$$
$$0 = a^2 - 2a - 3$$
$$0 = (a-3)(a+1)$$

$a = 3$ or $a = -1$

Substitute these numbers in equation (2) and solve for b.

$b = 4 - 3^2 = -5$

$b = 4 - (-1)^2 = 3$

The pairs $(3, -5)$ and $(-1, 3)$ check. They are the solutions.

40. $4x^2 + 9y^2 = 36,$

$x + 3y = 3$

$x = -3y + 3$

$$4(-3y+3)^2 + 9y^2 = 36$$
$$4(9y^2 - 18y + 9) + 9y^2 = 36$$
$$36y^2 - 72y + 36 + 9y^2 = 36$$
$$45y^2 - 72y = 0$$
$$5y^2 - 8y = 0$$
$$y(5y - 8) = 0$$

$y = 0$ or $y = \dfrac{8}{5}$

$x = -3 \cdot 0 + 3 = 3$

$x = -3\left(\dfrac{8}{5}\right) + 3 = -\dfrac{9}{5}$

The pairs $(3, 0)$ and $\left(-\dfrac{9}{5}, \dfrac{8}{5}\right)$ check.

41. $a^2 + b^2 = 89,$ (1)

$a - b = 3$ (2)

First solve equation (2) for a.

$a = b + 3$ (3)

Then substitute $b+3$ for a in equation (1) and solve for b.

$$(b+3)^2 + b^2 = 89$$
$$b^2 + 6b + 9 + b^2 = 89$$
$$2b^2 + 6b - 80 = 0$$
$$b^2 + 3b - 40 = 0$$
$$(b+8)(b-5) = 0$$

$b = -8$ or $b = 5$

Substitute these numbers in equation (3) and solve for a.

$a = -8 + 3 = -5$

$a = 5 + 3 = 8$

The pairs $(-5, -8)$ and $(8, 5)$ check. They are the solutions.

42. $xy = 4,$

$x + y = 5$

$x = -y + 5$

$$(-y+5)y = 4$$
$$-y^2 + 5y = 4$$
$$0 = y^2 - 5y + 4$$
$$0 = (y-4)(y-1)$$

$y = 4$ or $y = 1$

$x = -4 + 5 = 1$

$x = -1 + 5 = 4$

The pairs $(1, 4)$ and $(4, 1)$ check.

43. $xy - y^2 = 2,$ (1)

$2xy - 3y^2 = 0$ (2)

$$\begin{array}{rl} -2xy + 2y^2 = -4 & \text{Multiplying (1) by } -2 \\ 2xy - 3y^2 = 0 & \\ \hline -y^2 = -4 & \text{Adding} \\ y^2 = 4 & \\ y = \pm 2 & \end{array}$$

We substitute for y in equation (1) and solve for x.

When $y = 2$: $\quad x \cdot 2 - 2^2 = 2$
$$2x - 4 = 2$$
$$2x = 6$$
$$x = 3$$

When $y = -2$: $\quad x(-2) - (-2)^2 = 2$
$$-2x - 4 = 2$$
$$-2x = 6$$
$$x = -3$$

The pairs $(3, 2)$ and $(-3, -2)$ check. They are the solutions.

44. $4a^2 - 25b^2 = 0,$ (1)

$2a^2 - 10b^2 = 3b + 4$ (2)

$$\begin{array}{rl} 4a^2 - 25b^2 = 0 & \\ -4a^2 + 20b^2 = -6b - 8 & \text{Multiplying (2) by } -2 \\ \hline -5b^2 = -6b - 8 & \\ 0 = 5b^2 - 6b - 8 & \\ 0 = (5b+4)(b-2) & \end{array}$$

$b = -\dfrac{4}{5}$ or $b = 2$

Substitute for b in equation (1) and solve for a.

When $b = -\dfrac{4}{5}$: $\quad 4a^2 - 25\left(-\dfrac{4}{5}\right)^2 = 0$
$$4a^2 = 16$$
$$a^2 = 4$$
$$a = \pm 2$$

When $b = 2$:
$$\begin{aligned} 4a^2 - 25(2)^2 &= 0 \\ 4a^2 &= 100 \\ a^2 &= 25 \\ a &= \pm 5 \end{aligned}$$

The pairs $\left(2, -\frac{4}{5}\right)$, $\left(-2, -\frac{4}{5}\right)$, $(5, 2)$ and $(-5, 2)$ check.

45. $m^2 - 3mn + n^2 + 1 = 0$, (1)
$3m^2 - mn + 3n^2 = 13$ (2)

$m^2 - 3mn + n^2 = -1$ (3) Rewriting (1)
$3m^2 - mn + 3n^2 = 13$ (2)

$-3m^2 + 9mn - 3n^2 = 3$ Multiplying (3) by -3
$$\begin{aligned} 3m^2 - mn + 3n^2 &= 13 \\ \hline 8mn &= 16 \\ mn &= 2 \\ n &= \frac{2}{m} \quad (4) \end{aligned}$$

Substitute $\frac{2}{m}$ for n in equation (1) and solve for m.
$$\begin{aligned} m^2 - 3m\left(\frac{2}{m}\right) + \left(\frac{2}{m}\right)^2 + 1 &= 0 \\ m^2 - 6 + \frac{4}{m^2} + 1 &= 0 \\ m^2 - 5 + \frac{4}{m^2} &= 0 \\ m^4 - 5m^2 + 4 &= 0 \quad \text{Multiplying by } m^2 \end{aligned}$$

Substitute u for m^2.
$$\begin{aligned} u^2 - 5u + 4 &= 0 \\ (u-4)(u-1) &= 0 \\ u = 4 \quad &or \quad u = 1 \\ m^2 = 4 \quad &or \quad m^2 = 1 \\ m = \pm 2 \quad &or \quad m = \pm 1 \end{aligned}$$

Substitute for m in equation (4) and solve for n.

When $m = 2$, $n = \frac{2}{2} = 1$.

When $m = -2$, $n = \frac{2}{-2} = -1$.

When $m = 1$, $n = \frac{2}{1} = 2$.

When $m = -1$, $n = \frac{2}{-1} = -2$.

The pairs $(2, 1)$, $(-2, -1)$, $(1, 2)$, and $(-1, -2)$ check. They are the solutions.

46. $ab - b^2 = -4$, (1)
$ab - 2b^2 = -6$ (2)

$$\begin{aligned} ab - b^2 &= -4 \\ -ab + 2b^2 &= 6 \quad \text{Multiplying (2) by } -1 \\ \hline b^2 &= 2 \\ b &= \pm\sqrt{2} \end{aligned}$$

Substitute for b in equation (1) and solve for a.

When $b = \sqrt{2}$:
$$\begin{aligned} a(\sqrt{2}) - (\sqrt{2})^2 &= -4 \\ a\sqrt{2} &= -2 \\ a &= -\frac{2}{\sqrt{2}} = -\sqrt{2} \end{aligned}$$

When $b = -\sqrt{2}$:
$$\begin{aligned} a(-\sqrt{2}) - (-\sqrt{2})^2 &= -4 \\ -a\sqrt{2} &= -2 \\ a &= \frac{-2}{-\sqrt{2}} = \sqrt{2} \end{aligned}$$

The pairs $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$ check.

47. $x^2 + y^2 = 5$, (1)
$x - y = 8$ (2)

First solve equation (2) for x.

$x = y + 8$ (3)

Then substitute $y + 8$ for x in equation (1) and solve for y.
$$\begin{aligned} (y+8)^2 + y^2 &= 5 \\ y^2 + 16y + 64 + y^2 &= 5 \\ 2y^2 + 16y + 59 &= 0 \end{aligned}$$
$$y = \frac{-16 \pm \sqrt{(16)^2 - 4(2)(59)}}{2 \cdot 2}$$
$$y = \frac{-16 \pm \sqrt{-216}}{4}$$
$$y = \frac{-16 \pm 6i\sqrt{6}}{4}$$
$$y = -4 \pm \frac{3}{2}i\sqrt{6}$$

Now substitute these numbers in equation (3) and solve for x.
$$x = -4 + \frac{3}{2}i\sqrt{6} + 8 = 4 + \frac{3}{2}i\sqrt{6}$$
$$x = -4 - \frac{3}{2}i\sqrt{6} + 8 = 4 - \frac{3}{2}i\sqrt{6}$$

The pairs $\left(4 + \frac{3}{2}i\sqrt{6}, -4 + \frac{3}{2}i\sqrt{6}\right)$ and $\left(4 - \frac{3}{2}i\sqrt{6}, -4 - \frac{3}{2}i\sqrt{6}\right)$ check. They are the solutions.

48. $4x^2 + 9y^2 = 36$,
$y - x = 8$

$y = x + 8$
$$\begin{aligned} 4x^2 + 9(x+8)^2 &= 36 \\ 4x^2 + 9(x^2 + 16x + 64) &= 36 \\ 4x^2 + 9x^2 + 144x + 576 &= 36 \\ 13x^2 + 144x + 540 &= 0 \end{aligned}$$

$$x = \frac{-144 \pm \sqrt{(144)^2 - 4(13)(540)}}{2 \cdot 13}$$

$$x = \frac{-72 \pm 6i\sqrt{51}}{13} = -\frac{72}{13} \pm \frac{6}{13}i\sqrt{51}$$

$$y = -\frac{72}{13} + \frac{6}{13}i\sqrt{51} + 8 = \frac{32}{13} + \frac{6}{13}i\sqrt{51}$$

$$y = -\frac{72}{13} - \frac{6}{13}i\sqrt{51} + 8 = \frac{32}{13} - \frac{6}{13}i\sqrt{51}$$

The pairs $\left(-\frac{72}{13} + \frac{6}{13}i\sqrt{51}, \frac{32}{13} + \frac{6}{13}i\sqrt{51}\right)$ and $\left(-\frac{72}{13} - \frac{6}{13}i\sqrt{51}, \frac{32}{13} - \frac{6}{13}i\sqrt{51}\right)$ check.

49. $a^2 + b^2 = 14,\quad (1)$

$ab = 3\sqrt{5}\quad (2)$

Solve equation (2) for b.

$$b = \frac{3\sqrt{5}}{a}$$

Substitute $\frac{3\sqrt{5}}{a}$ for b in equation (1) and solve for a.

$$a^2 + \left(\frac{3\sqrt{5}}{a}\right)^2 = 14$$

$$a^2 + \frac{45}{a^2} = 14$$

$$a^4 + 45 = 14a^2$$

$$a^4 - 14a^2 + 45 = 0$$

$$u^2 - 14u + 45 = 0 \qquad \text{Letting } u = a^2$$

$$(u-9)(u-5) = 0$$

$$u = 9 \quad or \quad u = 5$$

$$a^2 = 9 \quad or \quad a^2 = 5$$

$$a = \pm 3 \quad or \quad a = \pm\sqrt{5}$$

Since $b = 3\sqrt{5}/a$, if $a = 3$, $b = \sqrt{5}$; if $a = -3$, $b = -\sqrt{5}$; if $a = \sqrt{5}$, $b = 3$; and if $a = -\sqrt{5}$, $b = -3$. The pairs $(3, \sqrt{5})$, $(-3, -\sqrt{5})$, $(\sqrt{5}, 3)$, $(-\sqrt{5}, -3)$ check. They are the solutions.

50. $x^2 + xy = 5,\quad (1)$

$2x^2 + xy = 2\quad (2)$

$$\begin{array}{rcl} -x^2 - xy & = & -5 \quad \text{Multiplying (1) by } -1 \\ 2x^2 + xy & = & 2 \\ \hline x^2 & = & -3 \\ x & = & \pm i\sqrt{3} \end{array}$$

Substitute for x in equation (1) and solve for y.

When $x = i\sqrt{3}$: $(i\sqrt{3})^2 + (i\sqrt{3})y = 5$

$$i\sqrt{3}y = 8$$

$$y = \frac{8}{i\sqrt{3}}$$

$$y = -\frac{8i\sqrt{3}}{3}$$

When $x = -i\sqrt{3}$: $(-i\sqrt{3})^2 + (-i\sqrt{3})y = 5$

$$-i\sqrt{3}y = 8$$

$$y = -\frac{8}{i\sqrt{3}}$$

$$y = -\frac{8i\sqrt{3}}{3}$$

The pairs $\left(i\sqrt{3}, -\frac{8i\sqrt{3}}{3}\right)$ and $\left(-i\sqrt{3}, \frac{8i\sqrt{3}}{3}\right)$.

51. $x^2 + y^2 = 25,\quad (1)$

$9x^2 + 4y^2 = 36\quad (2)$

$$\begin{array}{rcl} -4x^2 - 4y^2 & = & -100 \qquad \text{Multiplying (1) by } -4 \\ 9x^2 + 4y^2 & = & 36 \\ \hline 5x^2 & = & -64 \end{array}$$

$$x^2 = -\frac{64}{5}$$

$$x = \pm\sqrt{\frac{-64}{5}} = \pm\frac{8i}{\sqrt{5}}$$

$$x = \pm\frac{8i\sqrt{5}}{5} \qquad \text{Rationalizing the denominator}$$

Substituting $\frac{8i\sqrt{5}}{5}$ or $-\frac{8i\sqrt{5}}{5}$ for x in equation (1) and solving for y gives us

$$-\frac{64}{5} + y^2 = 25$$

$$y^2 = \frac{189}{5}$$

$$y = \pm\sqrt{\frac{189}{5}} = \pm 3\sqrt{\frac{21}{5}}$$

$$y = \pm\frac{3\sqrt{105}}{5}. \qquad \text{Rationalizing the denominator}$$

The pairs $\left(\frac{8i\sqrt{5}}{5}, \frac{3\sqrt{105}}{5}\right)$, $\left(-\frac{8i\sqrt{5}}{5}, \frac{3\sqrt{105}}{5}\right)$, $\left(\frac{8i\sqrt{5}}{5}, -\frac{3\sqrt{105}}{5}\right)$, and $\left(-\frac{8i\sqrt{5}}{5}, -\frac{3\sqrt{105}}{5}\right)$ check.

They are the solutions.

52. $x^2 + y^2 = 1,\quad (1)$

$9x^2 - 16y^2 = 144\quad (2)$

$$\begin{array}{rcl} 16x^2 + 16y^2 & = & 16 \qquad \text{Multiplying (1) by 16} \\ 9x^2 - 16y^2 & = & 144 \\ \hline 25x^2 & = & 160 \end{array}$$

$$x^2 = \frac{160}{25}$$

$$x = \pm\frac{4\sqrt{10}}{5}$$

Substituting for x in equation (1) and solving for y gives us

$$\frac{160}{25}+y^2=1$$
$$y^2=-\frac{135}{25}$$
$$y=\pm\sqrt{-\frac{135}{25}}$$
$$y=\pm\frac{3i\sqrt{15}}{5}.$$

The pairs $\left(\frac{4\sqrt{10}}{5},\frac{3i\sqrt{15}}{5}\right)$, $\left(\frac{4\sqrt{10}}{5},-\frac{3i\sqrt{15}}{5}\right)$, $\left(-\frac{4\sqrt{10}}{5},\frac{3i\sqrt{15}}{5}\right)$, and $\left(-\frac{4\sqrt{10}}{5},-\frac{3i\sqrt{15}}{5}\right)$ check.

53. $5y^2-x^2=1,\quad (1)$
$xy=2\quad (2)$

Solve equation (2) for x.

$$x=\frac{2}{y}$$

Substitute $\frac{2}{y}$ for x in equation (1) and solve for y.

$$5y^2-\left(\frac{2}{y}\right)^2=1$$
$$5y^2-\frac{4}{y^2}=1$$
$$5y^4-4=y^2$$
$$5y^4-y^2-4=0$$
$$5u^2-u-4=0\quad \text{Letting } u=y^2$$
$$(5u+4)(u-1)=0$$
$$5u+4=0\quad or\quad u-1=0$$
$$u=-\frac{4}{5}\quad or\quad u=1$$
$$y^2=-\frac{4}{5}\quad or\quad y^2=1$$
$$y=\pm\frac{2i}{\sqrt{5}}\quad or\quad y=\pm1$$
$$y=\pm\frac{2i\sqrt{5}}{5}\quad or\quad y=\pm1$$

Since $x=2/y$, if $y=\frac{2i\sqrt{5}}{5}$, $x=\frac{2}{\frac{2i\sqrt{5}}{5}}=\frac{5}{i\sqrt{5}}=\frac{5}{i\sqrt{5}}\cdot\frac{-i\sqrt{5}}{-i\sqrt{5}}=-i\sqrt{5}$; if $y=-\frac{2i\sqrt{5}}{5}$, $x=\frac{2}{-\frac{2i\sqrt{5}}{5}}=i\sqrt{5}$; if $y=1$, $x=2/1=2$; if $y=-1$, $x=2/-1=-2$.

The pairs $\left(-i\sqrt{5},\frac{2i\sqrt{5}}{5}\right)$, $\left(i\sqrt{5},-\frac{2i\sqrt{5}}{5}\right)$, $(2,1)$ and $(-2,-1)$ check. They are the solutions.

54. $x^2-7y^2=6,$
$xy=1$

$$y=\frac{1}{x}$$
$$x^2-7\left(\frac{1}{x}\right)^2=6$$
$$x^2-\frac{7}{x^2}=6$$
$$x^4-7=6x^2$$
$$x^4-6x^2-7=0$$
$$u^2-6u-7=0\quad \text{Letting } u=x^2$$
$$(u-7)(u+1)=0$$
$$u=7\quad or\quad u=-1$$
$$x^2=7\quad or\quad x^2=-1$$
$$x=\pm\sqrt{7}\quad or\quad x=\pm i$$

Since $y=1/x$, if $x=\sqrt{7}$, $y=1/\sqrt{7}=\sqrt{7}/7$; if $x=-\sqrt{7}$, $y=1/(-\sqrt{7})=-\sqrt{7}/7$; if $x=i$, $y=1/i=-i$, if $x=-i$, $y=1/(-i)=i$.

The pairs $\left(\sqrt{7},\frac{\sqrt{7}}{7}\right)$, $\left(-\sqrt{7},-\frac{\sqrt{7}}{7}\right)$, $(i,-i)$, and $(-i,i)$ check.

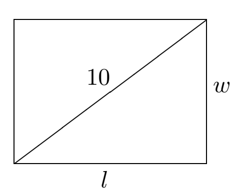
55. The statement is true. See Example 4, for instance.

56. The statement is false. See Example 2, for instance.

57. The statement is true because a line and a circle can intersect in at most two points.

58. The statement is true because it is possible for a line to be tangent to an ellipse (that is, to touch the ellipse at a single point).

59. ***Familiarize***. We first make a drawing. We let l and w represent the length and width, respectively.



Translate. The perimeter is 28 cm.

$2l+2w=28$, or $l+w=14$

Using the Pythagorean theorem we have another equation.

$l^2+w^2=10^2$, or $l^2+w^2=100$

Carry out. We solve the system:

$l+w=14,\quad (1)$
$l^2+w^2=100\quad (2)$

First solve equation (1) for w.

$w=14-l\quad (3)$

Then substitute $14-l$ for w in equation (2) and solve for l.

$$\begin{aligned} l^2 + w^2 &= 100 \\ l^2 + (14 - l)^2 &= 100 \\ l^2 + 196 - 28l + l^2 &= 100 \\ 2l^2 - 28l + 96 &= 0 \\ l^2 - 14l + 48 &= 0 \\ (l - 8)(l - 6) &= 0 \end{aligned}$$

$l = 8$ or $l = 6$

If $l = 8$, then $w = 14 - 8$, or 6. If $l = 6$, then $w = 14 - 6$, or 8. Since the length is usually considered to be longer than the width, we have the solution $l = 8$ and $w = 6$, or $(8, 6)$.

Check. If $l = 8$ and $w = 6$, then the perimeter is $2 \cdot 8 + 2 \cdot 6$, or 28. The length of a diagonal is $\sqrt{8^2 + 6^2}$, or $\sqrt{100}$, or 10. The numbers check.

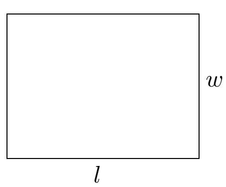
State. The length is 8 cm, and the width is 6 cm.

60. Let l and w represent the length and width, respectively. We solve the system:

$$lw = 2,$$
$$2l + 2w = 6$$

The solutions are $(1, 2)$ and $(2, 1)$. We choose the larger number to be the length, so the length is 2 yd and the width is 1 yd.

61. ***Familiarize***. We first make a drawing. Let $l =$ the length and $w =$ the width of the brochure.



Translate.

Area: $lw = 20$

Perimeter: $2l + 2w = 18$, or $l + w = 9$

Carry out. We solve the system:

Solve the second equation for l: $l = 9 - w$

Substitute $9 - w$ for l in the first equation and solve for w.

$$\begin{aligned} (9 - w)w &= 20 \\ 9w - w^2 &= 20 \\ 0 &= w^2 - 9w + 20 \\ 0 &= (w - 5)(w - 4) \end{aligned}$$

$w = 5$ or $w = 4$

If $w = 5$, then $l = 9 - w$, or 4. If $w = 4$, then $l = 9 - 4$, or 5. Since length is usually considered to be longer than width, we have the solution $l = 5$ and $w = 4$, or $(5, 4)$.

Check. If $l = 5$ and $w = 4$, the area is $5 \cdot 4$, or 20. The perimeter is $2 \cdot 5 + 2 \cdot 4$, or 18. The numbers check.

State. The length of the brochure is 5 in. and the width is 4 in.

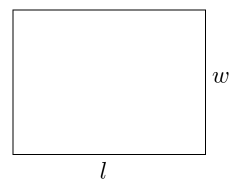
62. Let l and w represent the length and width, respectively. Solve the system:

$$2l + 2w = 6,$$
$$l^2 + w^2 = (\sqrt{5})^2, \text{ or}$$

$$l + w = 3$$
$$l^2 + w^2 = 5$$

The solutions are $(1, 2)$ and $(2, 1)$. Choosing the larger number as the length, we have the solution. The length is 2 m, and the width is 1 m.

63. ***Familiarize***. We make a drawing of the dog run. Let $l =$ the length and $w =$ the width.



Since it takes 210 yd of fencing to enclose the run, we know that the perimeter is 210 yd.

Translate.

Perimeter: $2l + 2w = 210$, or $l + w = 105$

Area: $lw = 2250$

Carry out. We solve the system:

Solve the first equation for l: $l = 105 - w$

Substitute $105 - w$ for l in the second equation and solve for w.

$$\begin{aligned} (105 - w)w &= 2250 \\ 105w - w^2 &= 2250 \\ 0 &= w^2 - 105w + 2250 \\ 0 &= (w - 30)(w - 75) \end{aligned}$$

$w = 30$ or $w = 75$

If $w = 30$, then $l = 105 - 30$, or 75. If $w = 75$, then $l = 105 - 75$, or 30. Since length is usually considered to be longer than width, we have the solution $l = 75$ and $w = 30$, or $(75, 30)$.

Check. If $l = 75$ and $w = 30$, the perimeter is $2 \cdot 75 + 2 \cdot 30$, or 210. The area is $75(30)$, or 2250. The numbers check.

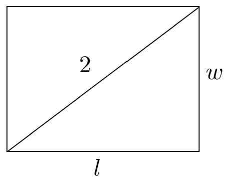
State. The length is 75 yd and the width is 30 yd.

64. Let $l =$ the length and $w =$ the width. Solve the system

$$lw = \sqrt{2},$$
$$l^2 + w^2 = (\sqrt{3})^2.$$

The solutions are $(\sqrt{2}, 1)$, $(-\sqrt{2}, -1)$, $(1, \sqrt{2})$, and $(-1, -\sqrt{2})$. Only the pairs $(\sqrt{2}, 1)$ and $(1, \sqrt{2})$ have meaning in this problem. Since length is usually considered to be longer than width, the length is $\sqrt{2}$ m, and the width is 1 m.

65. ***Familiarize***. We first make a drawing. Let $l =$ the length and $w =$ the width.



Translate.

Area: $lw = \sqrt{3}$ (1)

From the Pythagorean theorem: $l^2 + w^2 = 2^2$ (2)

Carry out. We solve the system of equations.

We first solve equation (1) for w.

$$lw = \sqrt{3}$$
$$w = \frac{\sqrt{3}}{l}$$

Then we substitute $\frac{\sqrt{3}}{l}$ for w in equation 2 and solve for l.

$$l^2 + \left(\frac{\sqrt{3}}{l}\right)^2 = 4$$
$$l^2 + \frac{3}{l^2} = 4$$
$$l^4 + 3 = 4l^2$$
$$l^4 - 4l^2 + 3 = 0$$
$$u^2 - 4u + 3 = 0 \quad \text{Letting } u = l^2$$
$$(u-3)(u-1) = 0$$
$$u = 3 \quad \text{or} \quad u = 1$$

We now substitute l^2 for u and solve for l.

$$l^2 = 3 \quad \text{or} \quad l^2 = 1$$
$$l = \pm\sqrt{3} \quad \text{or} \quad l = \pm 1$$

Measurements cannot be negative, so we only need to consider $l = \sqrt{3}$ and $l = 1$. Since $w = \sqrt{3}/l$, if $l = \sqrt{3}$, $w = 1$ and if $l = 1$, $w = \sqrt{3}$. Length is usually considered to be longer than width, so we have the solution $l = \sqrt{3}$ and $w = 1$, or $(\sqrt{3}, 1)$.

Check. If $l = \sqrt{3}$ and $w = 1$, the area is $\sqrt{3} \cdot 1 = \sqrt{3}$. Also $(\sqrt{3})^2 + 1^2 = 3 + 1 = 4 = 2^2$. The numbers check.

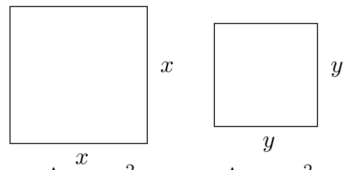
State. The length is $\sqrt{3}$ m, and the width is 1 m.

66. Let $p =$ the principal and $r =$ the interest rate. Solve the system:

$$pr = 7.5,$$
$$(p+25)(r-0.01) = 7.5$$

The solutions are $(125, 0.06)$ and $(-1.50, -0.05)$. Only $(125, 0.06)$ has meaning in this problem. The principal was \$125 and the interest rate was 0.06, or 6%.

67. ***Familiarize***. We let $x =$ the length of a side of one test plot and $y =$ the length of a side of the other plot. Make a drawing.



Area: x^2 Area: y^2

Translate.

$$\underbrace{\text{The sum of the areas}}_{x^2 + y^2} \quad \underbrace{\text{is}}_{=} \quad \underbrace{832 \text{ ft}^2}_{832}.$$

$$\underbrace{\text{The difference of the areas}}_{x^2 - y^2} \quad \underbrace{\text{is}}_{=} \quad \underbrace{320 \text{ ft}^2}_{320}.$$

Carry out. We solve the system of equations.

$$\begin{aligned} x^2 + y^2 &= 832 \\ x^2 - y^2 &= 320 \\ \hline 2x^2 \quad &= 1152 \quad \text{Adding} \\ x^2 &= 576 \\ x &= \pm 24 \end{aligned}$$

Since measurements cannot be negative, we consider only $x = 24$. Substitute 24 for x in the first equation and solve for y.

$$24^2 + y^2 = 832$$
$$576 + y^2 = 832$$
$$y^2 = 256$$
$$y = \pm 16$$

Again, we consider only the positive value, 16. The possible solution is $(24, 16)$.

Check. The areas of the test plots are 24^2, or 576, and 16^2, or 256. The sum of the areas is $576 + 256$, or 832. The difference of the areas is $576 - 256$, or 320. The values check.

State. The lengths of the test plots are 24 ft and 16 ft.

68. Let l and w represent the length and width, respectively. Solve the system:

$$\sqrt{l^2 + w^2} = l + 1,$$
$$\sqrt{l^2 + w^2} = 2w + 3$$

The solutions are $(12, 5)$ and $(0, -1)$. Only $(12, 5)$ has meaning in this problem. It checks. The length is 12 ft and the width is 5 ft.

69. The correct graph is (b).

70. The correct graph is (e).

71. The correct graph is (d).

72. The correct graph is (f).

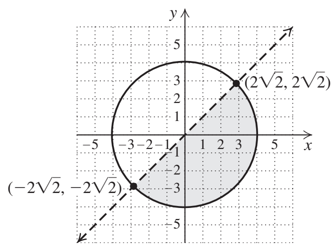
73. The correct graph is (a).

74. The correct graph is (c).

75. Graph: $x^2 + y^2 \leq 16$,

$y < x$

The solution set of $x^2 + y^2 \leq 16$ is the circle $x^2 + y^2 = 16$ and the region inside it. The solution set of $y < x$ is the half-plane below the line $y = x$. We shade the region common to the two solution sets.



To find the points of intersection of the graphs we solve the system of equations

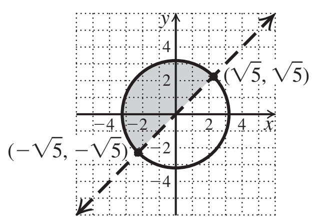
$x^2 + y^2 = 16$,

$y = x$.

The points of intersection are $(-2\sqrt{2}, -2\sqrt{2})$ and $(2\sqrt{2}, 2\sqrt{2})$.

76. Graph: $x^2 + y^2 \leq 10$,

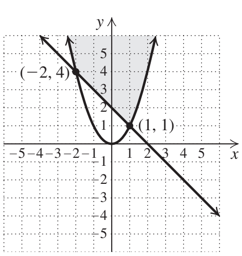
$y > x$



77. Graph: $x^2 \leq y$,

$x + y \geq 2$

The solution set of $x^2 \leq y$ is the parabola $x^2 = y$ and the region inside it. The solution set of $x + y \geq 2$ is the line $x + y = 2$ and the half-plane above the line. We shade the region common to the two solution sets.



To find the points of intersection of the graphs we solve the system of equations

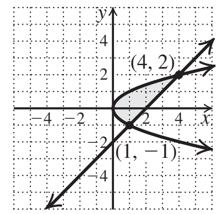
$x^2 = y$,

$x + y = 2$.

The points of intersection are $(-2, 4)$ and $(1, 1)$.

78. Graph: $x \geq y^2$,

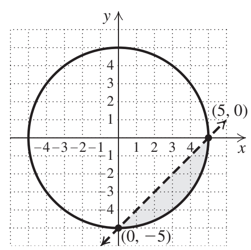
$x - y \leq 2$



79. Graph: $x^2 + y^2 \leq 25$,

$x - y > 5$

The solution set of $x^2 + y^2 \leq 25$ is the circle $x^2 + y^2 = 25$ and the region inside it. The solution set of $x - y > 5$ is the half-plane below the line $x - y = 5$. We shade the region common to the two solution sets.



To find the points of intersection of the graphs we solve the system of equations

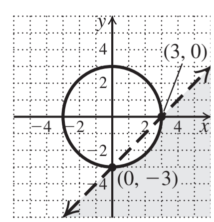
$x^2 + y^2 = 25$,

$x - y = 5$.

The points of intersection are $(0, -5)$ and $(5, 0)$.

80. Graph: $x^2 + y^2 \geq 9$,

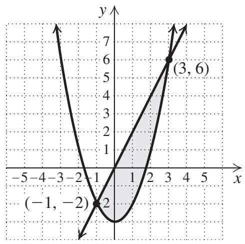
$x - y > 3$



81. Graph: $y \geq x^2 - 3$,

$y \leq 2x$

The solution set of $y \geq x^2 - 3$ is the parabola $y = x^2 - 3$ and the region inside it. The solution set of $y \leq 2x$ is the line $y = 2x$ and the half-plane below it. We shade the region common to the two solution sets.



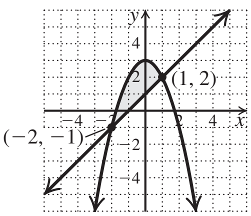
To find the points of intersection of the graphs we solve the system of equations

$y = x^2 - 3,$

$y = 2x.$

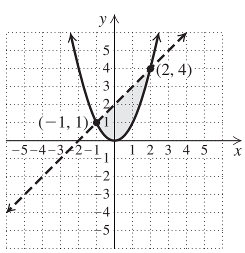
The points of intersection are $(-1,-2)$ and $(3,6)$.

82. Graph: $y \leq 3 - x^2,$
$y \geq x + 1$



83. Graph: $y \geq x^2,$
$y < x + 2$

The solution set of $y \geq x^2$ is the parabola $y = x^2$ and the region inside it. The solution set of $y < x + 2$ is the half-plane below the line $y = x + 2$. We shade the region common to the two solution sets.



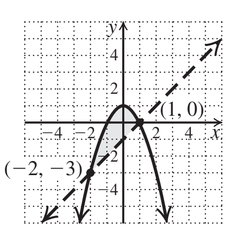
To find the points of intersection of the graphs we solve the system of equations

$y = x^2,$

$y = x + 2.$

The points of intersection are $(-1,1)$ and $(2,4)$.

84. Graph: $y \leq 1 - x^2,$
$y > x - 1$



85.
$$\begin{aligned} 2^{3x} &= 64 \\ 2^{3x} &= 2^6 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

The solution is 2.

86.
$$\begin{aligned} 5^x &= 27 \\ \ln 5^x &= \ln 27 \\ x \ln 5 &= \ln 27 \\ x &= \frac{\ln 27}{\ln 5} \\ x &\approx 2.048 \end{aligned}$$

87.
$$\begin{aligned} \log_3 x &= 4 \\ x &= 3^4 \\ x &= 81 \end{aligned}$$

The solution is 81.

88.
$$\begin{aligned} \log(x-3) + \log x &= 1 \\ \log(x-3)(x) &= 1 \\ x^2 - 3x &= 10 \\ x^2 - 3x - 10 &= 0 \\ (x-5)(x+2) &= 0 \end{aligned}$$

$x = 5$ *or* $x = -2$

Only 5 checks.

89. $(x-h)^2 + (y-k)^2 = r^2$

If $(2,4)$ is a point on the circle, then

$(2-h)^2 + (4-k)^2 = r^2.$

If $(3,3)$ is a point on the circle, then

$(3-h)^2 + (3-k)^2 = r^2.$

Thus

$$\begin{aligned} (2-h)^2 + (4-k)^2 &= (3-h)^2 + (3-k)^2 \\ 4 - 4h + h^2 + 16 - 8k + k^2 &= \\ &9 - 6h + h^2 + 9 - 6k + k^2 \\ -4h - 8k + 20 &= -6h - 6k + 18 \\ 2h - 2k &= -2 \\ h - k &= -1 \end{aligned}$$

If the center (h,k) is on the line $3x - y = 3$, then $3h - k = 3$.

Solving the system

$h - k = -1,$

$3h - k = 3$

we find that $(h,k) = (2,3)$.

Find r^2, substituting $(2,3)$ for (h,k) and $(2,4)$ for (x,y). We could also use $(3,3)$ for (x,y).

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ (2-2)^2 + (4-3)^2 &= r^2 \\ 0 + 1 &= r^2 \\ 1 &= r^2 \end{aligned}$$

The equation of the circle is $(x-2)^2 + (y-3)^2 = 1$.

90. Let (h,k) represent the point on the line $5x+8y=-2$ which is the center of a circle that passes through the points $(-2,3)$ and $(-4,1)$. The distance between (h,k) and $(-2,3)$ is the same as the distance between (h,k) and $(-4,1)$. This gives us one equation:

$$\begin{aligned}\sqrt{[h-(-2)]^2+(k-3)^2} &= \sqrt{[h-(-4)]^2+(k-1)^2}\\ (h+2)^2+(k-3)^2 &= (h+4)^2+(k-1)^2\\ h^2+4h+4+k^2-6k+9 &= h^2+8h+16+k^2-2k+1\\ 4h-6k+13 &= 8h-2k+17\\ -4h-4k &= 4\\ h+k &= -1\end{aligned}$$

We get a second equation by substituting (h,k) in $5x+8y=-2$.

$$5h+8k=-2$$

We now solve the following system:

$$\begin{aligned} h+\ \ k &= -1,\\ 5h+8k &= -2\end{aligned}$$

The solution, which is the center of the circle, is $(-2,1)$.

Next we find the length of the radius. We can find the distance between either $(-2,3)$ or $(-4,1)$ and the center $(-2,1)$. We use $(-2,3)$.

$$r=\sqrt{[-2-(-2)]^2+(1-3)^2}$$
$$r=\sqrt{0^2+(-2)^2}$$
$$r=\sqrt{4}=2$$

We can write the equation of the circle with center $(-2,1)$ and radius 2.

$$\begin{aligned}(x-h)^2+(y-k)^2 &= r^2\\ [x-(-2)]^2+(y-1)^2 &= 2^2\\ (x+2)^2+(y-1)^2 &= 4\end{aligned}$$

91. The equation of the ellipse is of the form $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Substitute $\left(1,\dfrac{\sqrt{3}}{2}\right)$ and $\left(\sqrt{3},\dfrac{1}{2}\right)$ for (x,y) to get two equations.

$$\frac{1^2}{a^2}+\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{b^2}=1, \ \ or\ \ \frac{1}{a^2}+\frac{3}{4b^2}=1$$

$$\frac{(\sqrt{3})^2}{a^2}+\frac{\left(\frac{1}{2}\right)^2}{b^2}=1, \ \ or\ \ \frac{3}{a^2}+\frac{1}{4b^2}=1$$

Substitute u for $\dfrac{1}{a^2}$ and v for $\dfrac{1}{b^2}$.

$$u+\frac{3}{4}v=1, \qquad 4u+3v=4,$$
$$or$$
$$3u+\frac{1}{4}v=1 \qquad 12u+v=4$$

Solving for u and v, we get $u=\dfrac{1}{4}$, $v=1$. Then $u=\dfrac{1}{a^2}=\dfrac{1}{4}$, so $a^2=4$; $v=\dfrac{1}{b^2}=1$, so $b^2=1$.

Then the equation of the ellipse is

$\dfrac{x^2}{4}+\dfrac{y^2}{1}=1$, or $\dfrac{x^2}{4}+y^2=1$.

92. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

Substitute each ordered pair for (x,y).

$$\frac{(-3)^2}{a^2}-\frac{\left(-\frac{3\sqrt{5}}{2}\right)^2}{b^2}=1,$$

$$\frac{(-3)^2}{a^2}-\frac{\left(\frac{3\sqrt{5}}{b^2}\right)^2}{b^2}=1,$$

$$\frac{\left(-\frac{3}{2}\right)^2}{a^2}-\frac{0^2}{b^2}=1$$

$$\frac{9}{a^2}-\frac{45}{4b^2}=1, \qquad (1)$$
$$\frac{9}{a^2}-\frac{45}{4b^2}=1, \qquad (2)$$
$$\frac{9}{4a^2}=1 \qquad (3)$$

Note that equation (1) and equation (2) are identical. Multiply both sides of equation (3) by 4:

$$\frac{9}{a^2}=4$$

Substitute 4 for $\dfrac{9}{a^2}$ in equation (1) and solve for b^2.

$$\begin{aligned}4-\frac{45}{4b^2} &= 1\\ 16b^2-45 &= 4b^2\\ 12b^2 &= 45\\ b^2 &= \frac{45}{12},\ \text{or}\ \frac{15}{4}\end{aligned}$$

Solve equation (3) for a^2.

$$\frac{9}{4a^2}=1$$
$$\frac{9}{4}=a^2$$

The equation of the hyperbola is $\dfrac{x^2}{9/4}-\dfrac{y^2}{15/4}=1$.

93. $(x-h)^2+(y-k)^2=r^2$ Standard form

Substitute $(4,6)$, $(6,2)$, and $(1,-3)$ for (x,y).

$$(4-h)^2+(6-k)^2=r^2 \qquad (1)$$
$$(6-h)^2+(2-k)^2=r^2 \qquad (2)$$
$$(1-h)^2+(-3-k)^2=r^2 \qquad (3)$$

Thus

$(4-h)^2+(6-k)^2=(6-h)^2+(2-k)^2$, or

$h-2k=-3$

and

$(4-h)^2+(6-k)^2=(1-h)^2+(-3-k)^2$, or

$h+3k=7$.

We solve the system

$$h-2k=-3,$$
$$h+3k=7.$$

Solving we get $h = 1$ and $k = 2$. Substituting these values in equation (1), (2), or (3), we find that $r^2 = 25$.

The equation of the circle is $(x-1)^2 + (y-2)^2 = 25$.

94. Using $(x-h)^2 + (y-k)^2 = r^2$ and the given points, we have

$$(2-h)^2 + (3-k)^2 = r^2 \quad (1)$$
$$(4-h)^2 + (5-k)^2 = r^2 \quad (2)$$
$$(0-h)^2 + (-3-k)^2 = r^2 \quad (3)$$

Then equation (1) − equation (2) gives $h + k = 7$ and equation (2) − equation (3) gives $h + 2k = 4$. We solve this system:

$$h + k = 7,$$
$$h + 2k = 4.$$

Then $h = 10$, $k = -3$, $r = 10$ and the equation of the circle is $(x-10)^2+[y-(-3)]^2 = 10^2$, or $(x-10)^2+(y+3)^2 = 100$.

95. See the answer section in the text.

96. Let x and y represent the numbers. Solve:

$$xy = 2,$$
$$\frac{1}{x} + \frac{1}{y} = \frac{33}{8}.$$

The solutions are $\left(\frac{1}{4}, 8\right)$ and $\left(8, \frac{1}{4}\right)$. In either case the numbers are $\frac{1}{4}$ and 8.

97. ***Familiarize***. Let x and y represent the numbers.

Translate.

The square of a certain number exceeds twice the square of another number by $\frac{1}{8}$.

$$x^2 = 2y^2 + \frac{1}{8}$$

The sum of the squares is $\frac{5}{16}$.

$$x^2 + y^2 = \frac{5}{16}$$

Carry out. We solve the system.

$$x^2 - 2y^2 = \frac{1}{8}, \quad (1)$$
$$x^2 + y^2 = \frac{5}{16} \quad (2)$$

$$x^2 - 2y^2 = \frac{1}{8},$$
$$2x^2 + 2y^2 = \frac{5}{8} \quad \text{Multiplying (2) by 2}$$
$$3x^2 = \frac{6}{8}$$
$$x^2 = \frac{1}{4}$$
$$x = \pm\frac{1}{2}$$

Substitute $\pm\frac{1}{2}$ for x in (2) and solve for y.

$$\left(\pm\frac{1}{2}\right)^2 + y^2 = \frac{5}{16}$$
$$\frac{1}{4} + y^2 = \frac{5}{16}$$
$$y^2 = \frac{1}{16}$$
$$y = \pm\frac{1}{4}$$

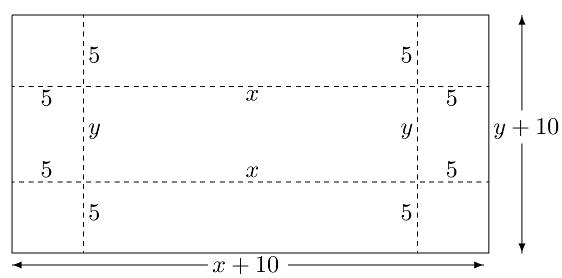
We get $\left(\frac{1}{2}, \frac{1}{4}\right)$, $\left(-\frac{1}{2}, \frac{1}{4}\right)$, $\left(\frac{1}{2}, -\frac{1}{4}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{4}\right)$.

Check. It is true that $\left(\pm\frac{1}{2}\right)^2$ exceeds twice $\left(\pm\frac{1}{4}\right)^2$ by $\frac{1}{8}$: $\frac{1}{4} = 2\left(\frac{1}{16}\right) + \frac{1}{8}$

Also $\left(\pm\frac{1}{2}\right)^2 + \left(\pm\frac{1}{4}\right)^2 = \frac{5}{16}$. The pairs check.

State. The numbers are $\frac{1}{2}$ and $\frac{1}{4}$ or $-\frac{1}{2}$ and $\frac{1}{4}$ or $\frac{1}{2}$ and $-\frac{1}{4}$ or $-\frac{1}{2}$ and $-\frac{1}{4}$.

98. Make a drawing.



We let x and y represent the length and width of the base of the box, respectively. Then the dimensions of the metal sheet are $x + 10$ and $y + 10$.

Solve the system

$$(x+10)(y+10) = 340,$$
$$x \cdot y \cdot 5 = 350.$$

The solutions are $(10, 7)$ and $(7, 10)$. The dimensions of the box are 10 in. by 7 in. by 5 in.

99. See the answer section in the text.

100. $x^2 - y^2 = a^2 - b^2, \quad (1)$

$x - y = a - b \quad (2)$

Solve equation (2) for x.

$x = y + a - b \quad (3)$

Substitute for x in equation (1) and solve for y.

$$(y+a-b)^2 - y^2 = a^2 - b^2$$
$$y^2 + 2ay - 2by + a^2 - 2ab + b^2 - y^2 = a^2 - b^2$$
$$2ay - 2by = 2ab - 2b^2$$
$$2y(a-b) = 2b(a-b)$$
$$y = b$$

Substitute for y in equation (3) and solve for x.

$x = b + a - b = a$

The pair (a, b) checks.

101. $x^3 + y^3 = 72, \quad (1)$

$x + y = 6 \quad (2)$

Solve equation (2) for y: $y = 6 - x$

Substitute for y in equation (1) and solve for x.

$$x^3 + (6-x)^3 = 72$$
$$x^3 + 216 - 108x + 18x^2 - x^3 = 72$$
$$18x^2 - 108x + 144 = 0$$
$$x^2 - 6x + 8 = 0 \quad \text{Multiplying by } \frac{1}{18}$$
$$(x-4)(x-2) = 0$$

$x = 4$ or $x = 2$

If $x = 4$, then $y = 6 - 4 = 2$.

If $x = 2$, then $y = 6 - 2 = 4$.

The pairs $(4, 2)$ and $(2, 4)$ check.

102. $a + b = \frac{5}{6}, \quad (1)$

$\frac{a}{b} + \frac{b}{a} = \frac{13}{6} \quad (2)$

$b = \frac{5}{6} - a = \frac{5-6a}{6} \quad$ Solving equation (1) for b

$$\frac{a}{\frac{5-6a}{6}} + \frac{\frac{5-6a}{6}}{a} = \frac{13}{6} \quad \text{Substituting for } b \text{ in equation (2)}$$
$$\frac{6a}{5-6a} + \frac{5-6a}{6a} = \frac{13}{6}$$
$$36a^2 + 25 - 60a + 36a^2 = 65a - 78a^2$$
$$150a^2 - 125a + 25 = 0$$
$$6a^2 - 5a + 1 = 0$$
$$(3a-1)(2a-1) = 0$$

$a = \frac{1}{3}$ or $a = \frac{1}{2}$

Substitute for a and solve for b.

When $a = \frac{1}{3}$, $b = \dfrac{5 - 6\left(\frac{1}{3}\right)}{6} = \frac{1}{2}$.

When $a = \frac{1}{2}$, $b = \dfrac{5 - 6\left(\frac{1}{2}\right)}{6} = \frac{1}{3}$.

The pairs $\left(\frac{1}{3}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{1}{3}\right)$ check. They are the solutions.

103. $p^2 + q^2 = 13, \quad (1)$

$\frac{1}{pq} = -\frac{1}{6} \quad (2)$

Solve equation (2) for p.

$$\frac{1}{q} = -\frac{p}{6}$$
$$-\frac{6}{q} = p$$

Substitute $-6/q$ for p in equation (1) and solve for q.

$$\left(-\frac{6}{q}\right)^2 + q^2 = 13$$
$$\frac{36}{q^2} + q^2 = 13$$
$$36 + q^4 = 13q^2$$
$$q^4 - 13q^2 + 36 = 0$$
$$u^2 - 13u + 36 = 0 \quad \text{Letting } u = q^2$$
$$(u-9)(u-4) = 0$$
$$u = 9 \quad \text{or} \quad u = 4$$
$$x^2 = 9 \quad \text{or} \quad x^2 = 4$$
$$x = \pm 3 \quad \text{or} \quad x = \pm 2$$

Since $p = -6/q$, if $q = 3$, $p = -2$; if $q = -3$, $p = 2$; if $q = 2$, $p = -3$; and if $q = -2$, $p = 3$. The pairs $(-2, 3)$, $(2, -3)$, $(-3, 2)$, and $(3, -2)$ check. They are the solutions.

104. $x^2 + y^2 = 4, \quad (1)$

$(x-1)^2 + y^2 = 4 \quad (2)$

Solve equation (1) for y^2.

$y^2 = 4 - x^2 \quad (3)$

Substitute for $4 - x^2$ for y^2 in equation (2) and solve for x.

$$(x-1)^2 + (4 - x^2) = 4$$
$$x^2 - 2x + 1 + 4 - x^2 = 4$$
$$-2x = -1$$
$$x = \frac{1}{2}$$

Substitute $\frac{1}{2}$ for x in equation (3) and solve for y.

$$y^2 = 4 - \left(\frac{1}{2}\right)^2$$
$$y^2 = 4 - \frac{1}{4}$$
$$y^2 = \frac{15}{4}$$
$$y = \pm\frac{\sqrt{15}}{2}$$

The pairs $\left(\frac{1}{2}, \frac{\sqrt{15}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{15}}{2}\right)$ check. They are the solutions.

105. $5^{x+y} = 100,$

$3^{2x-y} = 1000$

$(x+y)\log 5 = 2, \quad$ Taking logarithms and

$(2x-y)\log 3 = 3 \quad$ simplifying

$x\log 5 + y\log 5 = 2, \quad (1)$

$2x\log 3 - y\log 3 = 3 \quad (2)$

Multiply equation (1) by $\log 3$ and equation (2) by $\log 5$ and add.

$$\begin{aligned} x\log 3\cdot\log 5 + y\log 3\cdot\log 5 &= 2\log 3\\ 2x\log 3\cdot\log 5 - y\log 3\cdot\log 5 &= 3\log 5\\ \hline 3x\log 3\cdot\log 5 \qquad\qquad &= 2\log 3+\\ &\quad 3\log 5 \end{aligned}$$

$$x = \frac{2\log 3+3\log 5}{3\log 3\cdot\log 5}$$

Substitute in (1) to find y.

$$\frac{2\log 3+3\log 5}{3\log 3\cdot\log 5}\cdot\log 5 + y\log 5 = 2$$

$$y\log 5 = 2-\frac{2\log 3+3\log 5}{3\log 3}$$

$$y\log 5 = \frac{6\log 3-2\log 3-3\log 5}{3\log 3}$$

$$y\log 5 = \frac{4\log 3-3\log 5}{3\log 3}$$

$$y = \frac{4\log 3-3\log 5}{3\log 3\cdot\log 5}$$

The pair $\left(\dfrac{2\log 3+3\log 5}{3\log 3\cdot\log 5}, \dfrac{4\log 3-3\log 5}{3\log 3\cdot\log 5}\right)$ checks. It is the solution.

106. $e^x - e^{x+y} = 0,\quad (1)$
$e^y - e^{x-y} = 0\quad (2)$

Factor (1): $e^x(1-e^y)=0$

$e^x = 0\qquad$ or $\qquad 1-e^y = 0$
No solution $\qquad\qquad y = 0$

Substitute in (2).

$$\begin{aligned} e^0 - e^{x-0} &= 0\\ 1-e^x &= 0\\ x &= 0 \end{aligned}$$

The solution is $(0,0)$.

Chapter 10 Mid-Chapter Mixed Review

1. The equation $(x+3)^2 = 8(y-2)$ is equivalent to the equation $[x-(-3)]^2 = 4\cdot 2(y-2)$, so the given statement is true. See page 835 in the text.

2. The equation $(x-4)^2+(y+1)^2 = 9$ is equivalent to the equation $(x-4)^2+[y-(-1)]^2 = 3^2$. This is the equation of a circle with center $(4,-1)$ and radius 3, so the given statement is false.

3. True; see page 852 in the text.

4. False; see Example 2 on page 864 in the text.

5. Graph (b) is the graph of $x^2 = -4y$.

6. Graph (h) is the graph of $(y+2)^2 = 4(x-2)$.

7. Graph (d) is the graph of $16x^2+9y^2 = 144$.

8. Graph (a) is the graph of $x^2+y^2 = 16$.

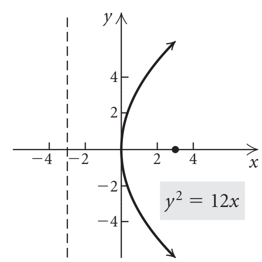
9. Graph (g) is the graph of $4(y-1)^2-9(x+2)^2 = 36$.

10. Graph (f) is the graph of $4(x+1)^2+9(y-2)^2 = 36$.

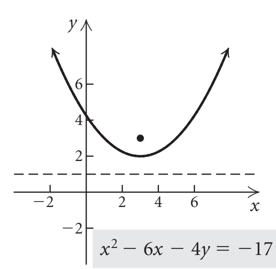
11. Graph (h) is the graph of $(x-2)^2+(y+3)^2 = 4$.

12. Graph (c) is the graph of $25x^2-4y^2 = 100$.

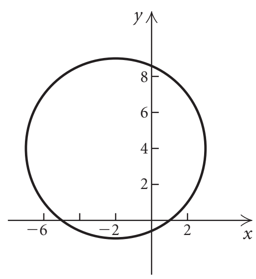
13. $y^2 = 12x$
$y^2 = 4\cdot 3\cdot x$
Vertex: $(0,0)$
Focus: $(3,0)$
Directrix: $x=-3$



14. $$\begin{aligned} x^2-6x-4y &= -17\\ x^2-6x &= 4y-17\\ x^2-6x+9 &= 4y-17+9\\ (x-3)^2 &= 4y-8\\ (x-3)^2 &= 4(y-2)\\ (x-3)^2 &= 4\cdot 1\cdot(y-2) \end{aligned}$$
$V:\ (3,2)$
$F:\ (3,2+1)$, or $(3,3)$
$D:\ y=2-1$, or $y=1$



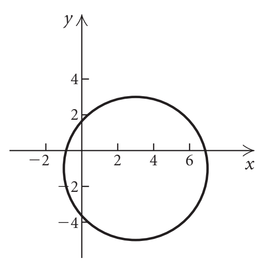
15. $$\begin{aligned} x^2+y^2+4x-8y &= 5\\ x^2+4x+y^2-8y &= 5\\ x^2+4x+4+y^2-8y+16 &= 5+4+16\\ (x+2)^2+(y-4)^2 &= 25\\ [x-(-2)]^2+(y-4)^2 &= 5^2 \end{aligned}$$
Center: $(-2,4)$; radius: 5



$x^2 + y^2 + 4x - 8y = 5$

16.
$$\begin{aligned} x^2+y^2-6x+2y-6&=0\\ x^2-6x+y^2+2y&=6\\ x^2-6x+9+y^2+2y+1&=6+9+1\\ (x-3)^2+(y+1)^2&=16\\ (x-3)^2+[y-(-1)]^2&=4^2 \end{aligned}$$

Center: $(3, -1)$; radius: 4



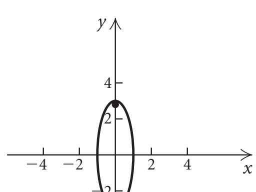
$x^2 + y^2 - 6x + 2y - 6 = 0$

17. $\dfrac{x^2}{1} + \dfrac{y^2}{9} = 1$

$\dfrac{x^2}{1^2} + \dfrac{y^2}{3^2} = 1$

$a = 3$, $b = 1$

The major axis is vertical, so the vertices are $(0, -3)$ and $(0, 3)$. Since $c^2 = a^2 - b^2$ we have $c^2 = 9 - 1 = 8$, so $c = \sqrt{8}$, or $2\sqrt{2}$, and the foci are $(0, -2\sqrt{2})$ and $(0, 2\sqrt{2})$.



$\dfrac{x^2}{1} + \dfrac{y^2}{9} = 1$

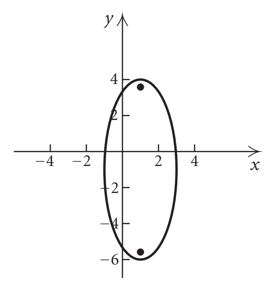
18.
$$\begin{aligned} 25x^2+4y^2-50x+8y&=71\\ 25(x^2-2x)+4(y^2+2y)&=71\\ 25(x^2-2x+1-1)+4(y^2+2y+1-1)&=71\\ 25(x-1)^2+4(y+1)^2&=71+25+4\\ 25(x-1)^2+4(y+1)^2&=100\\ \frac{(x-1)^2}{4}+\frac{(y+1)^2}{25}&=1\\ \frac{(x-1)^2}{2^2}+\frac{[y-(-1)]^2}{5^2}&=1 \end{aligned}$$

Center: $(1, -1)$; $a = 5$, $b = 2$

Vertices: $(1, -1-5)$ and $(1, -1+5)$, or $(1, -6)$ and $(1, 4)$

$c^2 = 25 - 4 = 21$, so $c = \sqrt{21}$

Foci: $(1, -1-\sqrt{21})$ and $(1, -1+\sqrt{21})$



$25x^2 + 4y^2 - 50x + 8y = 71$

19. $9y^2 - 16x^2 = 144$

$\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

$\dfrac{y^2}{4^2} - \dfrac{x^2}{3^2} = 1$

The center is $(0, 0)$; $a = 4$, $b = 3$. The transverse axis is vertical, so the vertices are $(0, -4)$ and $(0, 4)$. Since $c^2 = a^2 + b^2$, we have $c^2 = 16 + 9 = 25$, and $c = 5$. Then the foci are $(0, -5)$ and $(0, 5)$.

Find the asymptotes:

$$y = -\frac{a}{b}x \quad \text{and} \quad y = \frac{a}{b}x$$

$$y = -\frac{4}{3}x \quad \text{and} \quad y = \frac{4}{3}x$$

The graph is on page A-67 in the text.

20. $\dfrac{(x+3)^2}{1} - \dfrac{(y-2)^2}{4} = 1$

$\dfrac{[x-(-3)]^2}{1^2} - \dfrac{(y-2)^2}{2^2} = 1$

The center is $(-3, 2)$; $a = 1$, $b = 2$. The transverse axis is horizontal, so the vertices are 1 unit left and right of the center:

$$(-3-1, 2) \text{ and } (-3+1, 2), \text{ or } (-4, 2) \text{ and } (-2, 2).$$

Since $c^2 = a^2 + b^2$, we have $c^2 = 1 + 4 = 5$ and $c = \sqrt{5}$. Then the foci are $\sqrt{5}$ units left and right of the center:

$$(-3-\sqrt{5}, 2) \text{ and } (-3+\sqrt{5}, 2).$$

Find the asymptotes:

$$y-k=\frac{b}{a}(x-h) \quad \text{and} \quad y-k=-\frac{b}{a}(x-h)$$

$$y-2=\frac{2}{1}[x-(-3)] \quad \text{and} \quad y-2=-\frac{2}{1}[x-(-3)]$$

$$y-2=2(x+3) \quad \text{and} \quad y-2=-2(x+3)$$

The graph is on page A-67 in the text.

21. $x^2+y^2=29$, (1)
$x-y=3$ (2)

First solve equation (2) for x.

$$x-y=3$$
$$x=y+3$$

Substitute $y+3$ for x in equation (1) and solve for y.

$$(y+3)^2+y^2=29$$
$$y^2+6y+9+y^2=29$$
$$2y^2+6y+9=29$$
$$2y^2+6y-20=0$$
$$y^2+3y-10=0 \quad \text{Dividing by 2}$$
$$(y+5)(y-2)=0$$
$$y+5=0 \quad \textit{or} \quad y-2=0$$
$$y=-5 \quad \textit{or} \quad y=2$$

When $y=-5$, $x=y+3=-5+3=-2$.

When $y=2$, $x=y+3=2+3=5$.

The solutions are $(-2,-5)$ and $(5,2)$.

22. $x^2+y^2=8$, (1)
$xy=4$ (2)

Solve equation (2) for y.

$$y=\frac{4}{x}$$

Substitute $\frac{4}{x}$ for y in equation (1) and solve for x.

$$x^2+\left(\frac{4}{x}\right)^2=8$$
$$x^2+\frac{16}{x^2}=8$$
$$x^4+16=8x^2$$
$$x^4-8x^2+16=0$$

Let $u=x^2$.

$$u^2-8u+16=0$$
$$(u-4)^2=0$$
$$u-4=0 \quad \textit{or} \quad u-4=0$$
$$u=4 \quad \textit{or} \quad u=4$$
$$x^2=4 \quad \textit{or} \quad x^2=4$$
$$x=\pm 2 \quad \textit{or} \quad x=\pm 2$$

When $x=2$, $y=\frac{4}{2}=2$.

When $x=-2$, $y=\frac{4}{-2}=-2$.

The solutions are $(2,2)$ and $(-2,-2)$.

23. $x^2+2y^2=20$, (1)
$y^2-x^2=28$ (2)

$$\begin{array}{r} x^2+2y^2=20 \\ -x^2+y^2=28 \\ \hline 3y^2=48 \end{array} \quad \text{Adding}$$
$$y^2=16$$
$$y=\pm 4$$

Substitute in equation (1) to find the x-values that correspond to these y-values.

$$x^2+2(\pm 4)^2=20$$
$$x^2+2\cdot 16=20$$
$$x^2+32=20$$
$$x^2=-12$$
$$x=\pm 2\sqrt{3}i$$

The solutions are $(2\sqrt{3}i,4)$, $(2\sqrt{3}i,-4)$, $(-2\sqrt{3}i,4)$, and $(-2\sqrt{3}i,-4)$.

24. $2x-y=-4$, (1)
$3x^2+2y=7$ (2)

$$\begin{array}{r} 4x-2y=-8 \\ 3x^2+2y=7 \\ \hline 3x^2+4x=-1 \end{array}$$
$$3x^2+4x+1=0$$
$$(3x+1)(x+1)=0$$
$$3x+1=0 \quad \textit{or} \quad x+1=0$$
$$x=-\frac{1}{3} \quad \textit{or} \quad x=-1$$

When $x=-\frac{1}{3}$, we have

$$2\left(-\frac{1}{3}\right)-y=-4$$
$$-\frac{2}{3}-y=-4$$
$$-y=-\frac{10}{3}$$
$$y=\frac{10}{3}.$$

When $x=-1$, we have

$$2(-1)-y=-4$$
$$-2-y=-4$$
$$-y=-2$$
$$y=2.$$

The solutions are $\left(-\frac{1}{3},\frac{10}{3}\right)$ and $(-1,2)$.

25. ***Familiarize***. Let x and y represent the numbers.

Translate. The sum of the numbers is 1, so we have one equation:

$$x+y=1.$$

The sum of the squares of the numbers is 13, so we have a second equation:

$x^2 + y^2 = 13.$

Carry out. We solve the system of equations

$$x + y = 1, \quad (1)$$
$$x^2 + y^2 = 13. \quad (2)$$

First solve equation (1) for y.

$$x + y = 1$$
$$y = -x + 1$$

Substitute $-x + 1$ for y in equation (2) and solve for x.

$$x^2 + (-x + 1)^2 = 13$$
$$x^2 + x^2 - 2x + 1 = 13$$
$$2x^2 - 2x + 1 = 13$$
$$2x^2 - 2x - 12 = 0$$
$$x^2 - x - 6 = 0 \quad \text{Dividing by 2}$$
$$(x - 3)(x + 2) = 0$$
$$x - 3 = 0 \quad or \quad x + 2 = 0$$
$$x = 3 \quad or \quad x = -2$$

When $x = 3$, $y = -x + 1 = -3 + 1 = -2$.

When $x = -2$, $y = -x + 1 = -(-2) + 1 = 3$.

In either case, we find that the numbers are 3 and -2.

Check. $3 + (-2) = 1$ and $3^2 + (-2)^2 = 9 + 4 = 13$, so the solution checks.

State. The numbers are 3 and -2.

26. Graph: $x^2 + y^2 \le 8$,
$x > y$

The graph is on page A-67 in the text.

27. Graph: $y \ge x^2 - 1$,
$y \le x + 1$

The solution set of $y \ge x^2 - 1$ is the parabola $y = x^2 - 1$ and the region inside it. The solution set of $y \le x + 1$ is the line $y = x + 1$ and the half-plane below it. We shade the region common to the two solution sets.

The graph is on page A-67 in the text.

To find the points of intersection of the graphs we solve the system of equations

$$y = x^2 - 1,$$
$$y = x + 1.$$

The points of intersection are $(-1, 0)$ and $(2, 3)$.

28. No; parabolas with a horizontal axis of symmetry fail the vertical-line test.

29. No, the center of an ellipse is not part of the graph of the ellipse. Its coordinates do not satisfy the equation of the ellipse.

30. No; the asymptotes of a hyperbola are not part of the graph of the hyperbola. The coordinates of points on the asymptotes do not satisfy the equation of the hyperbola.

31. Although we can always visualize the real-number solutions, we cannot visualize the imaginary-number solutions.

Exercise Set 10.5

1. We use the rotation of axes formulas to find x' and y'.

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ &= \sqrt{2}\cos 45^\circ - \sqrt{2}\sin 45^\circ \\ &= \sqrt{2}\cdot\frac{\sqrt{2}}{2} - \sqrt{2}\cdot\frac{\sqrt{2}}{2} \\ &= 1 - 1 = 0 \end{aligned}$$

$$\begin{aligned} y' &= -x\sin\theta + y\cos\theta \\ &= -\sqrt{2}\sin 45^\circ - \sqrt{2}\cos 45^\circ \\ &= -\sqrt{2}\cdot\frac{\sqrt{2}}{2} - \sqrt{2}\cdot\frac{\sqrt{2}}{2} \\ &= -1 - 1 = -2 \end{aligned}$$

The coordinates are $(0, -2)$.

2.
$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ &= -1\cdot\cos 45^\circ + 3\sin 45^\circ \\ &= -\frac{\sqrt{2}}{2} + 3\cdot\frac{\sqrt{2}}{2} \\ &= \frac{2\sqrt{2}}{2} = \sqrt{2} \end{aligned}$$

$$\begin{aligned} y' &= -x\sin\theta + y\cos\theta \\ &= -(-1)\sin 45^\circ + 3\cos 45^\circ \\ &= \frac{\sqrt{2}}{2} + 3\cdot\frac{\sqrt{2}}{2} \\ &= \frac{4\sqrt{2}}{2} = 2\sqrt{2} \end{aligned}$$

The coordinates are $(\sqrt{2}, 2\sqrt{2})$.

3. We use the rotation of axes formulas to find x' and y'.

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ &= 0\cdot\cos 30^\circ + 2\sin 30^\circ \\ &= 0 + 2\cdot\frac{1}{2} = 1 \end{aligned}$$

$$\begin{aligned} y' &= -x\sin\theta + y\cos\theta \\ &= -0\cdot\sin 30^\circ + 2\cos 30^\circ \\ &= 0 + 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3} \end{aligned}$$

The coordinates are $(1, \sqrt{3})$.

4.
$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ &= 0\cdot\cos 60^\circ + \sqrt{3}\sin 60^\circ \\ &= 0 + \sqrt{3}\cdot\frac{\sqrt{3}}{2} = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} y' &= -x\sin\theta + y\cos\theta \\ &= -0\cdot\sin 60^\circ + \sqrt{3}\cos 60^\circ \\ &= 0 + \sqrt{3}\cdot\frac{1}{2} = \frac{\sqrt{3}}{2} \end{aligned}$$

The coordinates are $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$.

5. We use the rotation of axes formulas to find x and y.

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ &= 1\cdot\cos 45^\circ - (-1)\sin 45^\circ \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\ &= \frac{2\sqrt{2}}{2} = \sqrt{2} \end{aligned}$$

$$\begin{aligned} y &= x'\sin\theta + y'\cos\theta \\ &= 1\cdot\sin 45^\circ - 1\cdot\cos 45^\circ \\ &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0 \end{aligned}$$

The coordinates are $(\sqrt{2}, 0)$.

6.
$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ &= -3\sqrt{2}\cos 45^\circ - \sqrt{2}\sin 45^\circ \\ &= -3\sqrt{2}\cdot\frac{\sqrt{2}}{2} - \sqrt{2}\cdot\frac{\sqrt{2}}{2} \\ &= -3 - 1 = -4 \end{aligned}$$

$$\begin{aligned} y &= x'\sin\theta + y'\cos\theta \\ &= -3\sqrt{2}\sin 45^\circ + \sqrt{2}\cos 45^\circ \\ &= -3\sqrt{2}\cdot\frac{\sqrt{2}}{2} + \sqrt{2}\cdot\frac{\sqrt{2}}{2} \\ &= -3 + 1 = -2 \end{aligned}$$

The coordinates are $(-4, -2)$.

7. We use the rotation of axes formulas to find x and y.

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ &= 2\cos 30^\circ - 0\cdot\sin 30^\circ \\ &= 2\cdot\frac{\sqrt{3}}{2} - 0 = \sqrt{3} \end{aligned}$$

$$\begin{aligned} y &= x'\sin\theta + y'\cos\theta \\ &= 2\sin 30^\circ + 0\cdot\cos 30^\circ \\ &= 2\cdot\frac{1}{2} + 0 = 1 \end{aligned}$$

The coordinates are $(\sqrt{3}, 1)$.

8.
$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ &= -1\cdot\cos 60^\circ - (-\sqrt{3})\sin 60^\circ \\ &= -\frac{1}{2} + \sqrt{3}\cdot\frac{\sqrt{3}}{2} \\ &= -\frac{1}{2} + \frac{3}{2} = 1 \end{aligned}$$

$$\begin{aligned} y &= x'\sin\theta + y'\cos\theta \\ &= -1\cdot\sin 60^\circ - \sqrt{3}\cos 60^\circ \\ &= -\frac{\sqrt{3}}{2} - \sqrt{3}\cdot\frac{1}{2} \\ &= -\frac{2\sqrt{3}}{2} = -\sqrt{3} \end{aligned}$$

The coordinates are $(1, -\sqrt{3})$.

9. $3x^2 - 5xy + 3y^2 - 2x + 7y = 0$

$A = 3,\ B = -5,\ C = 3$

$B^2 - 4AC = (-5)^2 - 4\cdot 3\cdot 3 = 25 - 36 = -11$

Since the discriminant is negative, the graph is an ellipse (or circle).

10. $5x^2 + 6xy - 4y^2 + x - 3y + 4 = 0$

$B^2 - 4AC = 6^2 - 4\cdot 5\cdot(-4) = 36 + 80 = 116$

Since the discriminant is positive, the graph is a hyperbola.

11. $x^2 - 3xy - 2y^2 + 12 = 0$

$A = 1,\ B = -3,\ C = -2$

$B^2 - 4AC = (-3)^2 - 4\cdot 1\cdot(-2) = 9 + 8 = 17$

Since the discriminant is positive, the graph is a hyperbola.

12. $4x^2 + 7xy + 2y^2 - 3x + y = 0$

$B^2 - 4AC = 7^2 - 4\cdot 4\cdot 2 = 49 - 32 = 17$

Since the discriminant is positive, the graph is a hyperbola.

13. $4x^2 - 12xy + 9y^2 - 3x + y = 0$

$A = 4,\ B = -12,\ C = 9$

$B^2 - 4AC = (-12)^2 - 4\cdot 4\cdot 9 = 144 - 144 = 0$

Since the discriminant is zero, the graph is a parabola.

14. $6x^2 + 5xy + 6y^2 + 15 = 0$

$B^2 - 4AC = 5^2 - 4\cdot 6\cdot 6 = 25 - 144 = -119$

Since the discriminant is negative, the graph is an ellipse (or circle).

15. $2x^2 - 8xy + 7y^2 + x - 2y + 1 = 0$

$A = 2,\ B = -8,\ C = 7$

$B^2 - 4AC = (-8)^2 - 4\cdot 2\cdot 7 = 64 - 56 = 8$

Since the discriminant is positive, the graph is a hyperbola.

16. $x^2 + 6xy + 9y^2 - 3x + 4y = 0$

$B^2 - 4AC = 6^2 - 4\cdot 1\cdot 9 = 36 - 36 = 0$

Since the discriminant is zero, the graph is a parabola.

17. $8x^2 - 7xy + 5y^2 - 17 = 0$

$A = 8,\ B = -7,\ C = 5$

$B^2 - 4AC = (-7)^2 - 4\cdot 8\cdot 5 = 49 - 160 = -111$

Since the discriminant is negative, the graph is an ellipse (or circle).

18. $x^2 + xy - y^2 - 4x + 3y - 2 = 0$

$B^2 - 4AC = 1^2 - 4\cdot 1\cdot(-1) = 1 + 4 = 5$

Since the discriminant is positive, the graph is a hyperbola.

19. $4x^2 + 2xy + 4y^2 = 15$

$A = 4,\ B = 2,\ C = 4$

$B^2 - 4AC = 2^2 - 4\cdot 4\cdot 4 = 4 - 64 = -60$

Since the discriminant is negative, the graph is an ellipse (or circle). To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{4-4}{2} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'-y')$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'+y')$$

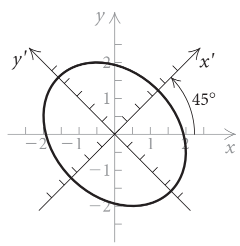
Substitute for x and y in the given equation.

$$4\left[\frac{\sqrt{2}}{2}(x'-y')\right]^2 + 2\left[\frac{\sqrt{2}}{2}(x'-y')\right]\left[\frac{\sqrt{2}}{2}(x'+y')\right] + 4\left[\frac{\sqrt{2}}{2}(x'+y')\right]^2 = 15$$

After simplifying we have

$$\frac{(x')^2}{3} + \frac{(y')^2}{5} = 1.$$

This is the equation of an ellipse with vertices $(0, -\sqrt{5})$ and $(0, \sqrt{5})$ on the y'-axis. The x'-intercepts are $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$. We sketch the graph.



$$\frac{(x')^2}{3} + \frac{(y')^2}{5} = 1$$

20. $3x^2 + 10xy + 3y^2 + 8 = 0$

$B^2 - 4AC = 10^2 - 4\cdot 3\cdot 3 = 100 - 36 = 64$

Since the discriminant is positive, the graph is a hyperbola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{3-3}{10} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

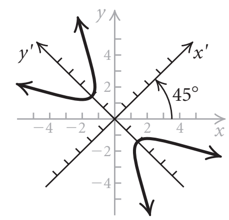
Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'-y')$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'+y')$$

After substituting for x and y in the given equation and simplifying, we have

$$\frac{(y')^2}{4} - \frac{(x')^2}{1} = 1.$$



$$\frac{(y')^2}{4} - \frac{(x')^2}{1} = 1$$

21. $x^2 - 10xy + y^2 + 36 = 0$

$A = 1,\ B = -10,\ C = 1$

$B^2 - 4AC = (-10)^2 - 4\cdot 1\cdot 1 = 100 - 4 = 96$

Since the discriminant is positive, the graph is a hyperbola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{1-1}{-10} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'-y')$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'+y')$$

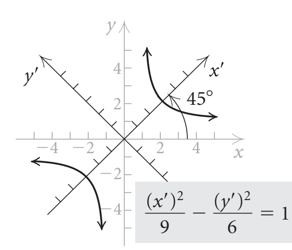
Substitute for x and y in the given equation.

$$\left[\frac{\sqrt{2}}{2}(x'-y')\right]^2 - 10\left[\frac{\sqrt{2}}{2}(x'-y')\right]\left[\frac{\sqrt{2}}{2}(x'+y')\right] + \left[\frac{\sqrt{2}}{2}(x'+y')\right]^2 + 36 = 0$$

After simplifying we have

$$\frac{(x')^2}{9} - \frac{(y')^2}{6} = 1.$$

This is the equation of a hyperbola with vertices $(-3, 0)$ and $(3, 0)$ on the x'-axis. The asymptotes are $y' = -\frac{\sqrt{6}}{3}x'$ and $y' = \frac{\sqrt{6}}{3}x'$. We sketch the graph.



22. $x^2 + 2xy + y^2 + 4\sqrt{2}x - 4\sqrt{2}y = 0$

$B^2 - 4AC = 2^2 - 4 \cdot 1 \cdot 1 = 4 - 4 = 0$

Since the discriminant is zero, the graph is a parabola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{1-1}{2} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

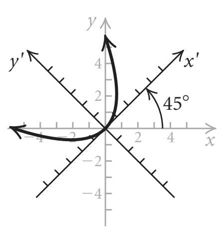
Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'-y')$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'+y')$$

After substituting for x and y in the given equation and simplifying, we have

$$(x')^2 = 4y'.$$



$(x')^2 = 4y'$

23. $x^2 - 2\sqrt{3}xy + 3y^2 - 12\sqrt{3}x - 12y = 0$

$A = 1,\ B = -2\sqrt{3},\ C = 3$

$B^2 - 4AC = (-2\sqrt{3})^2 - 4 \cdot 1 \cdot 3 = 12 - 12 = 0$

Since the discriminant is zero, the graph is a parabola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{1-3}{-2\sqrt{3}} = \frac{-2}{-2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Then $2\theta = 60°$ and $\theta = 30°$, so

$$\sin\theta = \frac{1}{2} \text{ and } \cos\theta = \frac{\sqrt{3}}{2}.$$

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x' \cdot \frac{\sqrt{3}}{2} - y' \cdot \frac{1}{2} = \frac{x'\sqrt{3}}{2} - \frac{y'}{2}$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x' \cdot \frac{1}{2} + y' \cdot \frac{\sqrt{3}}{2} = \frac{x'}{2} + \frac{y'\sqrt{3}}{2}$$

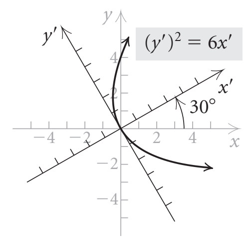
Substitute for x and y in the given equation.

$$\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^2 - 2\sqrt{3}\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right) +$$
$$3\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right)^2 - 12\sqrt{3}\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right) -$$
$$12\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right) = 0$$

After simplifying we have

$$(y')^2 = 6x'.$$

This is the equation of a parabola with vertex at $(0,0)$ of the $x'y'$-coordinate system and axis of symmetry $y' = 0$. We sketch the graph.



24. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$

$B^2 - 4AC = (6\sqrt{3})^2 - 4 \cdot 13 \cdot 7 = 108 - 364 = -256$

Since the discriminant is negative, the graph is an ellipse or a circle. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{13-7}{6\sqrt{3}} = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Then $2\theta = 60°$ and $\theta = 30°$, so

$$\sin\theta = \frac{1}{2} \text{ and } \cos\theta = \frac{\sqrt{3}}{2}.$$

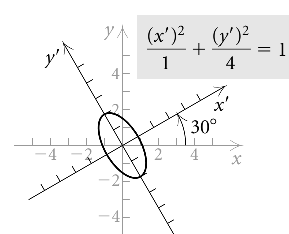
Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x' \cdot \frac{\sqrt{3}}{2} - y' \cdot \frac{1}{2} = \frac{x'\sqrt{3}}{2} - \frac{y'}{2}$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x' \cdot \frac{1}{2} + y' \cdot \frac{\sqrt{3}}{2} = \frac{x'}{2} + \frac{y'\sqrt{3}}{2}$$

After substituting for x and y in the given equation and simplifying, we have

$$\frac{(x')^2}{1} + \frac{(y')^2}{4} = 1.$$



25. $7x^2 + 6\sqrt{3}xy + 13y^2 - 32 = 0$

$A = 7,\ B = 6\sqrt{3},\ C = 13$

$B^2 - 4AC = (6\sqrt{3})^2 - 4 \cdot 7 \cdot 13 = 108 - 364 = -256$

Since the discriminant is negative, the graph is an ellipse or a circle. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{7-13}{6\sqrt{3}} = \frac{-6}{6\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

Then $2\theta = 120°$ and $\theta = 60°$, so

$\sin\theta = \dfrac{\sqrt{3}}{2}$ and $\cos\theta = \dfrac{1}{2}$.

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\cdot\frac{1}{2} - y'\cdot\frac{\sqrt{3}}{2} = \frac{x'}{2} - \frac{y'\sqrt{3}}{2}$$
$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\cdot\frac{\sqrt{3}}{2} + y'\cdot\frac{1}{2} = \frac{x'\sqrt{3}}{2} + \frac{y'}{2}$$

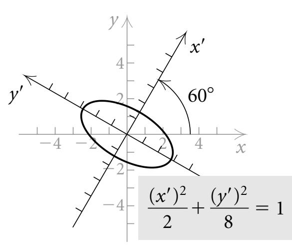
Substitute for x and y in the given equation.

$$7\left(\frac{x'}{2} - \frac{y'\sqrt{3}}{2}\right)^2 + 6\sqrt{3}\left(\frac{x'}{2} - \frac{y'\sqrt{3}}{2}\right)\left(\frac{x'\sqrt{3}}{2} + \frac{y'}{2}\right) + 13\left(\frac{x'\sqrt{3}}{2} + \frac{y'}{2}\right)^2 - 32 = 0$$

After simplifying we have

$$\frac{(x')^2}{2} + \frac{(y')^2}{8} = 1.$$

This is the equation of an ellipse with vertices $(0, -\sqrt{8})$ and $(0, \sqrt{8})$, or $(0, -2\sqrt{2})$ and $(0, 2\sqrt{2})$ on the y'-axis. The x'-intercepts are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$. We sketch the graph.



26. $x^2 + 4xy + y^2 - 9 = 0$

$B^2 - 4AC = 4^2 - 4\cdot 1\cdot 1 = 16 - 4 = 12$

Since the discriminant is positive, the graph is a hyperbola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{1-1}{4} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

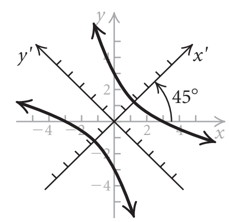
$\sin\theta = \dfrac{\sqrt{2}}{2}$ and $\cos\theta = \dfrac{\sqrt{2}}{2}$.

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y')$$
$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y')$$

After substituting for x and y in the given equation and simplifying, we have

$$\frac{(x')^2}{3} - \frac{(y')^2}{9} = 1.$$



$$\frac{(x')^2}{3} - \frac{(y')^2}{9} = 1$$

27. $11x^2 + 10\sqrt{3}xy + y^2 = 32$

$A = 11,\ B = 10\sqrt{3},\ C = 1$

$B^2 - 4AC = (10\sqrt{3})^2 - 4\cdot 11\cdot 1 = 300 - 44 = 256$

Since the discriminant is positive, the graph is a hyperbola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{11-1}{10\sqrt{3}} = \frac{10}{10\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Then $2\theta = 60°$ and $\theta = 30°$, so

$\sin\theta = \dfrac{1}{2}$ and $\cos\theta = \dfrac{\sqrt{3}}{2}$.

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\cdot\frac{\sqrt{3}}{2} - y'\cdot\frac{1}{2} = \frac{x'\sqrt{3}}{2} - \frac{y'}{2}$$
$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\cdot\frac{1}{2} + y'\cdot\frac{\sqrt{3}}{2} = \frac{x'}{2} + \frac{y'\sqrt{3}}{2}$$

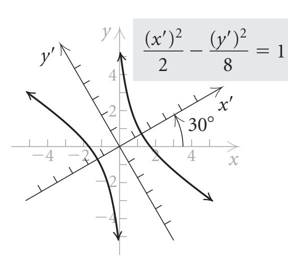
Substitute for x and y in the given equation.

$$11\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^2 + 10\sqrt{3}\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right) + \left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right)^2 = 32$$

After simplifying we have

$$\frac{(x')^2}{2} - \frac{(y')^2}{8} = 1.$$

This is the equation of a hyperbola with vertices $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$ on the x'-axis. The asymptotes are $y' = -\dfrac{\sqrt{8}}{\sqrt{2}}x'$ and $y' = \dfrac{\sqrt{8}}{\sqrt{2}}x'$, or $y' = -2x'$ and $y' = 2x'$. We sketch the graph.



28. $5x^2 - 8xy + 5y^2 = 8$

$B^2 - 4AC = (-8)^2 - 4\cdot 5\cdot 5 = 64 - 100 = -36$

Since the discriminant is negative, the graph is an ellipse or a circle. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{5-5}{-8} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

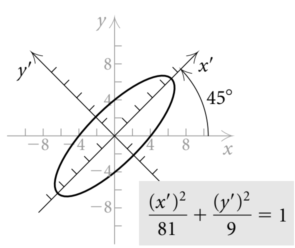
Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'-y')$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'+y')$$

After substituting for x and y in the given equation and simplifying, we have

$$\frac{(x')^2}{81} + \frac{(y')^2}{9} = 1.$$



29. $\sqrt{2}x^2 + 2\sqrt{2}xy + \sqrt{2}y^2 - 8x + 8y = 0$

$A = \sqrt{2},\ B = 2\sqrt{2},\ C = \sqrt{2}$

$B^2 - 4AC = (2\sqrt{2})^2 - 4\cdot\sqrt{2}\cdot\sqrt{2} = 8 - 8 = 0$

Since the discriminant is zero, the graph is a parabola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{\sqrt{2}-\sqrt{2}}{2\sqrt{2}} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'-y')$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x'+y')$$

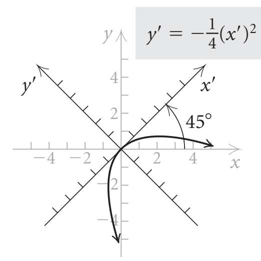
Substitute for x and y in the given equation.

$$\sqrt{2}\left[\frac{\sqrt{2}}{2}(x'-y')\right]^2 + 2\sqrt{2}\left[\frac{\sqrt{2}}{2}(x'-y')\right]\left[\frac{\sqrt{2}}{2}(x'+y')\right] +$$
$$\sqrt{2}\left[\frac{\sqrt{2}}{2}(x'+y')\right]^2 - 8\cdot\frac{\sqrt{2}}{2}(x'-y') + 8\cdot\frac{\sqrt{2}}{2}(x'+y') = 0$$

After simplifying we have

$$y' = -\frac{1}{4}(x')^2.$$

This is the equation of a parabola with vertex at $(0,0)$ of the $x'y'$-coordinate system and axis of symmetry $x' = 0$. We sketch the graph.



30. $x^2 + 2\sqrt{3}xy + 3y^2 - 8x + 8\sqrt{3}y = 0$

$B^2 - 4AC = (2\sqrt{3})^2 - 4\cdot 1\cdot 3 = 12 - 12 = 0$

Since the discriminant is zero, the graph is a parabola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{1-3}{2\sqrt{3}} = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

Then $2\theta = 120°$ and $\theta = 60°$, so

$$\sin\theta = \frac{\sqrt{3}}{2} \text{ and } \cos\theta = \frac{1}{2}.$$

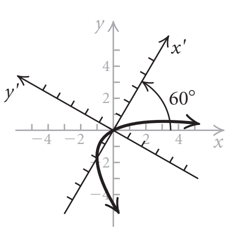
Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\cdot\frac{1}{2} - y'\cdot\frac{\sqrt{3}}{2} = \frac{x'}{2} - \frac{y'\sqrt{3}}{2}$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\cdot\frac{\sqrt{3}}{2} + y'\cdot\frac{1}{2} = \frac{x'\sqrt{3}}{2} + \frac{y'}{2}$$

After substituting for x and y in the given equation and simplifying, we have

$$(x'+1)^2 = -2\sqrt{3}y' + 1.$$



$(x' + 1)^2 = -2\sqrt{3}y' + 1$

31. $x^2 + 6\sqrt{3}xy - 5y^2 + 8x - 8\sqrt{3}y - 48 = 0$

$A = 1,\ B = 6\sqrt{3},\ C = -5$

$B^2 - 4AC = (6\sqrt{3})^2 - 4\cdot 1\cdot(-5) = 108 + 20 = 128$

Since the discriminant is positive, the graph is a hyperbola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{1-(-5)}{6\sqrt{3}} = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Then $2\theta = 60°$ and $\theta = 30°$, so

$$\sin\theta = \frac{1}{2} \text{ and } \cos\theta = \frac{\sqrt{3}}{2}.$$

Now substitute in the rotation of axes formulas.

$$x = x' \cos\theta - y' \sin\theta$$
$$= x' \cdot \frac{\sqrt{3}}{2} - y' \cdot \frac{1}{2} = \frac{x'\sqrt{3}}{2} - \frac{y'}{2}$$
$$y = x' \sin\theta + y' \cos\theta$$
$$= x' \cdot \frac{1}{2} + y' \cdot \frac{\sqrt{3}}{2} = \frac{x'}{2} + \frac{y'\sqrt{3}}{2}$$

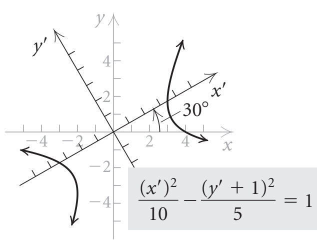
Substitute for x and y in the given equation.

$$\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^2 + 6\sqrt{3}\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right) -$$
$$5\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right)^2 + 8\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right) -$$
$$8\sqrt{3}\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right) - 48 = 0$$

After simplifying we have

$$\frac{(x')^2}{10} - \frac{(y'+1)^2}{5} = 1.$$

This is the equation of a hyperbola with vertices $(-\sqrt{10}, 0)$ and $(\sqrt{10}, 0)$ and asymptotes $y'+1 = -\frac{\sqrt{5}}{\sqrt{10}}x'$ and $y'+1 = \frac{\sqrt{5}}{\sqrt{10}}x'$, or $y'+1 = -\frac{1}{\sqrt{2}}x'$ and $y'+1 = \frac{1}{\sqrt{2}}x'$ We sketch the graph.



32. $3x^2 - 2xy + 3y^2 - 6\sqrt{2}x + 2\sqrt{2}y - 26 = 0$

$B^2 - 4AC = (-2)^2 - 4 \cdot 3 \cdot 3 = 4 - 36 = -32$

Since the discriminant is negative, the graph is an ellipse or a circle. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A - C}{B} = \frac{3-3}{-2} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

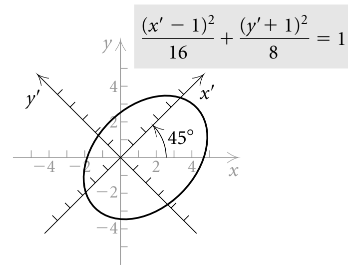
$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

Now substitute in the rotation of axes formulas.

$$x = x' \cos\theta - y' \sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y')$$
$$y = x' \sin\theta + y' \cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y')$$

After substituting for x and y in the given equation and simplifying, we have

$$\frac{(x'-1)^2}{16} + \frac{(y'+1)^2}{8} = 1.$$



33. $x^2 + xy + y^2 = 24$

$A = 1,\ B = 1,\ C = 1$

$B^2 - 4AC = 1^2 - 4 \cdot 1 \cdot 1 = 1 - 4 = -3$

Since the discriminant is negative, the graph is an ellipse or a circle. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A - C}{B} = \frac{1-1}{1} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

Now substitute in the rotation of axes formulas.

$$x = x' \cos\theta - y' \sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y')$$
$$y = x' \sin\theta + y' \cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y')$$

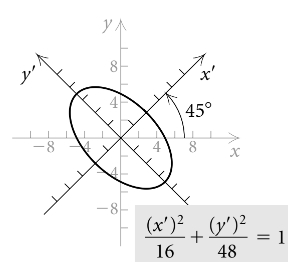
Substitute for x and y in the given equation.

$$\left[\frac{\sqrt{2}}{2}(x'-y')\right]^2 + \left[\frac{\sqrt{2}}{2}(x'-y')\right]\left[\frac{\sqrt{2}}{2}(x'+y')\right] +$$
$$\left[\frac{\sqrt{2}}{2}(x'+y')\right]^2 = 24$$

After simplifying we have

$$\frac{(x')^2}{16} + \frac{(y')^2}{48} = 1.$$

This is the equation of an ellipse with vertices $(0, -\sqrt{48})$ and $(0, \sqrt{48})$, or $(0, -4\sqrt{3})$ and $(0, 4\sqrt{3})$ on the y'-axis. The
x'-intercepts are $(-4, 0)$ and (4,0). We sketch the graph.



34. $4x^2 + 3\sqrt{3}xy + y^2 = 55$

$B^2 - 4AC = (3\sqrt{3})^2 - 4 \cdot 4 \cdot 1 = 27 - 16 = 11$

Since the discriminant is positive, the graph is a hyperbola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{4-1}{3\sqrt{3}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Then $2\theta = 60°$ and $\theta = 30°$, so

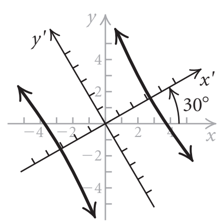
$$\sin\theta = \frac{1}{2} \text{ and } \cos\theta = \frac{\sqrt{3}}{2}.$$

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\cdot\frac{\sqrt{3}}{2} - y'\cdot\frac{1}{2} = \frac{x'\sqrt{3}}{2} - \frac{y'}{2}$$
$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\cdot\frac{1}{2} + y'\cdot\frac{\sqrt{3}}{2} = \frac{x'}{2} + \frac{y'\sqrt{3}}{2}$$

After substituting for x and y in the given equation and simplifying, we have

$$\frac{(x')^2}{10} - \frac{(y')^2}{110} = 1.$$



$$\frac{(x')^2}{10} - \frac{(y')^2}{110} = 1$$

35. $4x^2 - 4xy + y^2 - 8\sqrt{5}x - 16\sqrt{5}y = 0$

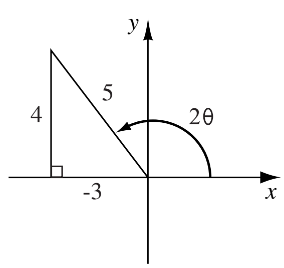
$A = 4,\ B = -4,\ C = 1$

$B^2 - 4AC = (-4)^2 - 4\cdot 4\cdot 1 = 16 - 16 = 0$

Since the discriminant is zero, the graph is a parabola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{4-1}{-4} = -\frac{3}{4}$$

Since $\cot 2\theta < 0$, we have $90° < 2\theta < 180°$. We make a sketch.



From the sketch we see that $\cos 2\theta = -\frac{3}{5}$. Using half-angle formulas, we have

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \frac{2}{\sqrt{5}}$$

and

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = \frac{1}{\sqrt{5}}.$$

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\cdot\frac{1}{\sqrt{5}} - y'\cdot\frac{2}{\sqrt{5}} = \frac{x'}{\sqrt{5}} - \frac{2y'}{\sqrt{5}}$$
$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\cdot\frac{2}{\sqrt{5}} + y'\cdot\frac{1}{\sqrt{5}} = \frac{2x'}{\sqrt{5}} + \frac{y'}{\sqrt{5}}$$

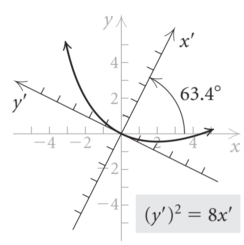
Substitute for x and y in the given equation.

$$4\left(\frac{x'}{\sqrt{5}} - \frac{2y'}{\sqrt{5}}\right)^2 - 4\left(\frac{x'}{\sqrt{5}} - \frac{2y'}{\sqrt{5}}\right)\left(\frac{2x'}{\sqrt{5}} + \frac{y'}{\sqrt{5}}\right) +$$
$$\left(\frac{2x'}{\sqrt{5}} + \frac{y'}{\sqrt{5}}\right)^2 - 8\sqrt{5}\left(\frac{x'}{\sqrt{5}} - \frac{2y'}{\sqrt{5}}\right) -$$
$$16\sqrt{5}\left(\frac{2x'}{\sqrt{5}} + \frac{y'}{\sqrt{5}}\right) = 0$$

After simplifying we have

$$(y')^2 = 8x'.$$

This is the equation of a parabola with vertex $(0,0)$ of the $x'y'$-coordinate system and axis of symmetry $y' = 0$. Since we know that $\sin\theta = \frac{2}{\sqrt{5}}$ and $0° < \theta < 90°$, we can use a calculator to find that $\theta \approx 63.4°$. Thus, the xy-axes are rotated through an angle of about $63.4°$ to obtain the $x'y'$-axes. We sketch the graph.



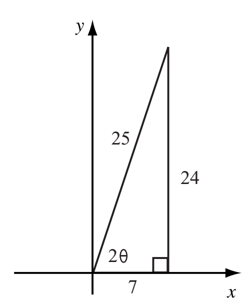
36. $9x^2 - 24xy + 16y^2 - 400x - 300y = 0$

$B^2 - 4AC = (-24)^2 - 4\cdot 9\cdot 16 = 576 - 576 = 0$

Since the discriminant is zero, the graph is a parabola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{9-16}{-24} = \frac{7}{24}$$

Since $\cot 2\theta > 0$, we have $0° < 2\theta < 90°$. We make a sketch.



From the sketch we see that $\cos 2\theta = \frac{7}{25}$. Then

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1-\frac{7}{25}}{2}} = \frac{3}{5}$$

and

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1+\frac{7}{25}}{2}} = \frac{4}{5}.$$

Now substitute in the rotation of axes formulas.

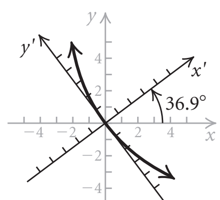
$x = x'\cos\theta - y'\sin\theta = \frac{4}{5}x' - \frac{3}{5}y'$

$y = x'\sin\theta + y'\cos\theta = \frac{3}{5}x' + \frac{4}{5}y'$

After substituting for x and y in the given equation and simplifying, we have

$$(y')^2 = 20x'.$$

Since we know that $\sin\theta = \frac{3}{5}$ and $0° < \theta < 90°$, we can use a calculator to find that $\theta \approx 36.9°$.



$(y')^2 = 20x'$

37. $11x^2 + 7xy - 13y^2 = 621$

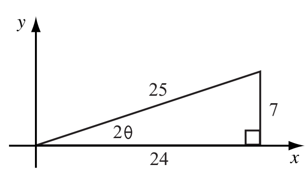
$A = 11,\ B = 7,\ C = -13$

$B^2 - 4AC = 7^2 - 4\cdot 11\cdot(-13) = 49 + 572 = 621$

Since the discriminant is positive, the graph is a hyperbola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{11-(-13)}{7} = \frac{24}{7}$$

Since $\cot 2\theta > 0$, we have $0° < 2\theta < 90°$. We make a sketch.



From the sketch we see that $\cos 2\theta = \frac{24}{25}$. Using half-angle formulas, we have

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1-\frac{24}{25}}{2}} = \frac{1}{\sqrt{50}}$$

and

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1+\frac{24}{25}}{2}} = \frac{7}{\sqrt{50}}.$$

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\cdot\frac{7}{\sqrt{50}} - y'\cdot\frac{1}{\sqrt{50}} = \frac{7x'}{\sqrt{50}} - \frac{y'}{\sqrt{50}}$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\cdot\frac{1}{\sqrt{50}} + y'\cdot\frac{7}{\sqrt{50}} = \frac{x'}{\sqrt{50}} + \frac{7y'}{\sqrt{50}}$$

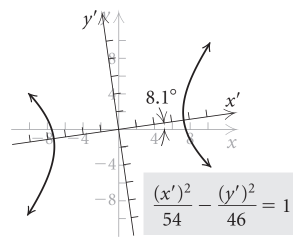
Substitute for x and y in the given equation.

$$11\left(\frac{7x'}{\sqrt{50}} - \frac{y'}{\sqrt{50}}\right)^2 + 7\left(\frac{7x'}{\sqrt{50}} - \frac{y'}{\sqrt{50}}\right)\left(\frac{x'}{\sqrt{50}} + \frac{7y'}{\sqrt{50}}\right) -$$
$$13\left(\frac{x'}{\sqrt{50}} + \frac{7y'}{\sqrt{50}}\right)^2 = 621$$

After simplifying we have

$$\frac{(x')^2}{54} - \frac{(y')^2}{46} = 1.$$

This is the equation of a hyperbola with vertices $(-\sqrt{54}, 0)$ and $(\sqrt{54}, 0)$, or $(-3\sqrt{6}, 0)$ and $(3\sqrt{6}, 0)$ on the x'-axis. The asymptotes are $y' = -\sqrt{\frac{23}{27}}x'$ and $y' = \sqrt{\frac{23}{27}}x'$. Since we know that $\sin\theta = \frac{1}{\sqrt{50}}$ and $0° < \theta < 90°$, we can use a calculator to find that $\theta \approx 8.1°$. Thus, the xy-axes are rotated through an angle of about 8.1° to obtain the $x'y'$-axes. We sketch the graph.



38. $3x^2 + 4xy + 6y^2 = 28$

$B^2 - 4AC = 4^2 - 4\cdot 3\cdot 6 = 16 - 72 = -56$

Since the discriminant is negative, the graph is an ellipse or a circle. To rotate the axes we first determine θ.

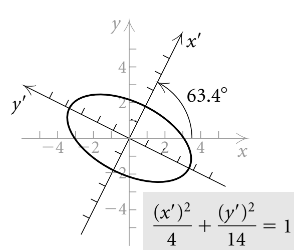
$$\cot 2\theta = \frac{A-C}{B} = \frac{3-6}{4} = -\frac{3}{4}$$

Proceeding as we did in Exercise 35, we find that

$$x = \frac{x'}{\sqrt{5}} - \frac{2y'}{\sqrt{5}},\ y' = \frac{2x'}{\sqrt{5}} + \frac{y'}{\sqrt{5}},\text{ and } \theta \approx 63.4°.$$

After substituting for x and y in the given equation and simplifying, we have

$$\frac{(x')^2}{4} + \frac{(y')^2}{14} = 1.$$



39. $$120° = 120° \cdot \frac{\pi \text{ radians}}{180°} = \frac{120°}{180°}\pi \text{ radians} = \frac{2\pi}{3} \text{ radians}$$

40. $$-315° = -315° \cdot \frac{\pi \text{ radians}}{180°} = \frac{-315°}{180°}\pi \text{ radians} = -\frac{7\pi}{4} \text{ radians}$$

41. $$\frac{\pi}{3} \text{ radians} = \frac{\pi}{3} \text{ radians} \cdot \frac{180°}{\pi \text{ radians}} = \frac{\pi}{3\pi} \cdot 180° = \frac{1}{3} \cdot 180° = 60°$$

42. $$\frac{3\pi}{4} \text{ radians} = \frac{3\pi}{4} \text{ radians} \cdot \frac{180°}{\pi \text{ radians}} = \frac{3\pi}{4\pi} \cdot 180° = \frac{3}{4} \cdot 180° = 135°$$

43. $x' = x\cos\theta + y\sin\theta$,
$y' = y\cos\theta - x\sin\theta$

First rewrite the system.

$$x\cos\theta + y\sin\theta = x', \quad (1)$$
$$-x\sin\theta + y\cos\theta = y' \quad (2)$$

Multiply Equation (1) by $\sin\theta$ and Equation (2) by $\cos\theta$ and add to eliminate x.

$$\begin{aligned} x\sin\theta\cos\theta + y\sin^2\theta &= x'\sin\theta \\ -x\sin\theta\cos\theta + y\cos^2\theta &= y'\cos\theta \\ \hline y(\sin^2\theta + \cos^2\theta) &= x'\sin\theta + y'\cos\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}$$

Substitute $x'\sin\theta + y'\cos\theta$ for y in Equation (1) and solve for x.

$$\begin{aligned} x\cos\theta + (x'\sin\theta + y'\cos\theta)\sin\theta &= x' \\ x\cos\theta + x'\sin^2\theta + y'\sin\theta\cos\theta &= x' \\ x\cos\theta &= x' - x'\sin^2\theta - y'\sin\theta\cos\theta \\ x\cos\theta &= x'(1 - \sin^2\theta) - y'\sin\theta\cos\theta \\ x\cos\theta &= x'\cos^2\theta - y'\sin\theta\cos\theta \\ x &= x'\cos\theta - y'\sin\theta \end{aligned}$$

Thus, we have $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.

44. $A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + y'\cos\theta)^2 + D(x'\cos\theta - y'\sin\theta) + E(x'\sin\theta + y'\cos\theta) + F = 0$

$A(x')^2\cos^2\theta - 2Ax'y'\cos\theta\sin\theta + A(y')^2\sin^2\theta + B(x')^2\cos\theta\sin\theta + Bx'y'(\cos^2\theta - \sin^2\theta) - B(y')^2\sin\theta\cos\theta + C(x')^2\sin^2\theta + 2Cx'y'\sin\theta\cos\theta + C(y')^2\cos^2\theta + Dx'\cos\theta - Dy'\sin\theta + Ex'\sin\theta + Ey'\cos\theta + F = 0$

$(x')^2(A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta) + x'y'(-2A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + 2C\sin\theta\cos\theta) + (y')^2(A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta) + x'(D\cos\theta + E\sin\theta) + y'(-D\sin\theta + E\cos\theta) + F = 0$

Thus we have an equation of the form

$A'(x')^2 + B'x'y' + C'(y')^2 + D'x' + E'y' + F' = 0$,

where

$A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta$,

$B' = -2A\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) + 2C\sin\theta\cos\theta$, or

$2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta)$,

$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta$,

$D' = D\cos\theta + E\sin\theta$,

$E' = -D\sin\theta + E\cos\theta$, and

$F' = F$.

45. $$\begin{aligned} &A' + C' \\ &= (A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta) + (A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta) \\ &= A(\cos^2\theta + \sin^2\theta) + B(\sin\theta\cos\theta - \sin\theta\cos\theta) + C(\sin^2\theta + \cos^2\theta) \\ &= A \cdot 1 + B \cdot 0 + C \cdot 1 \\ &= A + C \end{aligned}$$

46. $$\begin{aligned} x^2 + y^2 &= r^2 \\ (x'\cos\theta - y'\sin\theta)^2 + (x'\sin\theta + y'\cos\theta)^2 &= r^2 \\ (x')^2\cos^2\theta - 2x'y'\cos\theta\sin\theta + (y')^2\sin^2\theta + (x')^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + (y')^2\cos^2\theta &= r^2 \\ (x')^2(\cos^2\theta + \sin^2\theta) + x'y'(-2\sin\theta\cos\theta + 2\sin\theta\cos\theta) + (y')^2(\sin^2\theta + \cos^2\theta) &= r^2 \\ (x')^2 \cdot 1 + x'y' \cdot 0 + (y')^2 \cdot 1 &= r^2 \\ (x')^2 + (y')^2 &= r^2 \end{aligned}$$

Exercise Set 10.6

1. Graph (b) is the graph of $r = \dfrac{3}{1 + \cos\theta}$.
2. Graph (e) is the graph of $r = \dfrac{4}{1 + 2\sin\theta}$.
3. Graph (a) is the graph of $r = \dfrac{8}{4 - 2\cos\theta}$.
4. Graph (f) is the graph of $r = \dfrac{12}{4 + 6\sin\theta}$.
5. Graph (d) is the graph of $r = \dfrac{5}{3 - 3\sin\theta}$.

6. Graph (c) is the graph of $r = \dfrac{6}{3 + 2\cos\theta}$.

7. $r = \dfrac{1}{1 + \cos\theta}$

a) The equation is in the form $r = \dfrac{ep}{1 + e\cos\theta}$ with $e = 1$, so the graph is a parabola.

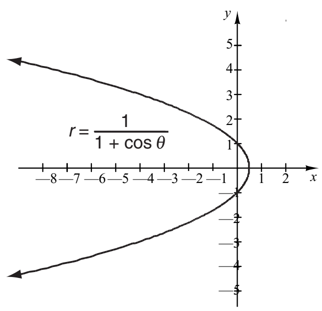
b) Since $e = 1$ and $ep = 1 \cdot p = 1$, we have $p = 1$. Thus the parabola has a vertical directrix 1 unit to the right of the pole.

c) We find the vertex by letting $\theta = 0$. When $\theta = 0$,

$$r = \frac{1}{1 + \cos 0} = \frac{1}{1 + 1} = \frac{1}{2}.$$

Thus, the vertex is $\left(\dfrac{1}{2}, 0\right)$.

d)



8. $r = \dfrac{4}{2 + \cos\theta}$

a) We first divide numerator and denominator by 2:

$$r = \frac{2}{1 + \frac{1}{2}\cos\theta}$$

The equation is in the form $r = \dfrac{ep}{1 + e\cos\theta}$ with $e = \dfrac{1}{2}$.

Since $0 < e < 1$, the graph is an ellipse.

b) Since $e = \dfrac{1}{2}$ and $ep = \dfrac{1}{2} \cdot p = 2$, we have $p = 4$. Thus the ellipse has a vertical directrix 4 units to the right of the pole.

c) We find the vertices by letting $\theta = 0$ and $\theta = \pi$. When $\theta = 0$,

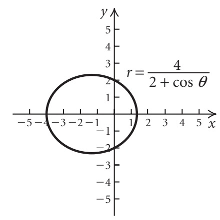
$$r = \frac{4}{2 + \cos 0} = \frac{4}{2 + 1} = \frac{4}{3}.$$

When $\theta = \pi$,

$$r = \frac{4}{2 + \cos\pi} = \frac{4}{2 - 1} = 4.$$

Thus, the vertices are $\left(\dfrac{4}{3}, 0\right)$ and $(4, \pi)$.

d)



9. $r = \dfrac{15}{5 - 10\sin\theta}$

a) We first divide numerator and denominator by 5:

$$r = \frac{3}{1 - 2\sin\theta}$$

The equation is in the form $r = \dfrac{ep}{1 - e\sin\theta}$ with $e = 2$.

Since $e > 1$, the graph is a hyperbola.

b) Since $e = 2$ and $ep = 2 \cdot p = 3$, we have $p = \dfrac{3}{2}$. Thus the hyperbola has a horizontal directrix $\dfrac{3}{2}$ units below the pole.

c) We find the vertices by letting $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$. When $\theta = \dfrac{\pi}{2}$,

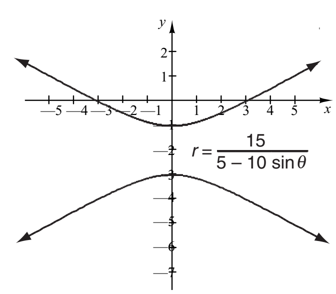
$$r = \frac{15}{5 - 10\sin\frac{\pi}{2}} = \frac{15}{5 - 10 \cdot 1} = \frac{15}{-5} = -3.$$

When $\theta = \dfrac{3\pi}{2}$,

$$r = \frac{15}{5 - 10\sin\frac{3\pi}{2}} = \frac{15}{5 - 10(-1)} = \frac{15}{15} = 1.$$

Thus, the vertices are $\left(-3, \dfrac{\pi}{2}\right)$ and $\left(1, \dfrac{3\pi}{2}\right)$.

d)



10. $r = \dfrac{12}{4 + 8\sin\theta}$

a) We first divide numerator and denominator by 4:

$$r = \frac{3}{1 + 2\sin\theta}$$

The equation is in the form $r = \dfrac{ep}{1 + e\sin\theta}$ with $e = 2$.

Since $e > 1$, the graph is a hyperbola.

b) Since $e = 2$ and $ep = 2 \cdot p = 3$, we have $p = \frac{3}{2}$. Thus the hyperbola has a horizontal directrix $\frac{3}{2}$ units above the pole.

c) We find the vertices by letting $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. When $\theta = \frac{\pi}{2}$,

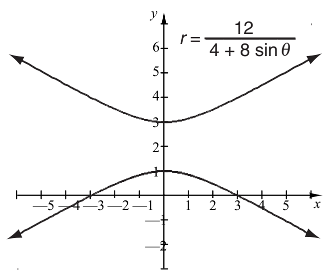
$$r = \frac{12}{4 + 8\sin\frac{\pi}{2}} = \frac{12}{4 + 8 \cdot 1} = \frac{12}{12} = 1.$$

When $\theta = \frac{3\pi}{2}$,

$$r = \frac{12}{4 + 8\sin\frac{3\pi}{2}} = \frac{12}{4 + 8(-1)} = \frac{12}{-4} = -3.$$

Thus, the vertices are $\left(1, \frac{\pi}{2}\right)$ and $\left(-3, \frac{3\pi}{2}\right)$.

d)



11. $r = \dfrac{8}{6 - 3\cos\theta}$

a) We first divide numerator and denominator by 6:

$$r = \frac{4/3}{1 - \frac{1}{2}\cos\theta}$$

The equation is in the form $r = \dfrac{ep}{1 - e\cos\theta}$ with $e = \frac{1}{2}$.

Since $0 < e < 1$, the graph is an ellipse.

b) Since $e = \frac{1}{2}$ and $ep = \frac{1}{2} \cdot p = \frac{4}{3}$, we have $p = \frac{8}{3}$. Thus the ellipse has a vertical directrix $\frac{8}{3}$ units to the left of the pole.

c) We find the vertices by letting $\theta = 0$ and $\theta = \pi$. When $\theta = 0$,

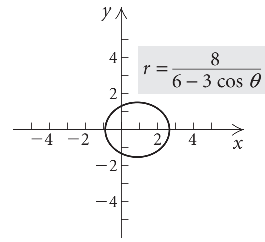
$$r = \frac{8}{6 - 3\cos 0} = \frac{8}{6 - 3 \cdot 1} = \frac{8}{3}.$$

When $\theta = \pi$,

$$r = \frac{8}{6 - 3\cos\pi} = \frac{8}{6 - 3(-1)} = \frac{8}{9}.$$

Thus, the vertices are $\left(\frac{8}{3}, 0\right)$ and $\left(\frac{8}{9}, \pi\right)$.

d)



12. $r = \dfrac{6}{2 + 2\sin\theta}$

a) We first divide numerator and denominator by 2:

$$r = \frac{3}{1 + \sin\theta}$$

The equation is in the form $r = \dfrac{ep}{1 + e\sin\theta}$ with $e = 1$, so the graph is a parabola.

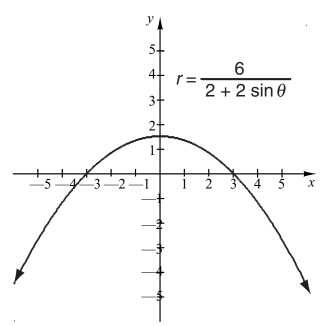
b) Since $e = 1$ and $ep = 1 \cdot p = 3$, we have $p = 3$. Thus the parabola has a horizontal directrix 3 units above the pole.

c) We find the vertex by letting $\theta = \frac{\pi}{2}$. When $\theta = \frac{\pi}{2}$,

$$r = \frac{6}{2 + 2\sin\frac{\pi}{2}} = \frac{6}{2 + 2 \cdot 1} = \frac{6}{4} = \frac{3}{2}.$$

Thus, the vertex is $\left(\frac{3}{2}, \frac{\pi}{2}\right)$.

d)



13. $r = \dfrac{20}{10 + 15\sin\theta}$

a) We first divide numerator and denominator by 10:

$$r = \frac{2}{1 + \frac{3}{2}\sin\theta}$$

The equation is in the form $r = \dfrac{ep}{1 + e\sin\theta}$ with $e = \frac{3}{2}$.

Since $e > 1$, the graph is a hyperbola.

b) Since $e = \frac{3}{2}$ and $ep = \frac{3}{2} \cdot p = 2$, we have $p = \frac{4}{3}$. Thus the hyperbola has a horizontal directrix $\frac{4}{3}$ units above the pole.

c) We find the vertices by letting $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. When $\theta = \frac{\pi}{2}$,

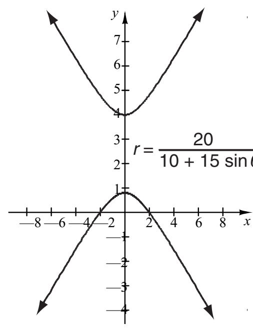
$$r = \frac{20}{10 + 15\sin\frac{\pi}{2}} = \frac{20}{10 + 15 \cdot 1} = \frac{20}{25} = \frac{4}{5}.$$

When $\theta = \frac{3\pi}{2}$,

$$r = \frac{20}{10 + 15\sin\frac{\pi}{2}} = \frac{20}{10 + 15(-1)} = \frac{20}{-5} = -4.$$

Thus, the vertices are $\left(\frac{4}{5}, \frac{\pi}{2}\right)$ and $\left(-4, \frac{3\pi}{2}\right)$.

d)



14. $r = \dfrac{10}{8 - 2\cos\theta}$

a) We first divide numerator and denominator by 8:

$$r = \frac{5/4}{1 - \frac{1}{4}\cos\theta}$$

The equation is in the form $r = \dfrac{ep}{1 - e\cos\theta}$ with $e = \frac{1}{4}$.

Since $0 < e < 1$, the graph is an ellipse.

b) Since $e = \frac{1}{4}$ and $ep = \frac{1}{4} \cdot p = \frac{5}{4}$, we have $p = 5$. Thus the ellipse has a vertical directrix 5 units to the left of the pole.

c) We find the vertices by letting $\theta = 0$ and $\theta = \pi$. When $\theta = 0$,

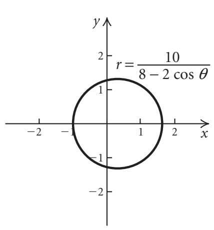
$$r = \frac{10}{8 - 2\cos 0} = \frac{10}{8 - 2 \cdot 1} = \frac{10}{6} = \frac{5}{3}.$$

When $\theta = \pi$,

$$r = \frac{10}{8 - 2\cos\pi} = \frac{10}{8 - 2(-1)} = \frac{10}{10} = 1.$$

Thus, the vertices are $\left(\frac{5}{3}, 0\right)$ and $(1, \pi)$.

d)



15. $r = \dfrac{9}{6 + 3\cos\theta}$

a) We first divide numerator and denominator by 6:

$$r = \frac{3/2}{1 + \frac{1}{2}\cos\theta}$$

The equation is in the form $r = \dfrac{ep}{1 + e\cos\theta}$ with $e = \frac{1}{2}$.

Since $0 < e < 1$, the graph is an ellipse.

b) Since $e = \frac{1}{2}$ and $ep = \frac{1}{2} \cdot p = \frac{3}{2}$, we have $p = 3$. Thus the ellipse has a vertical directrix 3 units to the right of the pole.

c) We find the vertices by letting $\theta = 0$ and $\theta = \pi$. When $\theta = 0$,

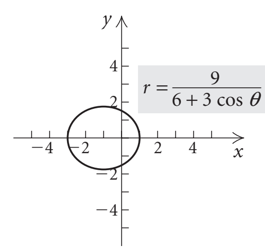
$$r = \frac{9}{6 + 3\cos 0} = \frac{9}{6 + 3 \cdot 1} = \frac{9}{9} = 1.$$

When $\theta = \pi$,

$$r = \frac{9}{6 + 3\cos\pi} = \frac{9}{6 + 3(-1)} = \frac{9}{3} = 3.$$

Thus, the vertices are $(1, 0)$ and $(3, \pi)$.

d)



16. $r = \dfrac{4}{3 - 9\sin\theta}$

a) We first divide numerator and denominator by 3:

$$r = \frac{4/3}{1 - 3\sin\theta}$$

The equation is in the form $r = \dfrac{ep}{1 - e\sin\theta}$ with $e = 3$.

Since $e > 1$, the graph is a hyperbola.

b) Since $e = 3$ and $ep = 3 \cdot p = \frac{4}{3}$, we have $p = \frac{4}{9}$. Thus the hyperbola has a horizontal directrix $\frac{4}{9}$ units below the pole.

c) We find the vertices by letting $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. When $\theta = \frac{\pi}{2}$,

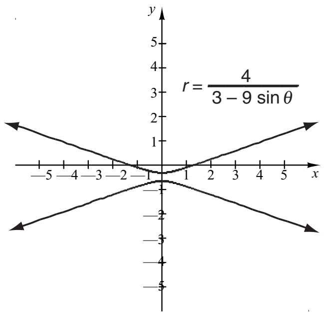
$$r = \frac{4}{3 - 9\sin\frac{\pi}{2}} = \frac{4}{3 - 9 \cdot 1} = \frac{4}{-6} = -\frac{2}{3}.$$

When $\theta = \dfrac{3\pi}{2}$,

$$r = \frac{4}{3 - 9\sin\frac{3\pi}{2}} = \frac{4}{3 - 9(-1)} = \frac{4}{12} = \frac{1}{3}.$$

Thus, the vertices are $\left(-\dfrac{2}{3}, \dfrac{\pi}{2}\right)$ and $\left(\dfrac{1}{3}, \dfrac{3\pi}{2}\right)$.

d)



17. $r = \dfrac{3}{2 - 2\sin\theta}$

a) We first divide numerator and denominator by 2:

$$r = \frac{3/2}{1 - \sin\theta}$$

The equation is in the form $r = \dfrac{ep}{1 - e\sin\theta}$ with $e = 1$, so the graph is a parabola.

b) Since $e = 1$ and $ep = 1 \cdot p = \dfrac{3}{2}$, we have $p = \dfrac{3}{2}$.

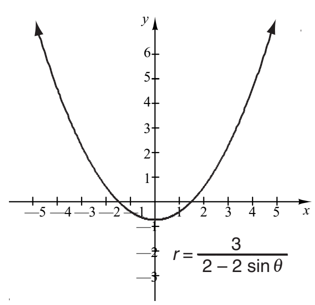
Thus the parabola has a horizontal directrix $\dfrac{3}{2}$ units below the pole.

c) We find the vertex by letting $\theta = \dfrac{3\pi}{2}$. When $\theta = \dfrac{3\pi}{2}$,

$$r = \frac{3}{2 - 2\sin\frac{3\pi}{2}} = \frac{3}{2 - 2(-1)} = \frac{3}{4}.$$

Thus, the vertex is $\left(\dfrac{3}{4}, \dfrac{3\pi}{2}\right)$.

d)



18. $r = \dfrac{12}{3 + 9\cos\theta}$

a) We first divide numerator and denominator by 3:

$$r = \frac{4}{1 + 3\cos\theta}$$

The equation is in the form $r = \dfrac{ep}{1 + e\cos\theta}$ with $e = 3$. Since $e > 1$, the graph is a hyperbola.

b) Since $e = 3$ and $ep = 3 \cdot p = 4$, we have $p = \dfrac{4}{3}$.

Thus the hyperbola has a vertical directrix $\dfrac{4}{3}$ units to the right of the pole.

c) We find the vertices by letting $\theta = 0$ and $\theta = \pi$. When $\theta = 0$,

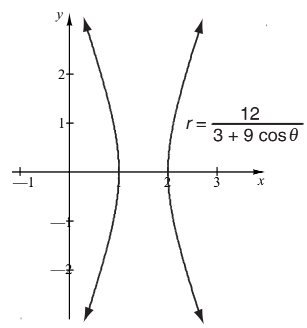
$$r = \frac{12}{3 + 9\cos 0} = \frac{12}{3 + 9 \cdot 1} = \frac{12}{12} = 1.$$

When $\theta = \pi$,

$$r = \frac{12}{3 + 9\cos\pi} = \frac{12}{3 + 9(-1)} = \frac{12}{-6} = -2.$$

Thus, the vertices are $(1, 0)$ and $(-2, \pi)$.

d)



19. $r = \dfrac{4}{2 - \cos\theta}$

a) We first divide numerator and denominator by 2:

$$r = \frac{2}{1 - \frac{1}{2}\cos\theta}$$

The equation is in the form $r = \dfrac{ep}{1 - e\cos\theta}$ with $e = \dfrac{1}{2}$.

Since $0 < e < 1$, the graph is an ellipse.

b) Since $e = \dfrac{1}{2}$ and $ep = \dfrac{1}{2} \cdot p = 2$, we have $p = 4$. Thus the ellipse has a vertical directrix 4 units to the left of the pole.

c) We find the vertices by letting $\theta = 0$ and $\theta = \pi$. When $\theta = 0$,

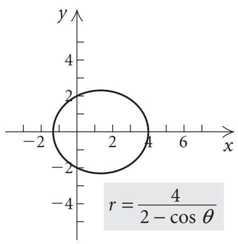
$$r = \frac{4}{2 - \cos 0} = \frac{4}{2 - 1} = 4.$$

When $\theta = \pi$,

$$r = \frac{4}{2 - \cos\pi} = \frac{4}{2 - (-1)} = \frac{4}{3}.$$

Thus, the vertices are $(4, 0)$ and $\left(\dfrac{4}{3}, \pi\right)$.

d)



20. $r = \dfrac{5}{1 - \sin\theta}$

a) The equation is in the form $r = \dfrac{ep}{1 - e\sin\theta}$ with $e = 1$,

so the graph is a parabola.

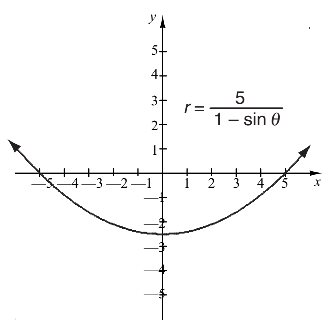
b) Since $e = 1$ and $ep = 1 \cdot p = 5$, we have $p = 5$. Thus the parabola has a horizontal directrix 5 units below the pole.

c) We find the vertex by letting $\theta = \dfrac{3\pi}{2}$. When $\theta = \dfrac{3\pi}{2}$,

$$r = \frac{5}{1 - \sin\dfrac{3\pi}{2}} = \frac{5}{1 - (-1)} = \frac{5}{2}.$$

Thus, the vertex is $\left(\dfrac{5}{2}, \dfrac{3\pi}{2}\right)$.

d)



21. $r = \dfrac{7}{2 + 10\sin\theta}$

a) We first divide numerator and denominator by 2:

$$r = \frac{7/2}{1 + 5\sin\theta}$$

The equation is in the form $r = \dfrac{ep}{1 + e\sin\theta}$ with $e = 5$.

Since $e > 1$, the graph is a hyperbola.

b) Since $e = 5$ and $ep = 5 \cdot p = \dfrac{7}{2}$, we have $p = \dfrac{7}{10}$. Thus the hyperbola has a horizontal directrix $\dfrac{7}{10}$ unit above the pole.

c) We find the vertices by letting $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$. When $\theta = \dfrac{\pi}{2}$,

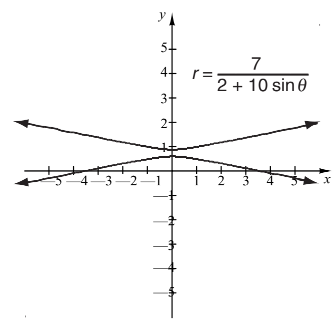
$$r = \frac{7}{2 + 10\sin\dfrac{\pi}{2}} = \frac{7}{2 + 10 \cdot 1} = \frac{7}{12}.$$

When $\theta = \dfrac{3\pi}{2}$,

$$r = \frac{7}{2 + 10\sin\dfrac{3\pi}{2}} = \frac{7}{2 + 10(-1)} = -\frac{7}{8}.$$

Thus, the vertices are $\left(\dfrac{7}{12}, \dfrac{\pi}{2}\right)$ and $\left(-\dfrac{7}{8}, \dfrac{3\pi}{2}\right)$.

d)



22. $r = \dfrac{3}{8 - 4\cos\theta}$

a) We first divide numerator and denominator by 8:

$$r = \frac{3/8}{1 - \dfrac{1}{2}\cos\theta}$$

The equation is in the form $r = \dfrac{ep}{1 - e\cos\theta}$ with $e = \dfrac{1}{2}$.

Since $0 < e < 1$, the graph is an ellipse.

b) Since $e = \dfrac{1}{2}$ and $ep = \dfrac{1}{2} \cdot p = \dfrac{3}{8}$, we have $p = \dfrac{3}{4}$.

Thus the ellipse has a vertical directrix $\dfrac{3}{4}$ unit to the left of the pole.

c) We find the vertices by letting $\theta = 0$ and $\theta = \pi$. When $\theta = 0$,

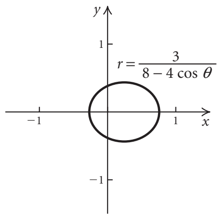
$$r = \frac{3}{8 - 4\cos 0} = \frac{3}{8 - 4 \cdot 1} = \frac{3}{4}.$$

When $\theta = \pi$,

$$r = \frac{3}{8 - 4\cos\pi} = \frac{3}{8 - 4(-1)} = \frac{3}{12} = \frac{1}{4}.$$

Thus, the vertices are $\left(\dfrac{3}{4}, 0\right)$ and $\left(\dfrac{1}{4}, \pi\right)$.

d)



23.
$$\begin{aligned} r &= \frac{1}{1+\cos\theta} \\ r + r\cos\theta &= 1 \\ r &= 1 - r\cos\theta \\ \sqrt{x^2+y^2} &= 1 - x \\ x^2 + y^2 &= 1 - 2x + x^2 \\ y^2 &= -2x + 1, \text{ or} \\ y^2 + 2x - 1 &= 0 \end{aligned}$$

24.
$$\begin{aligned} r &= \frac{4}{2+\cos\theta} \\ 2r + r\cos\theta &= 4 \\ 2r &= 4 - r\cos\theta \\ 2\sqrt{x^2+y^2} &= 4 - x \\ 4x^2 + 4y^2 &= 16 - 8x + x^2 \\ 3x^2 + 4y^2 + 8x - 16 &= 0 \end{aligned}$$

25.
$$\begin{aligned} r &= \frac{15}{5 - 10\sin\theta} \\ 5r - 10r\sin\theta &= 15 \\ 5r &= 10r\sin\theta + 15 \\ r &= 2r\sin\theta + 3 \\ \sqrt{x^2+y^2} &= 2y + 3 \\ x^2 + y^2 &= 4y^2 + 12y + 9 \\ x^2 - 3y^2 - 12y - 9 &= 0 \end{aligned}$$

26.
$$\begin{aligned} r &= \frac{12}{4 + 8\sin\theta} \\ 4r + 8r\sin\theta &= 12 \\ 4r &= 12 - 8r\sin\theta \\ r &= 3 - 2r\sin\theta \\ \sqrt{x^2+y^2} &= 3 - 2y \\ x^2 + y^2 &= 9 - 12y + 4y^2 \\ x^2 - 3y^2 + 12y - 9 &= 0 \end{aligned}$$

27.
$$\begin{aligned} r &= \frac{8}{6 - 3\cos\theta} \\ 6r - 3r\cos\theta &= 8 \\ 6r &= 3r\cos\theta + 8 \\ 6\sqrt{x^2+y^2} &= 3x + 8 \\ 36x^2 + 36y^2 &= 9x^2 + 48x + 64 \\ 27x^2 + 36y^2 - 48x - 64 &= 0 \end{aligned}$$

28.
$$\begin{aligned} r &= \frac{6}{2 + 2\sin\theta} \\ 2r + 2r\sin\theta &= 6 \\ 2r &= 6 - 2r\sin\theta \\ r &= 3 - r\sin\theta \\ \sqrt{x^2+y^2} &= 3 - y \\ x^2 + y^2 &= 9 - 6y + y^2 \\ x^2 &= -6y + 9, \text{ or} \\ x^2 + 6y - 9 &= 0 \end{aligned}$$

29.
$$\begin{aligned} r &= \frac{20}{10 + 15\sin\theta} \\ 10r + 15r\sin\theta &= 20 \\ 10r &= 20 - 15r\sin\theta \\ 2r &= 4 - 3r\sin\theta \\ 2\sqrt{x^2+y^2} &= 4 - 3y \\ 4x^2 + 4y^2 &= 16 - 24y + 9y^2 \\ 4x^2 - 5y^2 + 24y - 16 &= 0 \end{aligned}$$

30.
$$\begin{aligned} r &= \frac{10}{8 - 2\cos\theta} \\ 8r - 2r\cos\theta &= 10 \\ 8r &= 2r\cos\theta + 10 \\ 4r &= r\cos\theta + 5 \\ 4\sqrt{x^2+y^2} &= x + 5 \\ 16x^2 + 16y^2 &= x^2 + 10x + 25 \\ 15x^2 + 16y^2 - 10x - 25 &= 0 \end{aligned}$$

31.
$$\begin{aligned} r &= \frac{9}{6 + 3\cos\theta} \\ 6r + 3r\cos\theta &= 9 \\ 6r &= 9 - 3r\cos\theta \\ 2r &= 3 - r\cos\theta \\ 2\sqrt{x^2+y^2} &= 3 - x \\ 4x^2 + 4y^2 &= 9 - 6x + x^2 \\ 3x^2 + 4y^2 + 6x - 9 &= 0 \end{aligned}$$

32.
$$\begin{aligned} r &= \frac{4}{3 - 9\sin\theta} \\ 3r - 9r\sin\theta &= 4 \\ 3r &= 9r\sin\theta + 4 \\ 3\sqrt{x^2+y^2} &= 9y + 4 \\ 9x^2 + 9y^2 &= 81y^2 + 72y + 16 \\ 9x^2 - 72y^2 - 72y - 16 &= 0 \end{aligned}$$

33.
$$\begin{aligned} r &= \frac{3}{2 - 2\sin\theta} \\ 2r - 2r\sin\theta &= 3 \\ 2r &= 2r\sin\theta + 3 \\ 2\sqrt{x^2+y^2} &= 2y + 3 \\ 4x^2 + 4y^2 &= 4y^2 + 12y + 9 \\ 4x^2 &= 12y + 9, \text{ or} \\ 4x^2 - 12y - 9 &= 0 \end{aligned}$$

34.
$$\begin{aligned} r &= \frac{12}{3+9\cos\theta} \\ 3r+9r\cos\theta &= 12 \\ 3r &= 12-9r\cos\theta \\ r &= 4-3r\cos\theta \\ \sqrt{x^2+y^2} &= 4-3x \\ x^2+y^2 &= 16-24x+9x^2 \\ -8x^2+y^2+24x-16 &= 0 \end{aligned}$$

35.
$$\begin{aligned} r &= \frac{4}{2-\cos\theta} \\ 2r-r\cos\theta &= 4 \\ 2r &= r\cos\theta+4 \\ 2\sqrt{x^2+y^2} &= x+4 \\ 4x^2+4y^2 &= x^2+8x+16 \\ 3x^2+4y^2-8x-16 &= 0 \end{aligned}$$

36.
$$\begin{aligned} r &= \frac{5}{1-\sin\theta} \\ r-r\sin\theta &= 5 \\ r &= r\sin\theta+5 \\ \sqrt{x^2+y^2} &= y+5 \\ x^2+y^2 &= y^2+10y+25 \\ x^2 &= 10y+25, \text{ or} \\ x^2-10y-25 &= 0 \end{aligned}$$

37.
$$\begin{aligned} r &= \frac{7}{2+10\sin\theta} \\ 2r+10r\sin\theta &= 7 \\ 2r &= 7-10r\sin\theta \\ 2\sqrt{x^2+y^2} &= 7-10y \\ 4x^2+4y^2 &= 49-140y+100y^2 \\ 4x^2-96y^2+140y-49 &= 0 \end{aligned}$$

38.
$$\begin{aligned} r &= \frac{3}{8-4\cos\theta} \\ 8r-4r\cos\theta &= 3 \\ 8r &= 4r\cos\theta+3 \\ 8\sqrt{x^2+y^2} &= 4x+3 \\ 64x^2+64y^2 &= 16x^2+24x+9 \\ 48x^2+64y^2-24x-9 &= 0 \end{aligned}$$

39. $e=2,\ r=3\csc\theta$

The equation of the directrix can be written

$$r=\frac{3}{\sin\theta}, \text{ or } r\sin\theta=3.$$

This corresponds to the equation $y=3$ in rectangular coordinates, so the directrix is a horizontal line 3 units above the polar axis. Using the table on page 868 of the text, we see that the equation is of the form

$$r=\frac{ep}{1+e\sin\theta}.$$

Substituting 2 for e and 3 for p, we have

$$r=\frac{2\cdot 3}{1+2\sin\theta}=\frac{6}{1+2\sin\theta}.$$

40. $e=\frac{2}{3},\ r=-\sec\theta$

The equation of the directrix can be written

$$r=\frac{-1}{\cos\theta}, \text{ or } r\cos\theta=-1.$$

This corresponds to the equation $x=-1$ in rectangular coordinates, so the directrix is a vertical line 1 unit to the left of the pole. Using the table on page 868 of the text, we see that the equation is of the form

$$r=\frac{ep}{1-e\cos\theta}.$$

Substituting $\frac{2}{3}$ for e and 1 for p, we have

$$r=\frac{\frac{2}{3}\cdot 1}{1-\frac{2}{3}\cos\theta}=\frac{\frac{2}{3}}{1-\frac{2}{3}\cos\theta}, \text{ or } \frac{2}{3-2\cos\theta}.$$

41. $e=1,\ r=4\sec\theta$

The equation of the directrix can be written

$$r=\frac{4}{\cos\theta}, \text{ or } r\cos\theta=4.$$

This corresponds to the equation $x=4$ in rectangular coordinates, so the directrix is a vertical line 4 units to the right of the pole. Using the table on page 868 of the text, we see that the equation is of the form

$$r=\frac{ep}{1+e\cos\theta}.$$

Substituting 1 for e and 4 for p, we have

$$r=\frac{1\cdot 4}{1+1\cdot\cos\theta}=\frac{4}{1+\cos\theta}.$$

42. $e=3,\ r=2\csc\theta$

The equation of the directrix can be written

$$r=\frac{2}{\sin\theta}, \text{ or } r\sin\theta=2.$$

This corresponds to the equation $y=2$ in rectangular coordinates, so the directrix is a horizontal line 2 units above the polar axis. Using the table on page 868 of the text, we see that the equation is of the form

$$r=\frac{ep}{1+e\sin\theta}.$$

Substituting 3 for e and 2 for p, we have

$$r=\frac{3\cdot 2}{1+3\sin\theta}=\frac{6}{1+3\sin\theta}.$$

43. $e=\frac{1}{2},\ r=-2\sec\theta$

The equation of the directrix can be written

$$r=\frac{-2}{\cos\theta}, \text{ or } r\cos\theta=-2.$$

This corresponds to the equation $x=-2$ in rectangular coordinates, so the directrix is a vertical line 2 units to the left of the pole. Using the table on page 868 of the text, we see that the equation is of the form

$$r=\frac{ep}{1-e\cos\theta}.$$

Substituting $\frac{1}{2}$ for e and 2 for p, we have

$$r = \frac{\frac{1}{2}\cdot 2}{1-\frac{1}{2}\cos\theta} = \frac{1}{1-\frac{1}{2}\cos\theta}, \text{ or } \frac{2}{2-\cos\theta}.$$

44. $e = 1,\ r = 4\csc\theta$

The equation of the directrix can be written

$$r = \frac{4}{\sin\theta}, \text{ or } r\sin\theta = 4.$$

This corresponds to the equation $y = 4$ in rectangular coordinates, so the directrix is a horizontal line 4 units above the polar axis. Using the table on page 868 of the text, we see that the equation is of the form

$$r = \frac{ep}{1+e\sin\theta}.$$

Substituting 1 for e and 4 for p, we have

$$r = \frac{1\cdot 4}{1+1\cdot\sin\theta} = \frac{4}{1+\sin\theta}.$$

45. $e = \frac{3}{4},\ r = 5\csc\theta$

The equation of the directrix can be written

$$r = \frac{5}{\sin\theta}, \text{ or } r\sin\theta = 5.$$

This corresponds to the equation $y = 5$ in rectangular coordinates, so the directrix is a horizontal line 5 units above the polar axis. Using the table on page 868 of the text, we see that the equation is of the form

$$r = \frac{ep}{1+e\sin\theta}.$$

Substituting $\frac{3}{4}$ for e and 5 for p, we have

$$r = \frac{\frac{3}{4}\cdot 5}{1+\frac{3}{4}\sin\theta} = \frac{15/4}{1+\frac{3}{4}\sin\theta} \text{ or } \frac{15}{4+3\sin\theta}.$$

46. $e = \frac{4}{5},\ r = 2\sec\theta$

The equation of the directrix can be written

$$r = \frac{2}{\cos\theta}, \text{ or } r\cos\theta = 2.$$

This corresponds to the equation $x = 2$ in rectangular coordinates, so the directrix is a vertical line 2 units to the right of the pole. Using the table on page 868 of the text, we see that the equation is of the form

$$r = \frac{ep}{1+e\cos\theta}.$$

Substituting $\frac{4}{5}$ for e and 2 for p, we have

$$r = \frac{\frac{4}{5}\cdot 2}{1+\frac{4}{5}\cos\theta} = \frac{8/5}{1+\frac{4}{5}\cos\theta}, \text{ or } \frac{8}{5+4\cos\theta}.$$

47. $e = 4,\ r = -2\csc\theta$

The equation of the directrix can be written

$$r = \frac{-2}{\sin\theta}, \text{ or } r\sin\theta = -2.$$

This corresponds to the equation $y = -2$ in rectangular coordinates, so the directrix is a horizontal line 2 units below the polar axis. Using the table on page 868 of the text, we see that the equation is of the form

$$r = \frac{ep}{1-e\sin\theta}.$$

Substituting 4 for e and 2 for p, we have

$$r = \frac{4\cdot 2}{1-4\sin\theta} = \frac{8}{1-4\sin\theta}.$$

48. $e = 3,\ r = 3\csc\theta$

The equation of the directrix can be written

$$r = \frac{3}{\sin\theta}, \text{ or } r\sin\theta = 3.$$

This corresponds to the equation $y = 3$ in rectangular coordinates, so the directrix is a horizontal line 3 units above the polar axis. Using the table on page 868 of the text, we see that the equation is of the form

$$r = \frac{ep}{1+e\sin\theta}.$$

Substituting 3 for e and 3 for p, we have

$$r = \frac{3\cdot 3}{1+3\sin\theta} = \frac{9}{1+3\sin\theta}.$$

49. $f(x) = (x-3)^2 + 4$

$f(t) = (t-3)^2 + 4 = t^2 - 6t + 9 + 4 = t^2 - 6t + 13$

Thus, $f(t) = (t-3)^2 + 4$, or $t^2 - 6t + 13$.

50. $f(2t) = (2t-3)^2 + 4 = 4t^2 - 12t + 9 + 4 = 4t^2 - 12t + 13$

Thus, $f(2t) = (2t-3)^2 + 4$, or $4t^2 - 12t + 13$.

51. $f(x) = (x-3)^2 + 4$

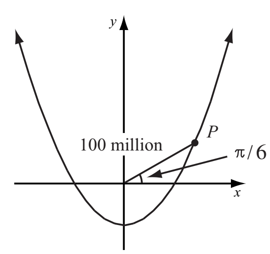
$f(t-1) = (t-1-3)^2 + 4 = (t-4)^2 + 4 = t^2 - 8t + 16 + 4 = t^2 - 8t + 20$

Thus, $f(t-1) = (t-4)^2 + 4$, or $t^2 - 8t + 20$.

52. $f(t+2) = (t+2-3)^2 + 4 = (t-1)^2 + 4 = t^2 - 2t + 1 + 4 = t^2 - 2t + 5$

Thus, $f(t+2) = (t-1)^2 + 4$, or $t^2 - 2t + 5$.

53.

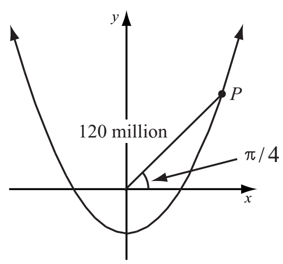


Since the directrix lies above the pole, the equation is of the form $r = \frac{ep}{1+e\sin\theta}$. The point P on the parabola has coordinates $(1\times 10^8, \pi/6)$. (Note that 100 million $= 1\times 10^8$.) Since the conic is a parabola, we know that $e = 1$. We substitute 1×10^8 for r, 1 for e, and $\pi/6$ for θ and then find p.

$$1 \times 10^8 = \frac{1 \cdot p}{1 + 1 \cdot \sin \frac{\pi}{6}}$$
$$1 \times 10^8 = \frac{p}{1 + 0.5}$$
$$1 \times 10^8 = \frac{p}{1.5}$$
$$1.5 \times 10^8 = p$$

The equation of the orbit is $r = \dfrac{1.5 \times 10^8}{1 + \sin\theta}$.

54.



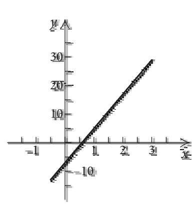
Since the directrix lies above the pole, the equation is of the form $r = \dfrac{ep}{1 + e\sin\theta}$. The point P on the parabola has coordinates $(1.2 \times 10^8, \pi/4)$. (Note that 120 million $= 1.2 \times 10^8$.) Since the conic is a parabola, we know that $e = 1$. We substitute 1.2×10^8 for r, 1 for e, and $\pi/4$ for θ and then find p.

$$1.2 \times 10^8 = \frac{1 \cdot p}{1 + 1 \cdot \sin \frac{\pi}{4}}$$
$$1.2 \times 10^8 = \frac{p}{1 + \frac{\sqrt{2}}{2}}$$
$$1.2 \times 10^8 = \frac{p}{\frac{2 + \sqrt{2}}{2}}$$
$$2.0 \times 10^8 \approx p$$

The equation of the orbit is $r = \dfrac{2.0 \times 10^8}{1 + \sin\theta}$.

Exercise Set 10.7

1.



$x = \frac{1}{2}t,\ y = 6t - 7;\ -1 \leqslant t \leqslant 6$

To find an equivalent rectangular equation, we first solve $x = \dfrac{1}{2}t$ for t.

$$x = \frac{1}{2}t$$
$$2x = t$$

Then we substitute $2x$ for t in $y = 6t - 7$.

$$y = 6(2x) - 7$$
$$y = 12x - 7$$

Given that $-1 \le t \le 6$, we find the corresponding restrictions on x:

For $t = -1:\ x = \dfrac{1}{2}t = \dfrac{1}{2}(-1) = -\dfrac{1}{2}$.

For $t = 6:\ x = \dfrac{1}{2}t = \dfrac{1}{2} \cdot 6 = 3$.

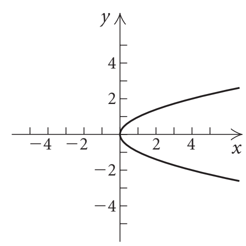
Then we have $y = 12x - 7,\ -\dfrac{1}{2} \le x \le 3$.

2. The graph is on page IA-62 in the text.

To find an equivalent rectangular equation, we substitute x for t in $y = 5 - t$: $y = 5 - x$

Given that $-2 \le t \le 3$ and $x = t$, the corresponding restrictions on x are $-2 \le x \le 3$. Then we have $y = 5 - x,\ -2 \le x \le 3$.

3.



$x = 4t^2,\ y = 2t,\ -1 \leqslant t \leqslant 1$

To find an equivalent rectangular equation, we first solve $y = 2t$ for t.

$$y = 2t$$
$$\frac{y}{2} = t$$

Then we substitue $\dfrac{y}{2}$ for t in $x = 4t^2$.

$$x = 4\left(\frac{y}{2}\right)^2$$
$$x = 4 \cdot \frac{y^2}{4}$$
$$x = y^2$$

Given that $-1 \le t \le 1$, we find the corresponding restrictions on y.

For $t = -1$: $y = 2t = 2(-1) = -2$.

For $t = 1$: $y = 2t = 2 \cdot 1 = 2$.

Then we have $x = y^2,\ -2 \le y \le 2$.

4. The graph is on page IA-62 in the text.

To find an equivalent rectangular equation, we first solve $x = \sqrt{t}$ for t:

$$x = \sqrt{t}$$
$$x^2 = t$$

Then we substitute x^2 for t in $y = 2t + 3$: $y = 2x^2 + 3$

Given that $0 \le t \le 8$, we find the corresponding restrictions on x:

For $t = 0:\ x = \sqrt{t} = \sqrt{0} = 0.$

For $t = 8:\ x = \sqrt{t} = \sqrt{8}$, or $2\sqrt{2}$.

Then we have $y = 2x^2 + 3,\ 0 \le x \le 2\sqrt{2}$.

5. To find an equivalent rectangular equation, we first solve $x = t^2$ for t.

$$x = t^2$$
$$\sqrt{x} = t$$

(We choose the nonnegative square root because $0 \le t \le 4$.)

Then we substitute $\sqrt{x}$ for t in $y = \sqrt{t}$:

$$y = \sqrt{\sqrt{x}} = (x^{1/2})^{1/2}$$
$$y = x^{1/4}, \text{ or } \sqrt[4]{x}$$

Given that $0 \le t \le 4$, we find the corresponding restrictions on x:

For $t = 0:\ x = t^2 = (0)^2 = 0.$

For $t = 4:\ x = t^2 = 4^2 = 16.$

Then we have $y = \sqrt[4]{x},\ 0 \le x \le 16$. (This result could also be expressed as $x = y^4,\ 0 \le y \le 2$.)

6. To find an equivalent rectangular equation, we first solve $x = t^3 + 1$ for t:

$$x = t^3 + 1$$
$$x - 1 = t^3$$
$$\sqrt[3]{x-1} = t$$

Then we substitute $\sqrt[3]{x-1}$ for t in $y = t$: $y = \sqrt[3]{x-1}$

Given that $-3 \le t \le 3$, we find the corresponding restrictions on x:

For $t = -3:\ x = t^3 + 1 = (-3)^3 + 1 = -26.$

For $t = 3:\ x = t^3 + 1 = 3^3 + 1 = 28.$

Then we have $y = \sqrt[3]{x-1},\ -26 \le x \le 28$.

7. To find an equivalent rectangular equation, we can substitute x for $t + 3$ in $y = \dfrac{1}{t+3}:\ y = \dfrac{1}{x}$.

Given that $-2 \le t \le 2$, we find the corresponding restrictions on x:

For $t = -2:\ x = t + 3 = -2 + 3 = 1.$

For $t = 2:\ x = t + 3 = 2 + 3 = 5.$

Then we have $y = \dfrac{1}{x},\ 1 \le x \le 5$.

8. To find an equivalent rectangular equation, we first solve $x = 2t^3 + 1$ for $2t^3$:

$$x = 2t^3 + 1$$
$$x - 1 = 2t^3$$

Then we substitute $x - 1$ for $2t^3$ in $y = 2t^3 - 1$:

$$y = (x - 1) - 1$$
$$y = x - 2$$

Given that $-4 \le t \le 4$, we find the corresponding restrictions on x:

For $t = -4:\ x = 2t^3 + 1 = 2(-4)^3 + 1 = -127.$

For $t = 4:\ x = 2t^3 + 1 = 2 \cdot 4^3 + 1 = 129.$

Then we have $y = x - 2,\ -127 \le x \le 129$.

9. To find an equivalent rectangular equation, we first solve $x = 2t - 1$ for t:

$$x = 2t - 1$$
$$x + 1 = 2t$$
$$\frac{1}{2}(x+1) = t$$

Then we substitute $\dfrac{1}{2}(x+1)$ for t in $y = t^2$:

$$y = \left[\frac{1}{2}(x+1)\right]^2$$
$$y = \frac{1}{4}(x+1)^2$$

Given that $-3 \le t \le 3$, we find the corresponding restrictions on x:

For $t = -3:\ x = 2t - 1 = 2(-3) - 1 = -7.$

For $t = 3:\ x = 2t - 1 = 2 \cdot 3 - 1 = 5.$

Then we have $y = \dfrac{1}{4}(x+1)^2,\ -7 \le x \le 5$.

10. To find an equivalent rectangular equation, we first solve $x = \dfrac{1}{3}t$ for t:

$$x = \frac{1}{3}t$$
$$3x = t$$

Then we substitute $3x$ for t in $y = t$: $y = 3x$.

Given that $-5 \le t \le 5$, we find the corresponding restrictions on x:

For $t = -5:\ x = \dfrac{1}{3}t = \dfrac{1}{3}(-5) = -\dfrac{5}{3}.$

For $t = 5:\ x = \dfrac{1}{3}t = \dfrac{1}{3} \cdot 5 = \dfrac{5}{3}.$

Then we have $y = 3x,\ -\dfrac{5}{3} \le x \le \dfrac{5}{3}$.

11. To find an equivalent rectangular equation, we first solve $x = e^{-t}$ for e^t:

$$x = e^{-t}$$
$$x = \frac{1}{e^t}$$
$$e^t = \frac{1}{x}$$

Then we substitute $\dfrac{1}{x}$ for e^t in $y = e^t$: $y = \dfrac{1}{x}$.

Given that $-\infty \le t \le \infty$, we find the corresponding restrictions on x:

As t approaches $-\infty$, e^{-t} approaches ∞. As t approaches ∞, e^{-t} approaches 0. Thus, we see that $x > 0$.

Then we have $y = \dfrac{1}{x},\ x > 0$.

12. To find an equivalent rectangular equation, we first solve $x = 2\ln t$ for t^2:

$$x = 2\ln t$$
$$x = \ln t^2$$
$$e^x = t^2$$

Then we substitute e^x for t^2 in $y = t^2$: $y = e^x$.

Given that $0 < t < \infty$, we find the corresponding restrictions on x. As t approaches 0, $2\ln t$ approaches $-\infty$. As t approaches ∞, $x = 2\ \ln\ t$ also approaches ∞.

Then we have $y = e^x,\ -\infty < x < \infty$.

13. To find an equivalent rectangular equation, we first solve for $\cos t$ and $\sin t$ in the parametric equations:

$$\frac{x}{3} = \cos t, \frac{y}{3} = \sin t$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can substitute to eliminate the parameter:

$$\sin^2 t + \cos^2 t = 1$$
$$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$$
$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$
$$x^2 + y^2 = 9$$

For $0 \le t \le 2\pi,\ -3 \le 3\cos t \le 3$.

Then we have $x^2 + y^2 = 9,\ -3 \le x \le 3$.

14. To find an equivalent rectangular equation, we first solve for $\cos t$ and $\sin t$ in the parametric equations:

$$\frac{x}{2} = \cos t, \frac{y}{4} = \sin t$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can substitute to eliminate the parameter:

$$\sin^2 t + \cos^2 t = 1$$
$$\left(\frac{y}{4}\right)^2 + \left(\frac{x}{2}\right)^2 = 1$$
$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

For $0 \le t \le 2\pi,\ -2 \le 2\cos t \le 2$.

Then we have $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1,\ -2 \le x \le 2$.

15. To find an equivalent rectangular equation, we first solve $y = 2\sin t$ for $\sin t$: $\dfrac{y}{2} = \sin t$.

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can substitute to eliminate the parameter:

$$\sin^2 t + \cos^2 t = 1$$
$$\left(\frac{y}{2}\right)^2 + x^2 = 1$$
$$x^2 + \frac{y^2}{4} = 1$$

For $0 \le t \le 2\pi,\ -1 \le \cos t \le 1$.

Then we have $x^2 + \dfrac{y^2}{4} = 1,\ -1 \le x \le 1$.

16. To find an equivalent rectangular equation, we first solve for $\cos t$ and $\sin t$ in the parametric equations:

$$\frac{x}{2} = \cos t, \frac{y}{2} = \sin t$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can substitute to eliminate the parameter:

$$\sin^2 t + \cos^2 t = 1$$
$$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = 1$$
$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$
$$x^2 + y^2 = 4$$

For $0 \le t \le 2\pi,\ -2 \le 2\cos t \le 2$.

Then we have $x^2 + y^2 = 4,\ -2 \le x \le 2$.

17. To find an equivalent rectangular equation, we first solve $x = \sec t$ for $\cos t$:

$$x = \sec t$$
$$x = \frac{1}{\cos t}$$
$$\cos t = \frac{1}{x}$$

Then we substitute $\dfrac{1}{x}$ for $\cos t$ in $y = \cos t$: $y = \dfrac{1}{x}$.

For $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2},\ 1 \le x < \infty$.

Then we have $y = \dfrac{1}{x},\ x \ge 1$.

18. To find an equivalent rectangular equation, we first solve $x = \sin t$ for $\csc t$:

$$x = \sin t$$
$$\frac{1}{x} = \frac{1}{\sin t}$$
$$\frac{1}{x} = \csc t$$

Then we substitute $\dfrac{1}{x}$ for $\csc t$ in $y = \csc t$: $y = \dfrac{1}{x}$.

For $0 < t < \pi,\ 0 < x \le 1$.

Then we have $y = \dfrac{1}{x},\ 0 < x \le 1$.

19. To find an equivalent rectangular equation, we first solve for $\cos t$ and $\sin t$ in the parametric equations:

$$x = 1 + 2\cos t \qquad y = 2 + 2\sin t$$
$$x - 1 = 2\cos t \qquad y - 2 = 2\sin t$$
$$\frac{x-1}{2} = \cos t \qquad \frac{y-2}{2} = \sin t$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can substitute to eliminate the parameter:

$$\sin^2 t + \cos^2 t = 1$$
$$\left(\frac{y-2}{2}\right)^2 + \left(\frac{x-1}{2}\right)^2 = 1$$
$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{4} = 1$$
$$(x-1)^2 + (y-2)^2 = 4$$

For $0 \le t \le 2\pi,\ -1 \le 1 + 2\cos t \le 3$.

Then we have $(x-1)^2 + (y-2)^2 = 4,\ -1 \le x \le 3$.

20. To find an equivalent rectangular equation, we first solve for $\sec t$ and $\tan t$ in the parametric equations:

$$x = 2 + \sec t \qquad y = 1 + 3\tan t$$
$$x - 2 = \sec t \qquad y - 1 = 3\tan t$$
$$\frac{y-1}{3} = \tan t$$

Using the identity $1+\tan^2\theta = \sec^2\theta$, we can substitute to eliminate the parameter:

$$1 + \tan^2 t = \sec^2 t$$
$$1 + \left(\frac{y-1}{3}\right)^2 = (x-2)^2$$
$$1 = (x-2)^2 - \frac{(y-1)^2}{9}$$

For $0 < t < \dfrac{\pi}{2}$, $3 < 2 + \sec t < \infty$.

Then we have $(x-2)^2 - \dfrac{(y-1)^2}{9} = 1$, $x > 3$.

21. $y = 4x - 3$

Answers may vary.

If $x = t$, then $y = 4t - 3$.

If $x = \dfrac{t}{4}$, then $y = 4\cdot\dfrac{t}{4} - 3 = t - 3$.

22. $y = x^2 - 1$

Answers may vary.

If $x = t$, then $y = t^2 - 1$.

If $x = t-2$, then $y = (t-2)^2 - 1 = t^2 - 4t + 4 - 1 = t^2 - 4t + 3$.

23. $y = (x-2)^2 - 6x$

Answers may vary.

If $x = t$, then $y = (t-2)^2 - 6t$.

If $x = t+2$, then $y = (t+2-2)^2 - 6(t+2) = t^2 - 6t - 12$.

24. $y = x^3 + 3$

Answers may vary.

If $x = t$, then $y = t^3 + 3$.

If $x = \sqrt[3]{t}$, then $y = (\sqrt[3]{t})^3 + 3 = t + 3$.

25. a) We substitute 7 for h, 80 for v_0, and 30°for θ in the parametric equations for projective motion.

$$\begin{aligned} x &= (v_0\cos\theta)t \\ &= (80\cos 30°)t \\ &= \left(80\cdot\frac{\sqrt{3}}{2}\right)t = 40\sqrt{3}t \end{aligned}$$

$$\begin{aligned} y &= h + (v_0\sin\theta)t - 16t^2 \\ &= 7 + (80\sin 30°)t - 16t^2 \\ &= 7 + \left(80\cdot\frac{1}{2}\right)t - 16t^2 \\ &= 7 + 40t - 16t^2 \end{aligned}$$

b) The height of the ball at time t is given by y.

When $t = 1$, $y = 7 + 40\cdot 1 - 16\cdot 1^2 = 31$ ft.

When $t = 2$, $y = 7 + 40\cdot 2 - 16\cdot 2^2 = 23$ ft.

c) The ball hits the ground when $y = 0$, so we solve the equation $y = 0$ using the quadratic formula.

$$7 + 40t - 16t^2 = 0$$
$$-16t^2 + 40t + 7 = 0$$
$$t = \frac{-40 \pm \sqrt{40^2 - 4(-16)(7)}}{2(-16)}$$
$$t \approx -0.2 \quad or \quad t \approx 2.7$$

The negative value for t has no meaning in this application. Thus the ball is in the air for about 2.7 sec.

d) Since the ball is in the air for about 2.7 sec, the horizontal distance it travels is given by

$$x = 40\sqrt{3}(2.7) \approx 187.1 \text{ ft.}$$

e) To find the maximum height of the ball, we find the maximum value of y. At the vertex of the quadratic function represented by y we have

$$t = -\frac{b}{2a} = -\frac{40}{2(-16)} = 1.25.$$

When $t = 1.25$,

$$y = 7 + 40(1.25) - 16(1.25)^2 = 32 \text{ ft.}$$

26. a) $x = (200\cos 60°)t = \left(200\cdot\dfrac{1}{2}\right)t = 100t$

$y = 0 + (200\sin 60°)t - 16t^2 = \left(200\cdot\dfrac{\sqrt{3}}{2}\right)t - 16t^2 = 100\sqrt{3}t - 16t^2$

b) When $t = 4$, $y = 100\sqrt{3}(4) - 16\cdot 4^2 \approx 436.8$ ft.

When $t = 8$, $y = 100\sqrt{3}(8) - 16\cdot 8^2 \approx 361.6$ ft.

c) Solve $y = 0$.

$$100\sqrt{3}t - 16t^2 = 0$$
$$t(100\sqrt{3} - 16t) = 0$$
$$t = 0 \quad or \quad t \approx 10.8$$

The projectile is in the air for about 10.8 sec.

d) When $t \approx 10.8$, $x \approx 100(10.8) \approx 1080$ ft.

e) At the vertex of the quadratic function represented by y we have

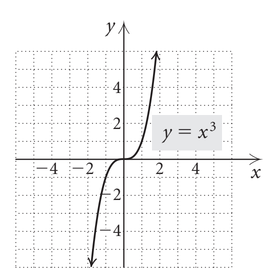
$$t = -\frac{b}{2a} = -\frac{100\sqrt{3}}{2(-16)} \approx 5.4$$

When $t \approx 5.4$,

$$y \approx 100\sqrt{3}(5.4) - 16(5.4)^2 \approx 468.7 \text{ ft.}$$

27. Graph $y = x^3$.

x	y
-2	-8
-1	-1
0	0
1	1
2	8



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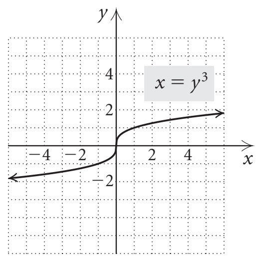
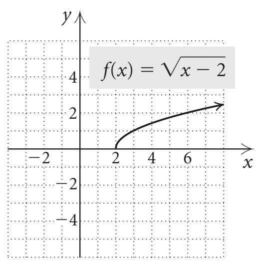


29. Graph $f(x) = \sqrt{x-2}$.

x	$f(x)$
2	0
3	1
6	2
11	3



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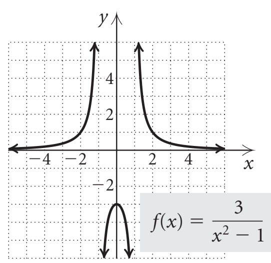


31. The curve is generated clockwise when the equations $x = 3\cos t,\ y = -3\sin t$ are used. Alternatively, the equations $x = 3\sin t,\ y = 3\cos t$ can be used.

32. To find the equivalent rectangular equation we first solve the given equations for $\cos^2 t$ and $\sin^2 t$, respectively.

$$x = \cos^3 t \qquad\qquad y = \sin^3 t$$
$$x^{2/3} = \cos^2 t \qquad\qquad y^{2/3} = \sin^2 t$$

Now substitute in the identity $\sin^2\theta + \cos^2\theta = 1$.

$$y^{2/3} + x^{2/3} = 1, \text{ or}$$
$$x^{2/3} + y^{2/3} = 1$$

Chapter 10 Review Exercises

1.
$$\begin{aligned} x + y^2 &= 1 \\ y^2 &= x - 1 \\ (y-0)^2 &= 4\cdot\frac{1}{4}(x-1) \end{aligned}$$

This parabola has a horizontal axis of symmetry, the focus is $\left(\frac{5}{4}, 0\right)$, and the directrix is $x = \frac{3}{4}$. Thus it opens to the left and the statement is true.

2.
$$\frac{(x-2)^2}{4} + \frac{(y+3)^2}{9} = 1$$
$$\frac{(x-2)^2}{2^2} + \frac{[y-(-3)]^2}{3^2} = 1 \quad \text{Standard form}$$

The graph is an ellipse with center $(2, -3)$, so the statement is false.

3. False; see pages 833 and 835 in the text.

4. The statement is true. See page 843 in the text.

5. The statement is false. See Example 4 on page 866 in the text.

6. Graph (d) is the graph of $y^2 = 5x$.

7. Graph (a) is the graph of $y^2 = 9 - x^2$.

8. Graph (e) is the graph of $3x^2 + 4y^2 = 12$.

9. Graph (g) is the graph of $9y^2 - 4x^2 = 36$.

10. Graph (b) is the graph of $x^2 + y^2 + 2x - 3y = 8$.

11. Graph (f) is the graph of $4x^2 + y^2 - 16x - 6y = 15$.

12. Graph (h) is the graph of $x^2 - 8x + 6y = 0$.

13. Graph (c) is the graph of $\dfrac{(x+3)^2}{16} - \dfrac{(y-1)^2}{25} = 1$.

14.
$$\begin{aligned} (x-h)^2 &= 4p(y-k) \\ (x-0)^2 &= 4\left(-\frac{3}{2}\right)(y-0) \\ x^2 &= -6y \end{aligned}$$

15.
$$\begin{aligned} y^2 &= -12x \\ y^2 &= 4(-3)x \end{aligned}$$

F: $(-3, 0)$, V: $(0, 0)$, D: $x = 3$

16.
$$\begin{aligned} x^2 + 10x + 2y + 9 &= 0 \\ x^2 + 10x &= -2y - 9 \\ x^2 + 10x + 25 &= -2y - 9 + 25 \\ (x+5)^2 &= -2y + 16 \\ [x-(-5)]^2 &= 4\left(-\frac{1}{2}\right)(y-8) \end{aligned}$$

$V:\ (-5, 8)$

$F:\ \left(-5, 8 + \left(-\frac{1}{2}\right)\right)$ or $\left(-5, \frac{15}{2}\right)$

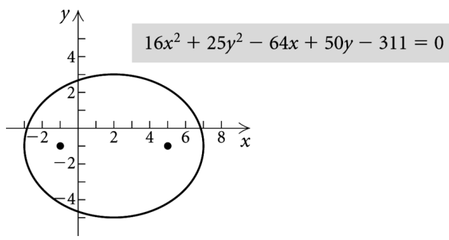
$D:\ y = 8 + \frac{1}{2} = \frac{17}{2}$

17. Begin by completing the square twice.

$$\begin{aligned} 16x^2 + 25y^2 - 64x + 50y - 311 &= 0 \\ 16(x^2 - 4x) + 25(y^2 + 2y) &= 311 \\ 16(x^2 - 4x + 4) + 25(y^2 + 2y + 1) &= 311 + 16\cdot 4 + 25\cdot 1 \\ 16(x-2)^2 + 25(y+1)^2 &= 400 \\ \frac{(x-2)^2}{25} + \frac{[y-(-1)]^2}{16} &= 1 \end{aligned}$$

The center is $(2, -1)$. Note that $a = 5$ and $b = 4$. The major axis is horizontal so the vertices are 5 units left and right of the center: $(2-5, -1)$ and $(2+5, -1)$, or $(-3, -1)$

and $(7,-1)$. We know that $c^2 = a^2 - b^2 = 25 - 16 = 9$ and $c = \sqrt{9} = 3$. Then the foci are 3 units left and right of the center: $(2-3,-1)$ and $(2+3,-1)$, or $(-1,-1)$ and $(5,-1)$.



18. The vertices $(0,-4)$ and $(0,4)$ are on the y-axis, so the major axis is vertical and $a = 4$. Since the vertices are equidistant from the origin, the center of the ellipse is at the origin. The length of the minor axis is 6, so $b = 6/2$, or 3. The equation is

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

19. Begin by completing the square twice.

$$x^2 - 2y^2 + 4x + y - \frac{1}{8} = 0$$
$$(x^2 + 4x) - 2\left(y^2 - \frac{1}{2}y\right) = \frac{1}{8}$$
$$(x^2 + 4x + 4) - 2\left(y^2 - \frac{1}{2}y + \frac{1}{16}\right) = \frac{1}{8} + 4 - 2 \cdot \frac{1}{16}$$
$$(x+2)^2 - 2\left(y - \frac{1}{4}\right)^2 = 4$$
$$\frac{[x-(-2)]^2}{4} - \frac{\left(y - \frac{1}{4}\right)^2}{2} = 1$$

The center is $\left(-2, \frac{1}{4}\right)$. The transverse axis is horizontal, so the vertices are 2 units left and right of the center: $\left(-2-2, \frac{1}{4}\right)$ and $\left(-2+2, \frac{1}{4}\right)$, or $\left(-4, \frac{1}{4}\right)$ and $\left(0, \frac{1}{4}\right)$. Since $c^2 = a^2 + b^2$, we have $c^2 = 4 + 2 = 6$ and $c = \sqrt{6}$. Then the foci are $\sqrt{6}$ units left and right of the center: $\left(-2-\sqrt{6}, \frac{1}{4}\right)$ and $\left(-2+\sqrt{6}, \frac{1}{4}\right)$.

Find the asymptotes:

$$y - k = \frac{b}{a}(x-h) \quad \text{and} \quad y - k = -\frac{b}{a}(x-h)$$
$$y - \frac{1}{4} = \frac{\sqrt{2}}{2}(x+2) \quad \text{and} \quad y - \frac{1}{4} = -\frac{\sqrt{2}}{2}(x+2)$$

20. The parabola is of the form $y^2 = 4px$. A point on the parabola is $\left(1.5, \frac{2}{2}\right)$, or $(1.5, 1)$.

$$y^2 = 4px$$
$$1^2 = 4 \cdot p \cdot 1.5$$
$$1 = 6p$$
$$\frac{1}{6} = p$$

Since the focus is at $(p,0) = \left(\frac{1}{6}, 0\right)$, the focus is $\frac{1}{6}$ ft, or 0.167 ft, from the vertex.

21. $x^2 - 16y = 0,\quad (1)$
$x^2 - y^2 = 64 \quad (2)$

From equation (1) we have $x^2 = 16y$. Substitute in equation (2).

$$16y - y^2 = 64$$
$$0 = y^2 - 16y + 64$$
$$0 = (y-8)^2$$
$$0 = y - 8$$
$$8 = y$$
$$x^2 - (8)^2 = 64 \qquad \text{Substituting in equation (2)}$$
$$x^2 = 128$$
$$x = \pm\sqrt{128} = \pm 8\sqrt{2}$$

The pairs $(-8\sqrt{2}, 8)$ and $(8\sqrt{2}, 8)$ check.

22. $4x^2 + 4y^2 = 65,\quad (1)$
$6x^2 - 4y^2 = 25 \quad (2)$

$$10x^2 = 90 \qquad \text{Adding}$$
$$x^2 = 9$$
$$x = \pm\sqrt{9} = \pm 3$$
$$4(\pm 3)^2 + 4y^2 = 65 \qquad \text{Substituting in (1)}$$
$$36 + 4y^2 = 65$$
$$4y^2 = 29$$
$$y = \pm\sqrt{\frac{29}{4}} = \frac{\pm\sqrt{29}}{2}$$

The pairs $\left(3, \frac{\sqrt{29}}{2}\right)$, $\left(-3, \frac{\sqrt{29}}{2}\right)$, $\left(3, -\frac{\sqrt{29}}{2}\right)$, and $\left(-3, -\frac{\sqrt{29}}{2}\right)$ check.

23. $x^2 - y^2 = 33,\quad (1)$
$x + y = 11 \quad (2)$

$$y = -x + 11$$
$$x^2 - (-x+11)^2 = 33 \qquad \text{Substituting in (1)}$$
$$x^2 - (x^2 - 22x + 121) = 33$$
$$x^2 - x^2 + 22x - 121 = 33$$
$$22x = 154$$
$$x = 7$$
$$y = -7 + 11 = 4$$

The pair $(7,4)$ checks.

24. $x^2 - 2x + 2y^2 = 8,\quad (1)$
$2x + y = 6 \quad (2)$

$$y = -2x + 6$$
$$x^2 - 2x + 2(-2x+6)^2 = 8 \qquad \text{Substituting in (1)}$$
$$x^2 - 2x + 2(4x^2 - 24x + 36) = 8$$
$$x^2 - 2x + 8x^2 - 48x + 72 = 8$$
$$9x^2 - 50x + 64 = 0$$
$$(x-2)(9x-32) = 0$$

$$x-2=0 \quad or \quad 9x-32=0$$
$$x=2 \quad or \quad 9x=32$$
$$x=2 \quad or \quad x=\frac{32}{9}$$
$$y=-2(2)+6=2$$
$$y=-2\left(\frac{32}{9}\right)+6=-\frac{10}{9}$$

The pairs $(2,2)$ and $\left(\frac{32}{9},-\frac{10}{9}\right)$ check.

25. $x^2-y=3, \quad (1)$
$2x-y=3 \quad (2)$

From equation (1) we have $y=x^2-3$. Substitute in equation (2).

$$2x-(x^2-3)=3$$
$$2x-x^2+3=3$$
$$0=x^2-2x$$
$$0=x(x-2)$$
$$x=0 \quad or \quad x=2$$
$$y=0^2-3=-3$$
$$y=2^2-3=1$$

The pairs $(0,-3)$ and $(2,1)$ check.

26.
$$\begin{array}{rll} x^2+y^2 &= 25, & (1) \\ x^2-y^2 &= 7 & (2) \\ \hline 2x^2 \quad &= 32 & \text{Adding} \end{array}$$
$$x^2=16$$
$$x=\pm 4$$
$$(\pm 4)^2+y^2=25$$
$$y^2=9$$
$$y=\pm 3$$

The pairs $(4,3)$, $(4,-3)$, $(-4,3)$ and $(-4,-3)$ check.

27. $x^2-y^2=3, \quad (1)$
$y=x^2-3 \quad (2)$

From equation (2) we have $x^2=y+3$. Substitute in equation (1).

$$y+3-y^2=3$$
$$0=y^2-y$$
$$0=y(y-1)$$
$$y=0 \quad or \quad y=1$$
$$x^2=0+3$$
$$x^2=3$$
$$x=\pm\sqrt{3}$$
$$x^2=1+3$$
$$x^2=4$$
$$x=\pm 2$$

The pairs $(\sqrt{3},0)$, $(-\sqrt{3},0)$, $(2,1)$, and $(-2,1)$ check.

28. $x^2+y^2=18, \quad (1)$
$2x+y=3 \quad (2)$

$$y=-2x+3 \qquad \text{Solving (2) for } y$$
$$x^2+(-2x+3)^2=18 \qquad \text{Substituting in (1)}$$
$$x^2+4x^2-12x+9=18$$
$$5x^2-12x-9=0$$
$$(5x+3)(x-3)=0$$
$$5x+3=0 \quad or \quad x-3=0$$
$$5x=-3 \quad or \quad x=3$$
$$x=-\frac{3}{5} \quad or \quad x=3$$
$$y=-2\left(-\frac{3}{5}\right)+3=\frac{21}{5}$$
$$y=-2\cdot 3+3=-3$$

The pairs $\left(-\frac{3}{5},\frac{21}{5}\right)$ and $(3,-3)$ check.

29.
$$\begin{array}{rll} x^2+y^2 &= 100, & (1) \\ 2x^2-3y^2 &= -120 & (2) \end{array}$$
$$\begin{array}{rll} 3x^2+3y^2 &= 300 & \text{Multiplying (1) by 3} \\ 2x^2-3y^2 &= -120 & \\ \hline 5x^2 \quad &= 180 & \text{Adding} \end{array}$$
$$x^2=36$$
$$x=\pm 6$$
$$(\pm 6)^2+y^2=100$$
$$y^2=64$$
$$y=\pm 8$$

The pairs $(6,8)$, $(-6,8)$, $(6,-8)$, and $(-6,-8)$ check.

30. $x^2+2y^2=12, \quad (1)$
$xy=4 \quad (2)$

$$y=\frac{4}{x} \qquad \text{Solving (2) for } y.$$
$$x^2+2\left(\frac{4}{x}\right)^2=12 \qquad \text{Substituting in (1)}$$
$$x^2+\frac{32}{x^2}=12$$
$$x^4+32=12x^2$$
$$x^4-12x^2+32=0$$
$$(x^2-4)(x^2-8)=0$$
$$x^2-4=0 \quad or \quad x^2-8=0$$
$$x^2=4 \quad or \quad x^2=8$$
$$x=\pm 2 \quad or \quad x=\pm 2\sqrt{2}$$
$$y=\frac{4}{2}=2$$
$$y=\frac{4}{-2}=-2$$
$$y=\frac{4}{2\sqrt{2}}=\sqrt{2}$$
$$y=\frac{4}{-2\sqrt{2}}=-\sqrt{2}$$

The pairs $(2,2)$, $(-2,-2)$, $(2\sqrt{2},\sqrt{2})$ and $(-2\sqrt{2},-\sqrt{2})$ check.

31. ***Familiarize***. Let x and y represent the numbers.

Translate. The sum of the numbers is 11.

$$x+y=11$$

The sum of the squares of the numbers is 65.

$$x^2+y^2=65$$

Carry out. We solve the system of equations.

$$x+y=11, \quad (1)$$
$$x^2+y^2=65 \quad (2)$$

First we solve equation (1) for y.

$$y=11-x$$

Then substitute $11-x$ for y in equation (2) and solve for x.

$$\begin{aligned} x^2+(11-x)^2&=65\\ x^2+121-22x+x^2&=65\\ 2x^2-22x+121&=65\\ 2x^2-22x+56&=0\\ x^2-11x+28&=0 \quad \text{Dividing by 2}\\ (x-4)(x-7)&=0 \end{aligned}$$

$$x-4=0 \quad or \quad x-7=0$$
$$x=4 \quad or \quad x=7$$

If $x=4$, then $y=11-4=7$.

If $x=7$, then $y=11-7=4$.

In either case, the possible numbers are 4 and 7.

Check. $4+7=11$ and $4^2+7^2=16+49=65$. The answer checks.

State. The numbers are 4 and 7.

32. ***Familiarize***. Let l and w represent the length and width of the rectangle, in meters, respectively.

Translate. The perimeter is 38 m.

$$2l+2w=38$$

The area is 84 m^2.

$$lw=84$$

Carry out. We solve the system of equations:

$$2l+2w=38, \quad (1)$$
$$lw=84 \quad (2)$$

First solve equation (2) for w.

$$w=\frac{84}{l}$$

Then substitute $\frac{84}{l}$ for w in equation (1) and solve for l.

$$\begin{aligned} 2l+2\cdot\frac{84}{l}&=38\\ 2l+\frac{168}{l}&=38\\ 2l^2+168&=38l \quad \text{Multiplying by } l\\ 2l^2-38l+168&=0\\ 2(l-7)(l-12)&=0 \end{aligned}$$

$$l-7=0 \quad or \quad l-12=0$$
$$l=7 \quad or \quad l=12$$

If $l=7$, then $w=\frac{84}{7}=12$. If $l=12$, then $w=\frac{84}{12}=7$. Since length is usually considered to be longer than width, we have $l=12$ and $w=7$, or $(12,7)$.

Check. If $l=12$ and $w=7$, then the perimeter is $2\cdot12+2\cdot7$, or 38, and the area is $12\cdot7$, or 84. The solution checks.

State. The length of the rectangle is 12 m, and the width is 7 m.

33. ***Familiarize***. Let x and y represent the positive integers.

Translate. The sum of the numbers is 12.

$$x+y=12$$

The sum of the reciprocals is $\frac{3}{8}$.

$$\frac{1}{x}+\frac{1}{y}=\frac{3}{8}$$

Carry out. We solve the system of equations.

$$x+y=12, \quad (1)$$
$$\frac{1}{x}+\frac{1}{y}=\frac{3}{8} \quad (2)$$

First solve equation (1) for y.

$$y=12-x$$

Then substitute $12-x$ for y in equation (2) and solve for x.

$$\begin{aligned} \frac{1}{x}+\frac{1}{12-x}&=\frac{3}{8}, \text{ LCD is } 8x(12-x)\\ 8x(12-x)\left(\frac{1}{x}+\frac{1}{12-x}\right)&=8x(12-x)\cdot\frac{3}{8}\\ 8(12-x)+8x&=x(12-x)\cdot3\\ 96-8x+8x&=36x-3x^2\\ 96&=36x-3x^2\\ 3x^2-36x+96&=0\\ x^2-12x+32&=0 \quad \text{Dividing by 3}\\ (x-4)(x-8)&=0 \end{aligned}$$

$$x-4=0 \quad or \quad x-8=0$$
$$x=4 \quad or \quad x=8$$

If $x=4$, $y=12-4=8$.

If $x=8$, $y=12-8=4$.

In either case, the possible numbers are 4 and 8.

Check. $4+8=12$; $\frac{1}{4}+\frac{1}{8}=\frac{2}{8}+\frac{1}{8}=\frac{3}{8}$. The answer checks.

State. The numbers are 4 and 8.

34. ***Familiarize***. Let x and y represent the length of a side of the larger and smaller squares, in centimeters, respectively.

Translate. The perimeter of the larger square is 12 cm more than the perimeter of the smaller square.

$$4x=4y+12$$

The area of the larger square exceeds the area of the other square by 39 cm^2.

$x^2 = y^2 + 39$

We have a system of equations.

$4x = 4y + 12,\quad (1)$

$x^2 = y^2 + 39 \quad (2)$

Carry out. We solve the system of equations. First solve equation (1) for x.

$x = y + 3$

Then substitute $y + 3$ for x in equation (2) and solve for y.

$$\begin{aligned}(y+3)^2 &= y^2 + 39\\ y^2 + 6y + 9 &= y^2 + 39\\ 6y + 9 &= 39\\ 6y &= 30\\ y &= 5\end{aligned}$$

If $y = 5$, then $x = 5 + 3 = 8$.

Check. The perimeters of the square are $4 \cdot 8$, or 32 cm, and $4 \cdot 5$, or 20 cm. The perimeter of the larger square is 12 cm more than the perimeter of the smaller square. The areas are 8^2, or 64 cm^2, and 5^2, or 25 cm^2. The area of the larger square exceeds the area of the smaller square by 39 cm^2. The solution checks.

State. The perimeters of the squares are 32 cm and 20 cm.

35. ***Familiarize***. Let x = the radius of the larger circle and let y = the radius of the smaller circle. We will use the formula for the area of a circle, $A = \pi r^2$.

Translate. The sum of the areas is 130π ft^2.

$\pi x^2 + \pi y^2 = 130\pi$

The difference of the areas is 112π ft^2.

$\pi x^2 - \pi y^2 = 112\pi$

We have a system of equations.

$\pi x^2 + \pi y^2 = 130\pi,\quad (1)$

$\pi x^2 - \pi y^2 = 112\pi \quad (2)$

Carry out. We add.

$$\begin{array}{rl}\pi x^2 + \pi y^2 &= 130\pi\\ \underline{\pi x^2 - \pi y^2} &\underline{= 112\pi}\\ 2\pi x^2 \qquad\quad &= 242\pi\\ x^2 &= 121 \quad \text{Dividing by } 2\pi\\ x &= \pm 11\end{array}$$

Since the length of a radius cannot be negative, we consider only $x = 11$. Substitute 11 for x in equation (1) and solve for y.

$$\begin{aligned}\pi \cdot 11^2 + \pi y^2 &= 130\pi\\ 121\pi + \pi y^2 &= 130\pi\\ \pi y^2 &= 9\pi\\ y^2 &= 9\\ y &= \pm 3\end{aligned}$$

Again, we consider only the positive solution.

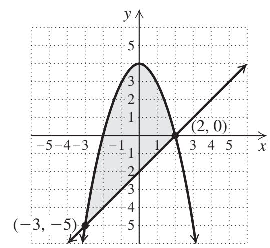
Check. If the radii are 11 ft and 3 ft, the sum of the areas is $\pi \cdot 11^2 + \pi \cdot 3^2 = 121\pi + 9\pi = 130\pi$ ft^2. The difference of the areas is $121\pi - 9\pi = 112\pi$ ft^2. The answer checks.

State. The radius of the larger circle is 11 ft, and the radius of the smaller circle is 3 ft.

36. Graph: $y \le 4 - x^2$,

$x - y \le 2$

The solution set of $y \le 4 - x^2$ is the parabola $y = 4 - x^2$ and the region inside it. The solution set of $x - y \le 2$ is the line $x - y = 2$ and the half-plane above it. We shade the region common to the two solution sets.



To find the points of intersection of the graphs we solve the system of equations

$y = 4 - x^2,$

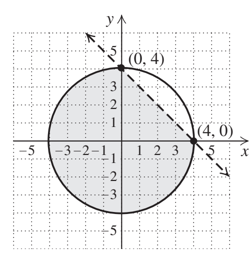
$x - y = 2.$

The points of intersection are $(-3, -5)$ and $(2, 0)$.

37. Graph: $x^2 + y^2 \le 16$,

$x + y < 4$

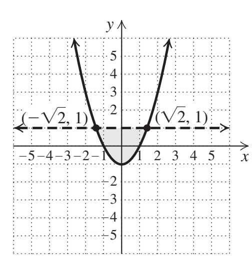
The solution set of $x^2 + y^2 \le 16$ is the circle $x^2 + y^2 = 16$ and the region inside it. The solution set of $x + y < 4$ is the half-plane below the line $x + y = 4$. We shade the region common to the two solution sets.



38. Graph: $y \ge x^2 - 1$,

$y < 1$

The solution set of $y \ge x^2 - 1$ is the parabola $y = x^2 - 1$ and the region inside it. The solution set of $y < 1$ is the half-plane below the line $y = 1$. We shade the region common to the two solution sets.



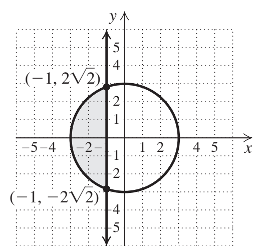
To find the points of intersection of the graphs of the related equations we solve the system of equations

$$y = x^2 - 1,$$
$$y = 1.$$

The points of intersection are $(-\sqrt{2}, 1)$ and $(\sqrt{2}, 1)$.

39. Graph: $x^2 + y^2 \le 9,$
$$x \le -1$$

The solution set of $x^2 + y^2 \le 9$ is the circle $x^2 + y^2 = 9$ and the region inside it. The solution set of $x \le -1$ is the line $x = -1$ and the half-plane to the left of it. We shade the region common to the two solution sets.



40. $5x^2 - 2xy + 5y^2 - 24 = 0$

$A = 5,\ B = -2,\ C = 5$

$B^2 - 4AC = (-2)^2 - 4 \cdot 5 \cdot 5 = 4 - 100 = -96$

Since the discriminant is negative, the graph is an ellipse (or circle). To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A - C}{B} = \frac{5 - 5}{-2} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y')$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y')$$

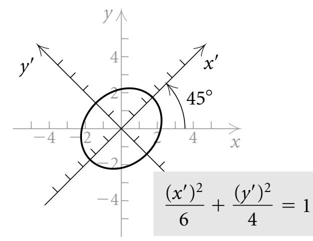
Substitute for x and y in the given equation.

$$5\left[\frac{\sqrt{2}}{2}(x' - y')\right]^2 - 2\left[\frac{\sqrt{2}}{2}(x' - y')\right]\left[\frac{\sqrt{2}}{2}(x' + y')\right] +$$
$$5\left[\frac{\sqrt{2}}{2}(x' + y')\right]^2 - 24 = 0$$

After simplifying we have

$$4(x')^2 + 6(y')^2 - 24 = 0, \text{ or}$$
$$\frac{(x')^2}{6} + \frac{(y')^2}{4} = 1.$$

This is the equation of an ellipse with vertices $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$ on the x'-axis. The y'-intercepts are $(0, -2)$ and $(0,2)$.



41. $x^2 - 10xy + y^2 + 12 = 0$

$B^2 - 4AC = (-10)^2 - 4 \cdot 1 \cdot 1 = 100 - 4 = 96$

Since the discriminant is positive, the graph is a hyperbola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 1}{-10} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

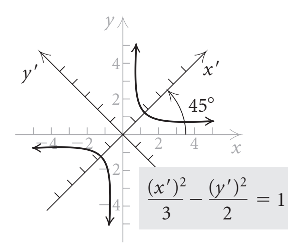
Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y')$$

$$y = x'\sin\theta + y'\cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y')$$

After substituting for x and y in the given equation and simplifying, we have

$$\frac{(x')^2}{3} - \frac{(y')^2}{2} = 1.$$



42. $5x^2 + 6\sqrt{3}xy - y^2 = 16$

$A = 5,\ B = 6\sqrt{3},\ C = -1$

$B^2 - 4AC = (6\sqrt{3})^2 - 4 \cdot 5 \cdot (-1) = 108 + 20 = 128$

Since the discriminant is positive, the graph is a hyperbola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A - C}{B} = \frac{5 - (-1)}{6\sqrt{3}} = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Then $2\theta = 60°$ and $\theta = 30°$, so

$$\sin\theta = \frac{1}{2} \text{ and } \cos\theta = \frac{\sqrt{3}}{2}.$$

Now substitute in the rotation of axes formulas.

$$x = x'\cos\theta - y'\sin\theta$$
$$= x' \cdot \frac{\sqrt{3}}{2} - y' \cdot \frac{1}{2} = \frac{x'\sqrt{3}}{2} - \frac{y'}{2}$$

$$y = x' \sin\theta + y' \cos\theta$$
$$= x' \cdot \frac{1}{2} + y' \cdot \frac{\sqrt{3}}{2} = \frac{x'}{2} + \frac{y'\sqrt{3}}{2}$$

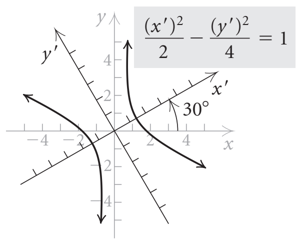
Substitute for x and y in the given equation.

$$5\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^2 + 6\sqrt{3}\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right) -$$
$$\left(\frac{x'}{2} + \frac{y'\sqrt{3}}{2}\right)^2 = 16$$

After simplifying we have

$$\frac{(x')^2}{2} - \frac{(y')^2}{4} = 1.$$

This is the equation of a hyperbola with vertices $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$ and asymptotes $y' = -\frac{2}{\sqrt{2}}x'$ and $y' = \frac{2}{\sqrt{2}}x'$, or $y' = -\sqrt{2}x'$ and $y' = \sqrt{2}x'$.



43. $x^2 + 2xy + y^2 - \sqrt{2}x + \sqrt{2}y = 0$

$B^2 - 4AC = 2^2 - 4 \cdot 1 \cdot 1 = 4 - 4 = 0$

Since the discriminant is zero, the graph is a parabola. To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 1}{2} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

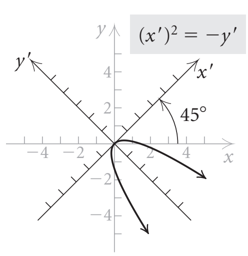
Now substitute in the rotation of axes formulas.

$$x = x' \cos\theta - y' \sin\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y')$$

$$y = x' \sin\theta + y' \cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y')$$

After substituting for x and y in the given equation and simplifying, we have

$$(x')^2 = -y'.$$



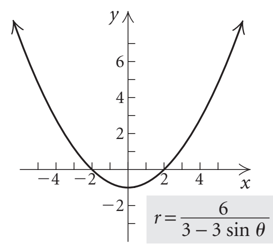
44. $r = \dfrac{6}{3 - 3\sin\theta}$

We first divide the numerator and denominator by 3:

$$r = \frac{2}{1 - \sin\theta}$$

This equation is in the form $r = \dfrac{ep}{1 - e\sin\theta}$ with $e = 1$.

Since $e = 1$, the graph is a parabola.



Since $e = 1$ and $ep = 1 \cdot p = 2$, we have $p = 2$.

Thus the parabola has a horizontal directrix 2 units below the pole.

We find the vertex by letting $\theta = \dfrac{3\pi}{2}$. When $\theta = \dfrac{3\pi}{2}$,

$$r = \frac{6}{3 - 3\sin\frac{3\pi}{2}} = \frac{6}{3 - 3(-1)} = 1.$$

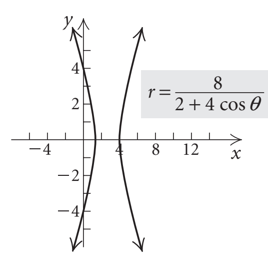
Thus the vertex is $\left(1, \dfrac{3\pi}{2}\right)$.

45. $r = \dfrac{8}{2 + 4\cos\theta}$

We first divide the numerator and denominator by 2:

$$r = \frac{4}{1 + 2\cos\theta}$$

Thus the equation is in the form $r = \dfrac{ep}{1 + e\cos\theta}$ with $e = 2$. Since $e > 1$, the graph is a hyperbola.



Since $e = 2$ and $ep = 2 \cdot p = 4$, we have $p = 2$. Thus the hyperbola has a vertical directrix 2 units to the right of the pole.

We find the vertices by letting $\theta = 0$ and $\theta = \pi$. When $\theta = 0$,

$$r = \frac{8}{2 + 4\cos 0} = \frac{8}{2 + 4} = \frac{4}{3}.$$

When $\theta = \pi$,

$$r = \frac{8}{2 + 4\cos\pi} = \frac{8}{2 - 4} = -4.$$

The vertices are $\left(\dfrac{4}{3}, 0\right)$ and $(-4, \pi)$.

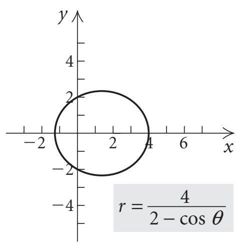
46. $r = \dfrac{4}{2 - \cos\theta}$

We first divide the numerator and denominator by 2:

$$r = \frac{2}{1 - \frac{1}{2}\cos\theta}$$

Then the equation is in the form $r = \dfrac{ep}{1 - e\cos\theta}$ with $e = \dfrac{1}{2}$.

Since $0 < e < 1$, the graph is an ellipse.



Since $e = \dfrac{1}{2}$ and $ep = \dfrac{1}{2} \cdot p = 2$, we have $p = 4$. Thus the ellipse has a vertical directrix 4 units to the left of the pole.

We find the vertices by letting $\theta = 0$ and $\theta = \pi$.

When $\theta = 0$,

$$r = \frac{4}{2 - \cos 0} = \frac{4}{2 - 1} = 4.$$

When $\theta = \pi$,

$$r = \frac{4}{2 - \cos\pi} = \frac{4}{2 - (-1)} = \frac{4}{3}.$$

The vertices are $(4, 0)$ and $\left(\dfrac{4}{3}, \pi\right)$.

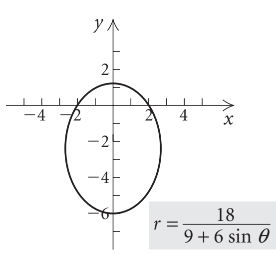
47. $r = \dfrac{18}{9 + 6\sin\theta}$

We first divide the numerator and denominator by 9:

$$r = \frac{2}{1 + \frac{2}{3}\sin\theta}$$

Thus the equation is in the form $r = \dfrac{ep}{1 + e\sin\theta}$ with $e = \dfrac{2}{3}$.

Since $0 < e < 1$, the graph is an ellipse.



Since $e = \dfrac{2}{3}$ and $ep = \dfrac{2}{3} \cdot p = 2$, we have $p = 3$. Thus the ellipse has a horizontal directrix 3 units above the pole.

We find the vertices by letting $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$.

When $\theta = \dfrac{\pi}{2}$,

$$r = \frac{18}{9 + 6\sin\frac{\pi}{2}} = \frac{18}{9 + 6} = \frac{6}{5}.$$

When $\theta = \dfrac{3\pi}{2}$,

$$r = \frac{18}{9 + 6\sin\frac{3\pi}{2}} = \frac{18}{9 - 6} = 6.$$

The vertices are $\left(\dfrac{6}{5}, \dfrac{\pi}{2}\right)$ and $\left(6, \dfrac{3\pi}{2}\right)$.

48.

$$\begin{aligned} r &= \frac{6}{3 - 3\sin\theta} \\ 3r - 3r\sin\theta &= 6 \\ 3r &= 3r\sin\theta + 6 \\ r &= r\sin\theta + 2 \\ \sqrt{x^2 + y^2} &= y + 2 \\ x^2 + y^2 &= y^2 + 4y + 4 \\ x^2 - 4y - 4 &= 0 \end{aligned}$$

49.

$$\begin{aligned} r &= \frac{8}{2 + 4\cos\theta} \\ 2r + 4r\cos\theta &= 8 \\ 2r &= -4r\cos\theta + 8 \\ r &= -2r\cos\theta + 4 \\ \sqrt{x^2 + y^2} &= -2x + 4 \\ x^2 + y^2 &= 4x^2 - 16x + 16 \\ 0 &= 3x^2 - y^2 - 16x + 16 \end{aligned}$$

50.

$$\begin{aligned} r &= \frac{4}{2 - \cos\theta} \\ 2r - r\cos\theta &= 4 \\ 2r &= r\cos\theta + 4 \\ 2\sqrt{x^2 + y^2} &= x + 4 \\ 4x^2 + 4y^2 &= x^2 + 8x + 16 \\ 3x^2 + 4y^2 - 8x - 16 &= 0 \end{aligned}$$

51.

$$\begin{aligned} r &= \frac{18}{9 + 6\sin\theta} \\ 9r + 6r\sin\theta &= 18 \\ 9r &= -6r\sin\theta + 18 \\ 9\sqrt{x^2 + y^2} &= -6y + 18 \\ 3\sqrt{x^2 + y^2} &= -2y + 6 \\ 9x^2 + 9y^2 &= 4y^2 - 24y + 36 \\ 9x^2 + 5y^2 + 24y - 36 &= 0 \end{aligned}$$

52. $e = \dfrac{1}{2}$, $r = 2\sec\theta$

The equation of the directrix can be written

$$r = \frac{2}{\cos\theta}, \text{ or } r\cos\theta = 2.$$

This corresponds to the equation $x = 2$ in rectangular coordinates, so the directrix is a vertical line 2 units to the right of the pole. Using the table on page 868 in the text, we see that the equation is of the form

$$r = \frac{ep}{1+e\cos\theta}.$$

Substituting $\frac{1}{2}$ for e and 2 for p, we have

$$r = \frac{\frac{1}{2}\cdot 2}{1+\frac{1}{2}\cos\theta} = \frac{1}{1+\frac{1}{2}\cos\theta}, \text{ or } \frac{2}{2+\cos\theta}.$$

53. $e = 3$, $r = -6\csc\theta$

The equation of the directrix can be written

$$r = -\frac{6}{\sin\theta}, \text{ or } r\sin\theta = -6.$$

This corresponds to the equation $y = -6$ in rectangular coordinates, so the directrix is a horizontal line 6 units below the pole. Using the table on page 868 in the text, we see that the equation is of the form

$$r = \frac{ep}{1-e\sin\theta}.$$

Substituting 3 for e and 6 for p, we have

$$r = \frac{3\cdot 6}{1-3\sin\theta} = \frac{18}{1-3\sin\theta}.$$

54. $e = 1$, $r = -4\sec\theta$

The equation of the directrix can be written

$$r = -\frac{4}{\cos\theta}, \text{ or } r\cos\theta = -4.$$

This corresponds to the equation $x = -4$ in rectangular coordinates, so the directrix is a vertical line 4 units to the left of the pole. Using the table on page 868 in the text, we see that the equation is of the form

$$r = \frac{ep}{1-e\cos\theta}.$$

Substituting 1 for e and 4 for p, we have

$$r = \frac{1\cdot 4}{1-1\cdot\cos\theta} = \frac{4}{1-\cos\theta}.$$

55. $e = 2$, $r = 3\csc\theta$

The equation of the directrix can be written

$$r = \frac{3}{\sin\theta}, \text{ or } r\sin\theta = 3.$$

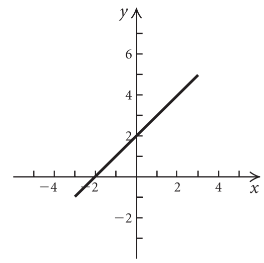
This corresponds to the equation $y = 3$ in rectangular coordinates, so the directrix is a horizontal line 3 units above the pole. Using the table on page 868 in the text, we see that the equation is of the form

$$r = \frac{ep}{1+e\sin\theta}.$$

Substituting 2 for e and 3 for p, we have

$$r = \frac{2\cdot 3}{1+2\sin\theta} = \frac{6}{1+2\sin\theta}.$$

56. $x = t$, $y = 2+t$; $-3 \le t \le 3$



$x = t,\ y = 2+t;\ -3 \leqslant t \leqslant 3$

To find an equivalent rectangular equation, we substitute x for t in $y = 2+t$: $y = 2+x$.

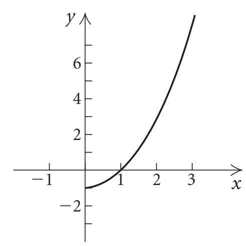
Given that $-3 \le t \le 3$, we find the corresponding restrictions on y:

For $t = -3$: $x = t = -3$.

For $t = 3$: $x = t = 3$.

Then we have $y = 2+x$, $-3 \le x \le 3$.

57. $x = \sqrt{t}$, $y = t-1$; $0 \le t \le 9$



$x = \sqrt{t},\ y = t-1;\ 0 \leqslant t \leqslant 9$

To find an equivalent rectangular equation, we first solve $x = \sqrt{t}$ for t.

$$x = \sqrt{t}$$
$$x^2 = t$$

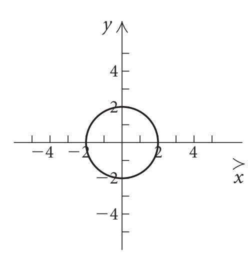
Then we substitute x^2 for t in $y = t-1$: $y = x^2 - 1$. Given that $0 \le t \le 9$, we find the corresponding restrictions on x:

For $t = 0$: $x = \sqrt{t} = \sqrt{0} = 0$.

For $t = 9$: $x = \sqrt{t} = \sqrt{9} = 3$.

Then we have $y = x^2 - 1$, $0 \le x \le 3$.

58. $x = 2\cos t$, $y = 2\sin t$; $0 \le t \le 2\pi$



$x = 2\cos t,\ y = 2\sin t;\ 0 \leqslant t \leqslant 2\pi$

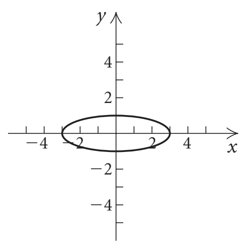
To find an equivalent rectangular equation, we first solve for $\cos t$ and $\sin t$ in the parametric equations:

$$\frac{x}{2} = \cos t,\ \frac{y}{2} = \sin t$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can eliminate the parameter:

$$\sin^2 t + \cos^2 t = 1$$
$$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = 1$$
$$\frac{y^2}{4} + \frac{x^2}{4} = 1$$
$$x^2 + y^2 = 4$$

59. $x = 3 \sin t,\ y = \cos t;\ 0 \le t \le 2\pi$



$x = 3 \sin t,\ y = 2 \cos t;\ 0 \leq t \leq 2\pi$

To find an equivalent rectangular equation, we first solve $x = 3 \sin t$ for $\sin t$: $\frac{x}{3} = \sin t$.

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can eliminate the parameter:

$$\sin^2 t + \cos^2 t = 1$$
$$\left(\frac{x}{3}\right)^2 + y^2 = 1$$
$$\frac{x^2}{9} + y^2 = 1$$

60. $y = 2x - 3$

Answers may vary.

If $x = t$, then $y = 2t - 3$.

If $x = t + 1$, then $y = 2(t + 1) - 3 = 2t - 1$.

61. $y = x^2 + 4$

Answers may vary.

If $x = t$, $y = t^2 + 4$.

If $x = t - 2$, $y = (t - 2)^2 + 4 = t^2 - 4t + 8$.

62. a) We substitute 0 for h, 150 for v_0, and 45°for θ in the parametric equations for projectile motion.

$$x = (v_0 \cos \theta)t$$
$$= (150 \cos 45°)t$$
$$= \left(150 \cdot \frac{\sqrt{2}}{2}\right)t = 75\sqrt{2}t$$
$$y = h + (v_0 \sin \theta)t - 16t^2$$
$$= 0 + (150 \sin 45°)t - 16t^2$$
$$= \left(150 \cdot \frac{\sqrt{2}}{2}\right)t - 16t^2 = 75\sqrt{2}t - 16t^2$$

b) The height of the projectile at time t is given by y.

When $t = 3$, $y = 75\sqrt{2}(3) - 16 \cdot 3^2 \approx 174.2$ ft.

When $t = 6$, $y = 75\sqrt{2}(6) - 16 \cdot 6^2 \approx 60.4$ ft.

c) The ball hits the ground when $y = 0$, so we solve the equation $y = 0$.

$$75\sqrt{2}t - 16t^2 = 0$$
$$t(75\sqrt{2} - 16t) = 0$$
$$t = 0 \quad or \quad t \approx 6.6$$

Time $t = 0$ corresponds to the time before the projectile is launched. The projectile hits the ground when $t \approx 6.6$, so it is in the air for about 6.6 sec.

d) Since the projectile is in the air for about 6.6 sec, the horizontal distance it travels is given by

$$x = 75\sqrt{2}(6.6) \approx 700 \text{ ft.}$$

e) The maximum height of the projectile is the maximum value of the quadratic function represented by y. At the vertex of that function we have

$$t = -\frac{b}{2a} = -\frac{75\sqrt{2}}{2(-16)} \approx 3.3$$

When $t \approx 3.3$,

$$y \approx 75\sqrt{2}(3.3) - 16(3.3)^2 \approx 175.8 \text{ ft.}$$

63. $y^2 - 4y - 12x - 8 = 0$

$$y^2 - 4y = 12x + 8$$
$$y^2 - 4y + 4 = 12x + 8 + 4$$
$$(y - 2)^2 = 12x + 12$$
$$(y - 2)^2 = 12(x + 1)$$
$$(y - 2)^2 = 12[x - (-1)]$$

The vertex of this parabola is $(-1, 2)$, so answer B is correct.

64. A straight line can intersect an ellipse at 0 points, 1 point, or 2 points but not at 4 points, so answer D is correct.

65. $x^2 + 4y^2 = 4$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \quad \text{Dividing by 4}$$

This represents an ellipse with center $(0, 0)$ and a horizontal major axis with vertices $(-2, 0)$ and $(2, 0)$ and y-intercepts $(0, -1)$ and $(0, 1)$. Graph C is the graph of this ellipse.

66. ***Familiarize***. Let x and y represent the numbers.

Translate. The product of the numbers is 4.

$$xy = 4$$

The sum of the reciprocals is $\frac{65}{56}$.

$$\frac{1}{x} + \frac{1}{y} = \frac{65}{56}$$

Carry out. We solve the system of equations.

$$xy = 4, \qquad (1)$$
$$\frac{1}{x} + \frac{1}{y} = \frac{65}{56} \qquad (2)$$

First solve equation (1) for y.

$$y = \frac{4}{x}$$

Then substitute $\frac{4}{x}$ for y in equation (2) and solve for x.

$$\frac{1}{x} + \frac{1}{4/x} = \frac{65}{56}$$
$$\frac{1}{x} + \frac{x}{4} = \frac{65}{56}$$
$$56x\left(\frac{1}{x} + \frac{x}{4}\right) = 56x \cdot \frac{65}{56}$$
$$56 + 14x^2 = 65x$$
$$14x^2 - 65x + 56 = 0$$
$$(2x-7)(7x-8) = 0$$
$$2x - 7 = 0 \quad or \quad 7x - 8 = 0$$
$$2x = 7 \quad or \quad 7x = 8$$
$$x = \frac{7}{2} \quad or \quad x = \frac{8}{7}$$

If $x = \frac{7}{2}$, $y = \frac{4}{7/2} = 4 \cdot \frac{2}{7} = \frac{8}{7}$.

If $x = \frac{8}{7}$, $y = \frac{4}{8/7} = 4 \cdot \frac{7}{8} = \frac{7}{2}$.

In either case the possible numbers are $\frac{7}{2}$ and $\frac{8}{7}$.

Check. $\frac{7}{2} \cdot \frac{8}{7} = 4$; $\frac{1}{7/2} + \frac{1}{8/7} = \frac{2}{7} + \frac{7}{8} = \frac{16}{56} + \frac{49}{56} = \frac{65}{56}$. The answer checks.

State. The numbers are $\frac{7}{2}$ and $\frac{8}{7}$.

67. Using $(x-h)^2 + (y-k)^2 = r^2$ and the given points, we have

$$(10-h)^2 + (7-k)^2 = r^2 \quad (1)$$
$$(-6-h)^2 + (7-k)^2 = r^2 \quad (2)$$
$$(-8-h)^2 + (1-k)^2 = r^2 \quad (3)$$

Subtract equation (2) from equation (1).

$$(10-h)^2 - (-6-h)^2 = 0$$
$$64 - 32h = 0$$
$$h = 2$$

Subtract equation (3) from equation (2).

$$(-6-h)^2 - (-8-h)^2 + (7-k)^2 - (1-k)^2 = 0$$
$$20 - 4h - 12k = 0$$
$$20 - 4(2) - 12k = 0$$
$$k = 1$$

Substitute $h = 2$, $k = 1$ into equation (1).

$$(10-2)^2 + (7-1)^2 = r^2$$
$$100 = r^2$$
$$10 = r$$

The equation is $(x-2)^2 + (y-1)^2 = 100$.

68. The vertices are $(0,-3)$ and $(0,3)$, so the center is $(0,0)$, and the major axis is vertical.

$$\frac{x^2}{a^2} + \frac{y^2}{3^2} = 1$$

Substitute $\left(-\frac{1}{2}, \frac{3\sqrt{3}}{2}\right)$ and solve for a.

$$\frac{\left(-\frac{1}{2}\right)^2}{a^2} + \frac{\left(\frac{3\sqrt{3}}{2}\right)^2}{3^2} = 1$$
$$\frac{1}{4a^2} + \frac{3}{4} = 1$$
$$1 + 3a^2 = 4a^2 \quad \text{Multiplying by } 4a^2$$
$$1 = a^2$$
$$1 = a$$

The equation of the ellipse is $x^2 + \frac{y^2}{9} = 1$.

69. A and B are the foci of the hyperbola, so $c = \frac{400}{2} = 200$.

300 microseconds $\cdot \frac{0.186 \text{ mi}}{1 \text{ microsecond}} = 55.8$ mi, the difference of the ship's distances from the foci. That is, $2a = 55.8$, so $a = 27.9$.

Find b^2.

$$c^2 = a^2 + b^2$$
$$200^2 = 27.9^2 + b^2$$
$$39,221.59 = b^2$$

Then the equation of the hyperbola is

$$\frac{x^2}{778.41} - \frac{y^2}{39,221.59} = 1.$$

70. See page 833 of the text.

71. Circles and ellipses are not functions.

72. The procedure for rotation of axes would be done first when $B \neq 0$. Then you would proceed as when $B = 0$.

73. Each graph is an ellipse. The value of e determines the location of the center and the lengths of the major and minor axes. The larger the value of e, the farther the center is from the pole and the longer the axes.

Chapter 10 Test

1. Graph (c) is the graph of $4x^2 - y^2 = 4$.

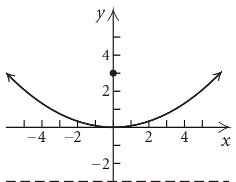
2. Graph (b) is the graph of $x^2 - 2x - 3y = 5$.

3. Graph (a) is the graph of $x^2 + 4x + y^2 - 2y - 4 = 0$.

4. Graph (d) is the graph of $9x^2 + 4y^2 = 36$.

5.
$$x^2 = 12y$$
$$x^2 = 4 \cdot 3y$$

$V:\ (0,0)$, $F:\ (0,3)$, $D:\ y = -3$



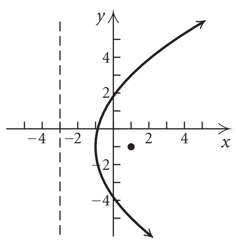
$x^2 = 12y$

6. $$\begin{aligned} y^2 + 2y - 8x - 7 &= 0 \\ y^2 + 2y &= 8x + 7 \\ y^2 + 2y + 1 &= 8x + 7 + 1 \\ (y+1)^2 &= 8x + 8 \\ [y-(-1)]^2 &= 4(2)[x-(-1)] \end{aligned}$$

$V:\ (-1,-1)$

$F:\ (-1+2,-1)\ \ or\ \ (1,-1)$

$D:\ x = -1 - 2 = -3$



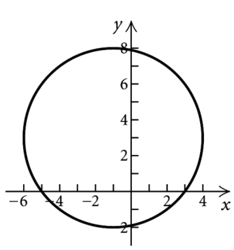
$y^2 + 2y - 8x - 7 = 0$

7. $$\begin{aligned} (x-h)^2 &= 4p(y-k) \\ (x-0)^2 &= 4 \cdot 2(y-0) \\ x^2 &= 8y \end{aligned}$$

8. Begin by completing the square twice.

$$\begin{aligned} x^2 + y^2 + 2x - 6y - 15 &= 0 \\ x^2 + 2x + y^2 - 6y &= 15 \\ (x^2 + 2x + 1) + (y^2 - 6y + 9) &= 15 + 1 + 9 \\ (x+1)^2 + (y-3)^2 &= 25 \\ [x-(-1)]^2 + (y-3)^2 &= 5^2 \end{aligned}$$

Center: $(-1,3)$, radius: 5

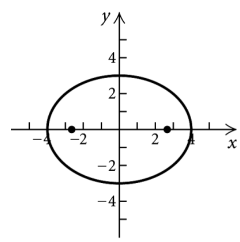


$x^2 + y^2 + 2x - 6y - 15 = 0$

9. $$\begin{aligned} 9x^2 + 16y^2 &= 144 \\ \frac{x^2}{16} + \frac{y^2}{9} &= 1 \\ \frac{x^2}{4^2} + \frac{y^2}{3^2} &= 1 \end{aligned}$$

$a = 4,\ b = 3$

The center is $(0,0)$. The major axis is horizontal, so the vertices are $(-4,0)$ and $(4,0)$. Since $c^2 = a^2 - b^2$, we have $c^2 = 16 - 9 = 7$, so $c = \sqrt{7}$ and the foci are $(-\sqrt{7},0)$ and $(\sqrt{7},0)$.



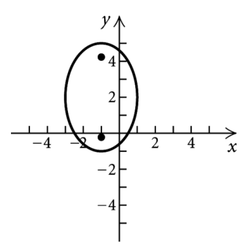
$9x^2 + 16y^2 = 144$

10. $$\begin{aligned} \frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} &= 1 \\ \frac{[x-(-1)]^2}{2^2} + \frac{(y-2)^2}{3^2} &= 1 \end{aligned}$$

The center is $(-1,2)$. Note that $a = 3$ and $b = 2$. The major axis is vertical, so the vertices are 3 units below and above the center:

$(-1,2-3)$ and $(-1,2+3)$ or $(-1,-1)$ and $(-1,5)$.

We know that $c^2 = a^2 - b^2$, so $c^2 = 9 - 4 = 5$ and $c = \sqrt{5}$. Then the foci are $\sqrt{5}$ units below and above the center: $(-1,2-\sqrt{5})$ and $(-1,2+\sqrt{5})$.



$\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1$

11. The vertices $(0,-5)$ and $(0,5)$ are on the y-axis, so the major axis is vertical and $a = 5$. Since the vertices are equidistant from the center, the center of the ellipse is at the origin. The length of the minor axis is 4, so $b = \frac{4}{2} = 2$. The equation is $\frac{x^2}{4} + \frac{y^2}{25} = 1$.

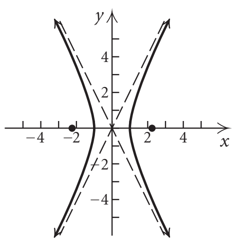
12. $$\begin{aligned} 4x^2 - y^2 &= 4 \\ \frac{x^2}{1} - \frac{y^2}{4} &= 1 \\ \frac{x^2}{1^2} - \frac{y^2}{2^2} &= 1 \end{aligned}$$

The center is $(0,0)$; $a = 1$ and $b = 2$.

The transverse axis is horizontal, so the vertices are $(-1,0)$ and $(1,0)$. Since $c^2 = a^2 + b^2$, we have $c^2 = 1 + 4 = 5$ and $c = \sqrt{5}$. Then the foci are $(-\sqrt{5},0)$ and $(\sqrt{5},0)$.

Find the asymptotes:

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$
$$y = \frac{2}{1}x \quad \text{and} \quad y = -\frac{2}{1}x$$
$$y = 2x \quad \text{and} \quad y = -2x$$



$4x^2 - y^2 = 4$

13. $$\frac{(y-2)^2}{4} - \frac{(x+1)^2}{9} = 1$$
$$\frac{(y-2)^2}{2^2} - \frac{[x-(-1)]^2}{3^2} = 1$$

The center is $(-1, 2)$; $a = 2$ and $b = 3$.

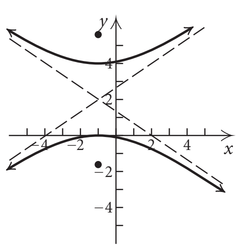
The transverse axis is vertical, so the vertices are 2 units below and above the center:

$(-1, 2-2)$ and $(-1, 2+2)$ or $(-1, 0)$ and $(-1, 4)$.

Since $c^2 = a^2 + b^2$, we have $c^2 = 4 + 9 = 13$ and $c = \sqrt{13}$. Then the foci are $\sqrt{13}$ units below and above the center: $(-1, 2-\sqrt{13})$ and $(-1, 2+\sqrt{13})$.

Find the asymptotes:

$$y - k = \frac{a}{b}(x-h) \quad \text{and} \quad y - k = -\frac{a}{b}(x-h)$$
$$y - 2 = \frac{2}{3}(x-(-1)) \quad \text{and} \quad y - 2 = -\frac{2}{3}(x-(-1))$$
$$y - 2 = \frac{2}{3}(x+1) \quad \text{and} \quad y - 2 = -\frac{2}{3}(x+1)$$
$$y = \frac{2}{3}x + \frac{8}{3} \quad \text{and} \quad y = -\frac{2}{3}x + \frac{4}{3}$$



$\frac{(y-2)^2}{4} - \frac{(x+1)^2}{9} = 1$

14. $$2y^2 - x^2 = 18$$
$$\frac{y^2}{9} - \frac{x^2}{18} = 1$$
$$\frac{y^2}{3^2} - \frac{x^2}{(3\sqrt{2})^2} = 1$$

$h = 0$, $k = 0$, $a = 3$, $b = 3\sqrt{2}$

$$y - k = \frac{a}{b}(x-h) \quad \text{and} \quad y - k = -\frac{a}{b}(x-h)$$
$$y - 0 = \frac{3}{3\sqrt{2}}(x-0) \quad \text{and} \quad y - 0 = -\frac{3}{3\sqrt{2}}(x-0)$$
$$y = \frac{\sqrt{2}}{2}x \quad \text{and} \quad y = -\frac{\sqrt{2}}{2}x$$

15. The parabola is of the form $y^2 = 4px$. A point on the parabola is $\left(6, \frac{18}{2}\right)$, or $(6, 9)$.

$$y^2 = 4px$$
$$9^2 = 4 \cdot p \cdot 6$$
$$81 = 24p$$
$$\frac{27}{8} = p$$

Since the focus is at $(p, 0) = \left(\frac{27}{8}, 0\right)$, the focus is $\frac{27}{8}$ in. from the vertex.

16. $2x^2 - 3y^2 = -10,\quad (1)$

$x^2 + 2y^2 = 9 \quad (2)$

$$\begin{array}{rll} 2x^2 - 3y^2 &= -10 & \\ -2x^2 - 4y^2 &= -18 & \text{Multiplying (2) by } -2 \\ \hline -7y^2 &= -28 & \text{Adding} \\ y^2 &= 4 & \\ y &= \pm 2 & \end{array}$$

$$x^2 + 2(\pm 2)^2 = 9 \quad \text{Substituting into (2)}$$
$$x^2 + 8 = 9$$
$$x^2 = 1$$
$$x = \pm 1$$

The pairs $(1, 2)$, $(1, -2)$, $(-1, 2)$ and $(-1, -2)$ check.

17. $x^2 + y^2 = 13,\quad (1)$

$x + y = 1 \quad (2)$

First solve equation (2) for y.

$$y = 1 - x$$

Then substitute $1 - x$ for y in equation (1) and solve for x.

$$x^2 + (1-x)^2 = 13$$
$$x^2 + 1 - 2x + x^2 = 13$$
$$2x^2 - 2x - 12 = 0$$
$$2(x^2 - x - 6) = 0$$
$$2(x-3)(x+2) = 0$$
$$x = 3 \quad \textit{or} \quad x = -2$$

If $x = 3$, $y = 1 - 3 = -2$. If $x = -2$, $y = 1 - (-2) = 3$.

The pairs $(3, -2)$ and $(-2, 3)$ check.

18. $x + y = 5,\quad (1)$

$xy = 6 \quad (2)$

First solve equation (1) for y.

$$y = -x + 5$$

Then substitute $-x+5$ for y in equation (2) and solve for x.

$$\begin{aligned} x(-x+5) &= 6 \\ -x^2+5x-6 &= 0 \\ -1(x^2-5x+6) &= 0 \\ -1(x-2)(x-3) &= 0 \end{aligned}$$

$x = 2 \quad or \quad x = 3$

If $x = 2$, $y = -2+5 = 3$. If $x = 3$, $y = -3+5 = 2$.

The pairs $(2, 3)$ and $(3, 2)$ check.

19. ***Familiarize***. Let l and w represent the length and width of the rectangle, in feet, respectively.

Translate. The perimeter is 18 ft.

$$2l + 2w = 18 \qquad (1)$$

From the Pythagorean theorem, we have

$$l^2 + w^2 = (\sqrt{41})^2 \qquad (2)$$

Carry out. We solve the system of equations. We first solve equation (1) for w.

$$\begin{aligned} 2l + 2w &= 18 \\ 2w &= 18 - 2l \\ w &= 9 - l \end{aligned}$$

Then substitute $9-l$ for w in equation (2) and solve for l.

$$\begin{aligned} l^2 + (9-l)^2 &= (\sqrt{41})^2 \\ l^2 + 81 - 18l + l^2 &= 41 \\ 2l^2 - 18l + 40 &= 0 \\ 2(l^2 - 9l + 20) &= 0 \\ 2(l-4)(l-5) &= 0 \end{aligned}$$

$l = 4 \quad or \quad l = 5$

If $l = 4$, then $w = 9-4 = 5$. If $l = 5$, then $w = 9-5 = 4$. Since length is usually considered to be longer than width, we have $l = 5$ and $w = 4$.

Check. The perimeter is $2 \cdot 5 + 2 \cdot 4$, or 18 ft. The length of a diagonal is $\sqrt{5^2+4^2}$, or $\sqrt{41}$ ft. The solution checks.

State. The dimensions of the garden are 5 ft by 4 ft.

20. ***Familiarize***. Let l and w represent the length and width of the playground, in feet, respectively.

Translate.

Perimeter: $2l + 2w = 210 \qquad (1)$

Area: $lw = 2700 \qquad (2)$

Carry out. We solve the system of equations. First solve equation (2) for w.

$$w = \frac{2700}{l}$$

Then substitute $\dfrac{2700}{l}$ for w in equation (1) and solve for l.

$$\begin{aligned} 2l + 2 \cdot \frac{2700}{l} &= 210 \\ 2l + \frac{5400}{l} &= 210 \\ 2l^2 + 5400 &= 210l \quad \text{Multiplying by } l \\ 2l^2 - 210l + 5400 &= 0 \\ 2(l^2 - 105l + 2700) &= 0 \\ 2(l-45)(l-60) &= 0 \end{aligned}$$

$l = 45 \quad or \quad l = 60$

If $l = 45$, then $w = \dfrac{2700}{45} = 60$. If $l = 60$, then $w = \dfrac{2700}{60} = 45$. Since length is usually considered to be longer than width, we have $l = 60$ and $w = 45$.

Check. Perimeter: $2 \cdot 60 + 2 \cdot 45 = 210$ ft

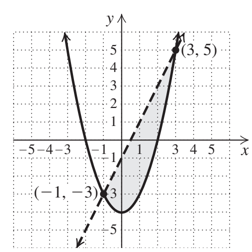
Area: $60 \cdot 45 = 2700$ ft^2

The solution checks.

State. The dimensions of the playground are 60 ft by 45 ft.

21. Graph: $y \geq x^2 - 4$,
$y < 2x - 1$

The solution set of $y \geq x^2 - 4$ is the parabola $y = x^2 - 4$ and the region inside it. The solution set of $y < 2x - 1$ is the half-plane below the line $y = 2x - 1$. We shade the region common to the two solution sets.



To find the points of intersection of the graphs of the related equations we solve the system of equations

$$y = x^2 - 4,$$
$$y = 2x - 1.$$

The points of intersection are $(-1, -3)$ and $(3, 5)$.

22. $5x^2 - 8xy + 5y^2 = 9$

$A = 5,\ B = -8,\ C = 5$

$B^2 - 4AC = (-8)^2 - 4 \cdot 5 \cdot 5 = 64 - 100 = -36$

Since the discriminant is negative, the graph is an ellipse (or a circle). To rotate the axes we first determine θ.

$$\cot 2\theta = \frac{A-C}{B} = \frac{5-5}{-8} = 0$$

Then $2\theta = 90°$ and $\theta = 45°$, so

$$\sin\theta = \frac{\sqrt{2}}{2} \text{ and } \cos\theta = \frac{\sqrt{2}}{2}.$$

Now substitute in the rotation of axes formulas.

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ &= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y') \end{aligned}$$

$$y = x' \sin\theta + y' \cos\theta$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y')$$

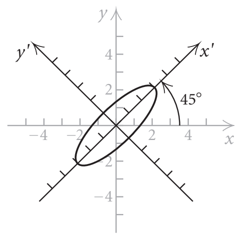
Substitute for x and y in the given equation.

$$5\left[\frac{\sqrt{2}}{2}(x' - y')\right]^2 - 8\left[\frac{\sqrt{2}}{2}(x' - y')\right]\left[\frac{\sqrt{2}}{2}(x' + y')\right] +$$
$$5\left[\frac{\sqrt{2}}{2}(x' + y')\right]^2 = 9$$

After simplifying we have

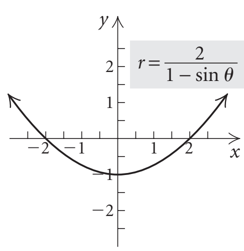
$$\frac{(x')^2}{9} + (y')^2 = 1.$$

This is the equation of an ellipse with vertices $(-3, 0)$ and $(3, 0)$ on the x'-axis. The y'-intercepts are $(0, -1)$ and $(0,1)$. We sketch the graph.



23. $r = \dfrac{2}{1 - \sin\theta}$

The equation is of the form $r = \dfrac{ep}{1 - e \sin\theta}$ with $e = 1$. Since $e = 1$, the graph is a parabola.



Since $e = 1$ and $ep = 1 \cdot p = 2$, we have $p = 2$. Thus the parabola has a horizontal directrix 2 units below the pole.

We find the vertex by letting $\theta = \dfrac{3\pi}{2}$.

When $\theta = \dfrac{3\pi}{2}$,

$$r = \frac{2}{1 - \sin\frac{3\pi}{2}} = \frac{2}{1 - (-1)} = 1.$$

The vertex is $\left(1, \dfrac{3\pi}{2}\right)$.

24. $e = 2$, $r = 3 \sec\theta$

The equation of the directrix can be written

$$r = \frac{3}{\cos\theta}, \text{ or } r\cos\theta = 3.$$

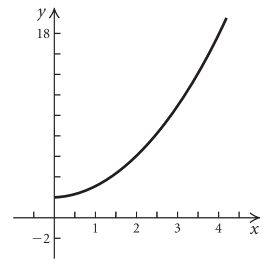
This corresponds to the equation $x = 3$ in rectangular coordinates, so the directrix is a vertical line 3 units to the right of the pole. Using the table on page 868 in the text, we see that the equation is of the form

$$r = \frac{ep}{1 + e\cos\theta}.$$

Substituting 2 for e and 3 for p, we have

$$r = \frac{2 \cdot 3}{1 + 2\cos\theta} = \frac{6}{1 + 2\cos\theta}.$$

25.



$x = \sqrt{t},\ y = t + 2;\ 0 \leqslant t \leqslant 16$

26. $x = 3\cos\theta,\ y = 3\sin\theta;\ 0 \le \theta \le 2\pi$

To find an equivalent rectangular equation, we first solve for $\cos\theta$ and $\sin\theta$ in the parametric equations:

$$\frac{x}{3} = \cos\theta, \frac{y}{3} = \sin\theta$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can substitute to eliminate the parameter:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$$
$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$
$$x^2 + y^2 = 9$$

For $0 \le \theta \le 2\pi,\ -3 \le 3\cos\theta \le 3$.

Then we have $x^2 + y^2 = 9,\ -3 \le x \le 3$.

27. $y = x - 5$

Answers may vary.

If $x = t$, $y = t - 5$.

If $x = t + 5$, $y = t + 5 - 5 = t$.

28. a) We substitute 10 for h, 250 for v_0, and 30°for θ in the parametric equations for projectile motion.

$$x = (v_0 \cos\theta)t$$
$$= (250 \cos 30°)t$$
$$= \left(250 \cdot \frac{\sqrt{3}}{2}\right)t = 125\sqrt{3}t$$
$$y = h + (v_0 \sin\theta)t - 16t^2$$
$$= 10 + (250 \sin 30°)t - 16t^2$$
$$= 10 + \left(250 \cdot \frac{1}{2}\right)t - 16t^2$$
$$= 10 + 125t - 16t^2$$

b) The height of the projectile at time t is given by y.

When $t = 1$, $y = 10 + 125 \cdot 1 - 16 \cdot 1^2 = 119$ ft.

When $t = 3$, $y = 10 + 125 \cdot 3 - 16 \cdot 3^2 = 241$ ft.

c) The ball hits the ground when $y = 0$, so we solve the equation $y = 0$.

$$10 + 125t - 16t^2 = 0$$
$$-16t^2 + 125t + 10 = 0$$
$$t = \frac{-125 \pm \sqrt{(125)^2 - 4(-16)(10)}}{2(-16)}$$
$$t \approx -0.1 \quad or \quad t \approx 7.9$$

The negative value for t has no meaning in this application. Thus we see that the projectile is in the air for about 7.9 sec.

d) Since the projectile is in the air for about 7.9 sec, the horizontal distance it travels is given by

$$x = 125\sqrt{3}(7.9) \approx 1710.4 \text{ ft.}$$

e) The maximum height of the projectile is the maximum value of the quadratic function represented by y. At the vertex of that function we have

$$t = -\frac{b}{2a} = -\frac{125}{2(-16)} = 3.90625.$$

When $t = 3.90625$,

$$y = 10 + 125(3.90625) - 16(3.90625)^2 \approx 254.1 \text{ ft.}$$

29. $(y-1)^2 = 4(x+1)$ represents a parabola with vertex $(-1, 1)$ that opens to the right. Thus the correct answer is A.

30. Use the midpoint formula to find the center.

$$(h, k) = \left(\frac{1+5}{2}, \frac{1+(-3)}{2}\right) = (3, -1)$$

Use the distance formula to find the radius.

$$r = \frac{1}{2}\sqrt{(1-5)^2 + (1-(-3))^2} = \frac{1}{2}\sqrt{(-4)^2 + (4)^2} = 2\sqrt{2}$$

Write the equation of the circle.

$$(x-h)^2 + (y-k)^2 = r^2$$
$$(x-3)^2 + [y-(-1)]^2 = (2\sqrt{2})^2$$
$$(x-3)^2 + (y+1)^2 = 8$$