Chapter 11

Sequences, Series, and Combinatorics

Exercise Set 11.1

1. $a_n = 4n - 1$

$$\begin{aligned} a_1 &= 4 \cdot 1 - 1 = 3, \\ a_2 &= 4 \cdot 2 - 1 = 7, \\ a_3 &= 4 \cdot 3 - 1 = 11, \\ a_4 &= 4 \cdot 4 - 1 = 15; \\ a_{10} &= 4 \cdot 10 - 1 = 39; \\ a_{15} &= 4 \cdot 15 - 1 = 59 \end{aligned}$$

2.

$$\begin{aligned} a_1 &= (1-1)(1-2)(1-3) = 0, \\ a_2 &= (2-1)(2-2)(2-3) = 0, \\ a_3 &= (3-1)(3-2)(3-3) = 0, \\ a_4 &= (4-1)(4-2)(4-3) = 3 \cdot 2 \cdot 1 = 6; \\ a_{10} &= (10-1)(10-2)(10-3) = 9 \cdot 8 \cdot 7 = 504; \\ a_{15} &= (15-1)(15-2)(15-3) = 14 \cdot 13 \cdot 12 = 2184 \end{aligned}$$

3. $a_n = \dfrac{n}{n-1}$, $n \geq 2$

The first 4 terms are a_2, a_3, a_4, and a_5:

$$\begin{aligned} a_2 &= \frac{2}{2-1} = 2, \\ a_3 &= \frac{3}{3-1} = \frac{3}{2}, \\ a_4 &= \frac{4}{4-1} = \frac{4}{3}, \\ a_5 &= \frac{5}{5-1} = \frac{5}{4}; \\ a_{10} &= \frac{10}{10-1} = \frac{10}{9}; \\ a_{15} &= \frac{15}{15-1} = \frac{15}{14} \end{aligned}$$

4.

$$\begin{aligned} a_3 &= 3^2 - 1 = 8, \\ a_4 &= 4^2 - 1 = 15, \\ a_5 &= 5^2 - 1 = 24, \\ a_6 &= 6^2 - 1 = 35; \\ a_{10} &= 10^2 - 1 = 99; \\ a_{15} &= 15^2 - 1 = 224 \end{aligned}$$

5. $a_n = \dfrac{n^2 - 1}{n^2 + 1}$,

$$\begin{aligned} a_1 &= \frac{1^2 - 1}{1^2 + 1} = 0, \\ a_2 &= \frac{2^2 - 1}{2^2 + 1} = \frac{3}{5}, \\ a_3 &= \frac{3^2 - 1}{3^2 + 1} = \frac{8}{10} = \frac{4}{5}, \\ a_4 &= \frac{4^2 - 1}{4^2 + 1} = \frac{15}{17}; \\ a_{10} &= \frac{10^2 - 1}{10^2 + 1} = \frac{99}{101}; \\ a_{15} &= \frac{15^2 - 1}{15^2 + 1} = \frac{224}{226} = \frac{112}{113} \end{aligned}$$

6.

$$\begin{aligned} a_1 &= \left(-\frac{1}{2}\right)^{1-1} = 1, \\ a_2 &= \left(-\frac{1}{2}\right)^{2-1} = -\frac{1}{2}, \\ a_3 &= \left(-\frac{1}{2}\right)^{3-1} = \frac{1}{4}, \\ a_4 &= \left(-\frac{1}{2}\right)^{4-1} = -\frac{1}{8}, \\ a_{10} &= \left(-\frac{1}{2}\right)^{10-1} = -\frac{1}{512}; \\ a_{15} &= \left(-\frac{1}{2}\right)^{15-1} = \frac{1}{16,384} \end{aligned}$$

7. $a_n = (-1)^n n^2$

$$\begin{aligned} a_1 &= (-1)^1 1^2 = -1, \\ a_2 &= (-1)^2 2^2 = 4, \\ a_3 &= (-1)^3 3^2 = -9, \\ a_4 &= (-1)^4 4^2 = 16; \\ a_{10} &= (-1)^{10} 10^2 = 100; \\ a_{15} &= (-1)^{15} 15^2 = -225 \end{aligned}$$

8.

$$\begin{aligned} a_1 &= (-1)^{1-1}(3 \cdot 1 - 5) = -2, \\ a_2 &= (-1)^{2-1}(3 \cdot 2 - 5) = -1, \\ a_3 &= (-1)^{3-1}(3 \cdot 3 - 5) = 4, \\ a_4 &= (-1)^{4-1}(3 \cdot 4 - 5) = -7; \\ a_{10} &= (-1)^{10-1}(3 \cdot 10 - 5) = -25, \\ a_{15} &= (-1)^{15-1}(3 \cdot 15 - 5) = 40 \end{aligned}$$

9. $a_n = 5 + \dfrac{(-2)^{n+1}}{2^n}$

$$\begin{aligned} a_1 &= 5 + \frac{(-2)^{1+1}}{2^1} = 5 + \frac{4}{2} = 7, \\ a_2 &= 5 + \frac{(-2)^{2+1}}{2^2} = 5 + \frac{-8}{4} = 3, \\ a_3 &= 5 + \frac{(-2)^{3+1}}{2^3} = 5 + \frac{16}{8} = 7, \\ a_4 &= 5 + \frac{(-2)^{4+1}}{2^4} = 5 + \frac{-32}{16} = 3; \end{aligned}$$

$a_{10} = 5 + \dfrac{(-2)^{10+1}}{2^{10}} = 5 + \dfrac{-1 \cdot 2^{11}}{2^{10}} = 3;$

$a_{15} = 5 + \dfrac{(-2)^{15+1}}{2^{15}} = 5 + \dfrac{2^{16}}{2^{15}} = 7$

10. $a_1 = \dfrac{2 \cdot 1 - 1}{1^2 + 2 \cdot 1} = \dfrac{1}{3},$

$a_2 = \dfrac{2 \cdot 2 - 1}{2^2 + 2 \cdot 2} = \dfrac{3}{8},$

$a_3 = \dfrac{2 \cdot 3 - 1}{3^2 + 2 \cdot 3} = \dfrac{5}{15} = \dfrac{1}{3},$

$a_4 = \dfrac{2 \cdot 4 - 1}{4^2 + 2 \cdot 4} = \dfrac{7}{24};$

$a_{10} = \dfrac{2 \cdot 10 - 1}{10^2 + 2 \cdot 10} = \dfrac{19}{120};$

$a_{15} = \dfrac{2 \cdot 15 - 1}{15^2 + 2 \cdot 15} = \dfrac{29}{255}$

11. $a_n = 5n - 6$

$a_8 = 5 \cdot 8 - 6 = 40 - 6 = 34$

12. $a_7 = (3 \cdot 7 - 4)(2 \cdot 7 + 5) = 17 \cdot 19 = 323$

13. $a_n = (2n + 3)^2$

$a_6 = (2 \cdot 6 + 3)^2 = 225$

14. $a_{12} = (-1)^{12-1}[4.6(12) - 18.3] = -36.9$

15. $a_n = 5n^2(4n - 100)$

$a_{11} = 5(11)^2(4 \cdot 11 - 100) = 5(121)(-56) =$
$-33,880$

16. $a_{80} = \left(1 + \dfrac{1}{80}\right)^2 = \left(\dfrac{81}{80}\right)^2 = \dfrac{6561}{6400}$

17. $a_n = \ln e^n$

$a_{67} = \ln e^{67} = 67$

18. $a_{100} = 2 - \dfrac{1000}{100} = 2 - 10 = -8$

19. $2, 4, 6, 8, 10, \ldots$

These are the even integers, so the general term might be $2n$.

20. 3^n

21. $-2,\ 6,\ -18,\ 54,\ \ldots$

We can see a pattern if we write the sequence as

$-1 \cdot 2 \cdot 1,\ 1 \cdot 2 \cdot 3,\ -1 \cdot 2 \cdot 9,\ 1 \cdot 2 \cdot 27,\ \ldots$

The general term might be $(-1)^n 2(3)^{n-1}$.

22. $5n - 7$

23. $\dfrac{2}{3},\ \dfrac{3}{4},\ \dfrac{4}{5},\ \dfrac{5}{6},\ \dfrac{6}{7},\ \ldots$

These are fractions in which the denominator is 1 greater than the numerator. Also, each numerator is 1 greater than the preceding numerator. The general term might be $\dfrac{n+1}{n+2}$.

24. $\sqrt{2n}$

25. $1 \cdot 2,\ 2 \cdot 3,\ 3 \cdot 4,\ 4 \cdot 5,\ \ldots$

These are the products of pairs of consecutive natural numbers. The general term might be $n(n + 1)$.

26. $-1 - 3(n - 1)$, or $-3n + 2$, or $-(3n - 2)$

27. $0,\ \log 10,\ \log 100,\ \log 1000,\ \ldots$

We can see a pattern if we write the sequence as

$\log 1,\ \log 10,\ \log 100,\ \log 1000,\ \ldots$

The general term might be $\log 10^{n-1}$. This is equivalent to $n - 1$.

28. $\ln e^{n+1}$, or $n + 1$

29. $1, 2, 3, 4, 5, 6, 7, \ldots$

$S_3 = 1 + 2 + 3 = 6$

$S_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$

30. $S_2 = 1 - 3 = -2$

$S_8 = 1 - 3 + 5 - 7 + 9 = 5$

31. $2, 4, 6, 8, \ldots$

$S_4 = 2 + 4 + 6 + 8 = 20$

$S_5 = 2 + 4 + 6 + 8 + 10 = 30$

32. $S_1 = 1$

$S_5 = 1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \dfrac{1}{25} = \dfrac{5269}{3600}$

33.
$$\begin{aligned}\sum_{k=1}^{5} \frac{1}{2k} &= \frac{1}{2 \cdot 1} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{2 \cdot 5} \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} \\ &= \frac{60}{120} + \frac{30}{120} + \frac{20}{120} + \frac{15}{120} + \frac{12}{120} \\ &= \frac{137}{120}\end{aligned}$$

34. $\dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \dfrac{1}{9} + \dfrac{1}{11} + \dfrac{1}{13} = \dfrac{43,024}{45,045}$

35.
$$\begin{aligned}\sum_{i=0}^{6} 2^i &= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 \\ &= 1 + 2 + 4 + 8 + 16 + 32 + 64 \\ &= 127\end{aligned}$$

36. $\sqrt{7} + \sqrt{9} + \sqrt{11} + \sqrt{13} \approx 12.5679$

37. $\displaystyle\sum_{k=7}^{10} \ln k = \ln 7 + \ln 8 + \ln 9 + \ln 10 =$

$\ln(7 \cdot 8 \cdot 9 \cdot 10) = \ln 5040 \approx 8.5252$

38. $\pi + 2\pi + 3\pi + 4\pi = 10\pi \approx 31.4159$

39.
$$\begin{aligned}\sum_{k=1}^{8} \frac{k}{k+1} &= \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} + \\ &\quad \frac{5}{5+1} + \frac{6}{6+1} + \frac{7}{7+1} + \frac{8}{8+1} \\ &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \frac{7}{8} + \frac{8}{9} \\ &= \frac{15,551}{2520}\end{aligned}$$

40. $\dfrac{1-1}{1+3}+\dfrac{2-1}{2+3}+\dfrac{3-1}{3+3}+\dfrac{4-1}{4+3}+\dfrac{5-1}{5+3}$

$= 0+\dfrac{1}{5}+\dfrac{2}{6}+\dfrac{3}{7}+\dfrac{4}{8}$

$= 0+\dfrac{1}{5}+\dfrac{1}{3}+\dfrac{3}{7}+\dfrac{1}{2}$

$= \dfrac{307}{210}$

41. $\displaystyle\sum_{i=1}^{5}(-1)^i$

$= (-1)^1+(-1)^2+(-1)^3+(-1)^4+(-1)^5$

$= -1+1-1+1-1$

$= -1$

42. $-1+1-1+1-1+1=0$

43. $\displaystyle\sum_{k=1}^{8}(-1)^{k+1}3k$

$= (-1)^2 3\cdot 1+(-1)^3 3\cdot 2+(-1)^4 3\cdot 3+$

$(-1)^5 3\cdot 4+(-1)^6 3\cdot 5+(-1)^7 3\cdot 6+$

$(-1)^8 3\cdot 7+(-1)^9 3\cdot 8$

$= 3-6+9-12+15-18+21-24$

$= -12$

44. $4-4^2+4^3-4^4+4^5-4^6+4^7-4^8=-52,428$

45. $\displaystyle\sum_{k=0}^{6}\frac{2}{k^2+1} = \frac{2}{0^2+1}+\frac{2}{1^2+1}+\frac{2}{2^2+1}+\frac{2}{3^2+1}+$

$\dfrac{2}{4^2+1}+\dfrac{2}{5^2+1}+\dfrac{2}{6^2+1}$

$= 2+1+\dfrac{2}{5}+\dfrac{2}{10}+\dfrac{2}{17}+\dfrac{2}{26}+\dfrac{2}{37}$

$= 2+1+\dfrac{2}{5}+\dfrac{1}{5}+\dfrac{2}{17}+\dfrac{1}{13}+\dfrac{2}{37}$

$= \dfrac{157,351}{40,885}$

46. $1\cdot2+2\cdot3+3\cdot4+4\cdot5+5\cdot6+6\cdot7+7\cdot8+8\cdot9+9\cdot10+10\cdot11=440$

47. $\displaystyle\sum_{k=0}^{5}(k^2-2k+3)$

$= (0^2-2\cdot0+3)+(1^2-2\cdot1+3)+$

$(2^2-2\cdot2+3)+(3^2-2\cdot3+3)+$

$(4^2-2\cdot4+3)+(5^2-2\cdot5+3)$

$= 3+2+3+6+11+18$

$= 43$

48. $\dfrac{1}{1\cdot2}+\dfrac{1}{2\cdot3}+\dfrac{1}{3\cdot4}+\dfrac{1}{4\cdot5}+\dfrac{1}{5\cdot6}+$

$\dfrac{1}{6\cdot7}+\dfrac{1}{7\cdot8}+\dfrac{1}{8\cdot9}+\dfrac{1}{9\cdot10}+\dfrac{1}{10\cdot11}$

$= \dfrac{1}{2}+\dfrac{1}{6}+\dfrac{1}{12}+\dfrac{1}{20}+\dfrac{1}{30}+\dfrac{1}{42}+\dfrac{1}{56}+$

$\dfrac{1}{72}+\dfrac{1}{90}+\dfrac{1}{110}$

$= \dfrac{10}{11}$

49. $\displaystyle\sum_{i=0}^{10}\frac{2i}{2^i+1}$

$= \dfrac{2^0}{2^0+1}+\dfrac{2^1}{2^1+1}+\dfrac{2^2}{2^2+1}+\dfrac{2^3}{2^3+1}+\dfrac{2^4}{2^4+1}+$

$\dfrac{2^5}{2^5+1}+\dfrac{2^6}{2^6+1}+\dfrac{2^7}{2^7+1}+\dfrac{2^8}{2^8+1}+\dfrac{2^9}{2^9+1}+$

$\dfrac{2^{10}}{2^{10}+1}$

$= \dfrac{1}{2}+\dfrac{2}{3}+\dfrac{4}{5}+\dfrac{8}{9}+\dfrac{16}{17}+\dfrac{32}{33}+\dfrac{64}{65}+\dfrac{128}{129}+$

$\dfrac{256}{257}+\dfrac{512}{513}+\dfrac{1024}{1025}$

≈ 9.736

50. $(-2)^0+(-2)^2+(-2)^4+(-2)^6=1+4+16+64=85$

51. $5+10+15+20+25+\ .\ .\ .$

This is a sum of multiples of 5, and it is an infinite series. Sigma notation is

$$\sum_{k=1}^{\infty}5k.$$

52. $\displaystyle\sum_{k=1}^{\infty}7k$

53. $2-4+8-16+32-64$

This is a sum of powers of 2 with alternating signs. Sigma notation is

$$\sum_{k=1}^{6}(-1)^{k+1}2k, \text{ or } \sum_{k=1}^{6}(-1)^{k-1}2k$$

54. $\displaystyle\sum_{k=1}^{5}3k$

55. $-\dfrac{1}{2}+\dfrac{2}{3}-\dfrac{3}{4}+\dfrac{4}{5}-\dfrac{5}{6}+\dfrac{6}{7}$

This is a sum of fractions in which the denominator is one greater than the numerator. Also, each numerator is 1 greater than the preceding numerator and the signs alternate. Sigma notation is

$$\sum_{k=1}^{6}(-1)^k\frac{k}{k+1}.$$

56. $\displaystyle\sum_{k=1}^{5}\frac{1}{k^2}$

57. $4-9+16-25+\ .\ .\ .\ +(-1)^n n^2$

This is a sum of terms of the form $(-1)^k k^2$, beginning with $k=2$ and continuing through $k=n$. Sigma notation is

$$\sum_{k=2}^{n}(-1)^k k^2.$$

58. $\displaystyle\sum_{k=3}^{n}(-1)^{k+1}k^2, \text{ or } \sum_{k=3}^{n}(-1)^{k-1}k^2$

59. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots$
This is a sum of fractions in which the numerator is 1 and the denominator is a product of two consecutive integers. The larger integer in each product is the smaller integer in the succeeding product. It is an infinite series. Sigma notation is

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}.$$

60. $\sum_{k=1}^{\infty}\frac{1}{k(k+1)^2}$

61. $a_1=4, \quad a_{k+1}=1+\frac{1}{a_k}$

$a_2=1+\frac{1}{4}=1\frac{1}{4}$, or $\frac{5}{4}$

$a_3=1+\frac{1}{\frac{5}{4}}=1+\frac{4}{5}=1\frac{4}{5}$, or $\frac{9}{5}$

$a_4=1+\frac{1}{\frac{9}{5}}=1+\frac{5}{9}=1\frac{5}{9}$, or $\frac{14}{9}$

62. $a_1=256$, $a_2=\sqrt{256}=16$, $a_3=\sqrt{16}=4$,
$a_4=\sqrt{4}=2$

63. $a_1=6561, \quad a_{k+1}=(-1)^k\sqrt{a_k}$
$a_2=(-1)^1\sqrt{6561}=-81$
$a_3=(-1)^2\sqrt{-81}=9i$
$a_4=(-1)^3\sqrt{9i}=-3\sqrt{i}$

64. $a_1=e^Q$, $a_2=\ln e^Q=Q$, $a_3=\ln Q$, $a_4=\ln(\ln Q)$

65. $a_1=2, \quad a_{k+1}=a_k+a_{k-1}$
$a_2=3$
$a_3=3+2=5$
$a_4=5+3=8$

66. $a_1=-10$, $a_2=8$, $a_3=8-(-10)=18$,
$a_4=18-8=10$

67. a) $a_1=\$1000(1.062)^1=\$1062$
$a_2=\$1000(1.062)^2\approx\$1127.84$
$a_3=\$1000(1.062)^3\approx\$1197.77$
$a_4=\$1000(1.062)^4\approx\$1272.03$
$a_5=\$1000(1.062)^5\approx\$1350.90$
$a_6=\$1000(1.062)^6\approx\$1434.65$
$a_7=\$1000(1.062)^7\approx\$1523.60$
$a_8=\$1000(1.062)^8\approx\$1618.07$
$a_9=\$1000(1.062)^9\approx\$1718.39$
$a_{10}=\$1000(1.062)^{10}\approx\$1824.93$
b) $a_{20}=\$1000(1.062)^{20}\approx\$3330.35$

68. Find each term by multiplying the preceding term by 0.75:
$5200, $3900, $2925, $2193.75, $1645.31,
$1233.98, $925.49, $694.12, $520.59, $390.44

69. Find each term by multiplying the preceding term by 2. Find 17 terms, beginning with $a_1=1$, since there are 16 fifteen minute periods in 4 hr.
1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024,
2048, 4096, 8192, 16,384, 32,768, 65,536

70. Find each term by adding $0.30 to the preceding term:
$8.30, $8.60, $8.90, $9.20, $9.50, $9.80,
$10.10, $10.40, $10.70, $11.00

71. $a_1=1$ (Given)
$a_2=1$ (Given)
$a_3=a_2+a_1=1+1=2$
$a_4=a_3+a_2=2+1=3$
$a_5=a_4+a_3=3+2=5$
$a_6=a_5+a_4=5+3=8$
$a_7=a_6+a_5=8+5=13$

72. $3x-2y=3$, (1)
$2x+3y=-11$ (2)
Multiply equation (1) by 3 and equation (2) by 2 and add.

$$\begin{array}{rcl} 9x-6y&=&9\\ 4x+6y&=&-22\\ \hline 13x&=&-13\\ x&=&-1\end{array}$$

Back-substitute to find y.
$2(-1)+3y=-11$ Using equation (2)
$-2+3y=-11$
$3y=-9$
$y=-3$
The solution is $(-1,-3)$.

73. ***Familiarize***. Let x and y represent the number of international visitors to New York City in 2008 and 2009, respectively, in millions.

Translate. The total number of visitors was 18.1 million.
$x+y=18.1$
There were 0.9 million fewer visitors in 2009 than in 2008.
$y=x-0.9$
We have a system of equations.
$x+y=18.1$, (1)
$y=x-0.9$ (2)

Carry out. We first substitute $x-0.9$ for y in equation (1) and solve for x.
$x+x-0.9=18.1$
$2x-0.9=18.1$
$2x=19$
$x=9.5$
Now substitute 9.5 for x in equation (2) to find y.
$y=9.5-0.9=8.6$

Check. 9.5 million + 8.6 million = 18.1 million, and 8.6 million is 0.9 million less than 9.5 million. The answer checks.

State. The number of international visitors to New York City was 9.5 million in 2008 and 8.6 million in 2009.

74.
$$x^2+y^2-6x+4y=3$$
$$x^2-6x+9+y^2+4y+4=3+9+4$$
$$(x-3)^2+(y+2)^2=16$$

Center: $(3,-2)$; radius: 4

75. We complete the square twice.
$$x^2+y^2+5x-8y=2$$
$$x^2+5x+y^2-8y=2$$
$$x^2+5x+\frac{25}{4}+y^2-8y+16=2+\frac{25}{4}+16$$
$$\left(x+\frac{5}{2}\right)^2+(y-4)^2=\frac{97}{4}$$
$$\left[x-\left(-\frac{5}{2}\right)\right]^2+(y-4)^2=\left(\frac{\sqrt{97}}{2}\right)^2$$

The center is $\left(-\frac{5}{2},4\right)$ and the radius is $\frac{\sqrt{97}}{2}$.

76. $a_n=\frac{1}{2^n}\log\ 1000^n$

$a_1=\frac{1}{2^1}\log\ 1000^1=\frac{1}{2}\log\ 10^3=\frac{1}{2}\cdot 3=\frac{3}{2}$

$a_2=\frac{1}{2^2}\log\ 1000^2=\frac{1}{4}\log\ (10^3)^2=\frac{1}{4}\log\ 10^6=$
$\frac{1}{4}\cdot 6=\frac{3}{2}$

$a_3=\frac{1}{2^3}\log\ 1000^3=\frac{1}{8}\log\ (10^3)^3=\frac{1}{8}\log\ 10^9=$
$\frac{1}{8}\cdot 9=\frac{9}{8}$

$a_4=\frac{1}{2^4}\log\ 1000^4=\frac{1}{16}\log\ (10^3)^4=$
$\frac{1}{16}\log\ 10^{12}=\frac{1}{16}\cdot 12=\frac{3}{4}$

$a_5=\frac{1}{2^5}\log\ 1000^5=\frac{1}{32}\log\ (10^3)^5=$
$\frac{1}{32}\log\ 10^{15}=\frac{1}{32}\cdot 15=\frac{15}{32}$

$S_5=\frac{3}{2}+\frac{3}{2}+\frac{9}{8}+\frac{3}{4}+\frac{15}{32}=\frac{171}{32}$

77. $a_n=i^n$

$a_1=i$

$a_2=i^2=-1$

$a_3=i^3=-i$

$a_4=i^4=1$

$a_5=i^5=i^4\cdot i=i;$

$S_5=i-1-i+1+i=i$

78. $a_n=\ln(1\cdot 2\cdot 3\cdots n)$

$a_1=\ln\ 1=0$

$a_2=\ln(1\cdot 2)=\ln\ 2$

$a_3=\ln(1\cdot 2\cdot 3)=\ln\ 6$

$a_4=\ln(1\cdot 2\cdot 3\cdot 4)=\ln\ 24$

$a_5=\ln(1\cdot 2\cdot 3\cdot 4\cdot 5)=\ln\ 120$

$S_5=0+\ln\ 2+\ln\ 6+\ln\ 24+\ln\ 120;$
$\quad=\ln(2\cdot 6\cdot 24\cdot 120)=\ln\ 34,560\approx 10.450$

79. $S_n=\ln 1+\ln 2+\ln 3+\cdots+\ln n$
$\quad=\ln(1\cdot 2\cdot 3\cdots n)$

80. $S_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+$
$\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$
$\quad=1-\frac{1}{n+1}=\frac{n}{n+1}$

Exercise Set 11.2

1. 3, 8, 13, 18, . . .

$a_1=3$

$d=5\quad(8-3=5,\ 13-8=5,\ 18-13=5)$

2. $a_1=\$1.08$, $d=\$0.08$, $(\$1.16-\$1.08=\$0.08)$

3. 9, 5, 1, −3, . . .

$a_1=9$

$d=-4\quad(5-9=-4,\ 1-5=-4,\ -3-1=-4)$

4. $a_1=-8$, $d=3$ $(-5-(-8)=3)$

5. $\frac{3}{2},\frac{9}{4},3,\frac{15}{4},\ldots$

$a_1=\frac{3}{2}$

$d=\frac{3}{4}\quad\left(\frac{9}{4}-\frac{3}{2}=\frac{3}{4},3-\frac{9}{4}=\frac{3}{4}\right)$

6. $a_1=\frac{3}{5}$, $d=-\frac{1}{2}$ $\left(\frac{1}{10}-\frac{3}{5}=-\frac{1}{2}\right)$

7. $a_1=\$316$

$d=-\$3$ $(\$313-\$316=-\$3,$
$\$310-\$313=-\$3,\ \$307-\$310=-\$3)$

8. $a_1=0.07,\ d=0.05,$ and $n=11$

$a_{11}=0.07+(11-1)(0.05)=0.07+0.5=0.57$

9. 2, 6, 10, . . .

$a_1=2,\ d=4,$ and $n=12$

$a_n=a_1+(n-1)d$

$a_{12}=2+(12-1)4=2+11\cdot 4=2+44=46$

10. $a_1=7,\ d=-3,$ and $n=17$

$a_{17}=7+(17-1)(-3)=7+16(-3)=$
$7-48=-41$

11. $3, \frac{7}{3}, \frac{5}{3}, \ldots$

$a_1 = 3,\ d = -\frac{2}{3},$ and $n = 14$

$a_n = a_1 + (n-1)d$

$a_{14} = 3 + (14-1)\left(-\frac{2}{3}\right) = 3 - \frac{26}{3} = -\frac{17}{3}$

12. $a_1 = \$1200,\ d = \$964.32 - \$1200 = -\$235.68,$ and $n = 13$

$a_{13} = \$1200 + (13-1)(-\$235.68) = \$1200 + 12(-\$235.68) = \$1200 - \$2828.16 = -\$1628.16$

13. \$2345.78, \$2967.54, \$3589.30, . . .

$a_1 = \$2345.78,\ d = \$621.76,$ and $n = 10$

$a_n = a_1 + (n-1)d$

$a_{10} = \$2345.78 + (10-1)(\$621.76) = \$7941.62$

14. $106 = 2 + (n-1)(4)$

$106 = 2 + 4n - 4$

$108 = 4n$

$27 = n$

The 27th term is 106.

15. $a_1 = 0.07,\ d = 0.05$

$a_n = a_1 + (n-1)d$

Let $a_n = 1.67$, and solve for n.

$1.67 = 0.07 + (n-1)(0.05)$

$1.67 = 0.07 + 0.05n - 0.05$

$1.65 = 0.05n$

$33 = n$

The 33rd term is 1.67.

16. $-296 = 7 + (n-1)(-3)$

$-296 = 7 - 3n + 3$

$-306 = -3n$

$102 = n$

The 102nd term is -296.

17. $a_1 = 3,\ d = -\frac{2}{3}$

$a_n = a_1 + (n-1)d$

Let $a_n = -27$, and solve for n.

$-27 = 3 + (n-1)\left(-\frac{2}{3}\right)$

$-81 = 9 + (n-1)(-2)$

$-81 = 9 - 2n + 2$

$-92 = -2n$

$46 = n$

The 46th term is -27.

18. $a_{20} = 14 + (20-1)(-3) = 14 + 19(-3) = -43$

19. $a_n = a_1 + (n-1)d$

$33 = a_1 + (8-1)4$ Substituting 33 for a_8, 8 for n, and 4 for d

$33 = a_1 + 28$

$5 = a_1$

(Note that this procedure is equivalent to subtracting d from a_8 seven times to get a_1: $33 - 7(4) = 33 - 28 = 5$)

20. $26 = 8 + (11-1)d$

$26 = 8 + 10d$

$18 = 10d$

$1.8 = d$

21. $a_n = a_1 + (n-1)d$

$-507 = 25 + (n-1)(-14)$

$-507 = 25 - 14n + 14$

$-546 = -14n$

$39 = n$

22. We know that $a_{17} = -40$ and $a_{28} = -73$. We would have to add d eleven times to get from a_{17} to a_{28}. That is,

$-40 + 11d = -73$

$11d = -33$

$d = -3.$

Since $a_{17} = -40$, we subtract d sixteen times to get to a_1.

$a_1 = -40 - 16(-3) = -40 + 48 = 8$

We write the first five terms of the sequence:

$8, 5, 2, -1, -4$

23. $\frac{25}{3} + 15d = \frac{95}{6}$

$15d = \frac{45}{6}$

$d = \frac{1}{2}$

$a_1 = \frac{25}{3} - 16\left(\frac{1}{2}\right) = \frac{25}{3} - 8 = \frac{1}{3}$

The first five terms of the sequence are $\frac{1}{3}, \frac{5}{6}, \frac{4}{3}, \frac{11}{6}, \frac{7}{3}$.

24. $a_{14} = 11 + (14-1)(-4) = 11 + 13(-4) = -41$

$S_{14} = \frac{14}{2}[11 + (-41)] = 7(-30) = -210$

25. $5 + 8 + 11 + 14 + \ldots$

Note that $a_1 = 5$, $d = 3$, and $n = 20$. First we find a_{20}:

$a_n = a_1 + (n-1)d$

$a_{20} = 5 + (20-1)3$

$= 5 + 19 \cdot 3 = 62$

Then

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_{20} = \frac{20}{2}(5 + 62)$

$= 10(67) = 670.$

26. $1+2+3+\ .\ .\ .\ +299+300.$

$$S_{300}=\frac{300}{2}(1+300)=150(301)=45,150$$

27. The sum is $2+4+6+\ .\ .\ .\ +798+800$. This is the sum of the arithmetic sequence for which $a_1=2$, $a_n=800$, and $n=400$.

$$S_n=\frac{n}{2}(a_1+a_n)$$

$$S_{400}=\frac{400}{2}(2+800)=200(802)=160,400$$

28. $1+3+5+\ .\ .\ .\ +197+199.$

$$S_{100}=\frac{100}{2}(1+199)=50(200)=10,000$$

29. The sum is $7+14+21+\ .\ .\ .\ +91+98$. This is the sum of the arithmetic sequence for which $a_1=7$, $a_n=98$, and $n=14$.

$$S_n=\frac{n}{2}(a_1+a_n)$$

$$S_{14}=\frac{14}{2}(7+98)=7(105)=735$$

30. $16+20+24+\ .\ .\ .\ +516+520$

$$S_{127}=\frac{127}{2}(16+520)=34,036$$

31. First we find a_{20}:

$$\begin{aligned}a_n&=a_1+(n-1)d\\a_{20}&=2+(20-1)5\\&=2+19\cdot 5=97\end{aligned}$$

Then

$$\begin{aligned}S_n&=\frac{n}{2}(a_1+a_n)\\S_{20}&=\frac{20}{2}(2+97)\\&=10(99)=990.\end{aligned}$$

32. $a_{32}=7+(32-1)(-3)=7+(31)(-3)=-86$

$$S_{32}=\frac{32}{2}[7+(-86)]=16(-79)=-1264$$

33. $\displaystyle\sum_{k=1}^{40}(2k+3)$

Write a few terms of the sum:

$5+7+9+\ .\ .\ .\ +83$

This is a series coming from an arithmetic sequence with $a_1=5$, $n=40$, and $a_{40}=83$. Then

$$\begin{aligned}S_n&=\frac{n}{2}(a_1+a_n)\\S_{40}&=\frac{40}{2}(5+83)\\&=20(88)=1760\end{aligned}$$

34. $\displaystyle\sum_{k=5}^{20}8k$

$40+48+56+64+\ .\ .\ .\ +160$

This is equivalent to a series coming from an arithmetic sequence with $a_1=40$, $n=16$, and $n_{16}=160$.

$$S_{16}=\frac{16}{2}(40+160)=1600$$

35. $\displaystyle\sum_{k=0}^{19}\frac{k-3}{4}$

Write a few terms of the sum:

$$-\frac{3}{4}-\frac{1}{2}-\frac{1}{4}+0+\frac{1}{4}+\ .\ .\ .\ +4$$

Since k goes from 0 through 19, there are 20 terms. Thus, this is equivalent to a series coming from an arithmetic sequence with $a_1=-\frac{3}{4}$, $n=20$, and $a_{20}=4$. Then

$$\begin{aligned}S_n&=\frac{n}{2}(a_1+a_n)\\S_{20}&=\frac{20}{2}\left(-\frac{3}{4}+4\right)\\&=10\cdot\frac{13}{4}=\frac{65}{2}.\end{aligned}$$

36. $\displaystyle\sum_{k=2}^{50}(2000-3k)$

$1994+1991+1988+\ .\ .\ .\ +1850$

This is equivalent to a series coming from an arithmetic sequence with $a_1=1994$, $n=49$, and $n_{49}=1850$.

$$S_{49}=\frac{49}{2}(1994+1850)=94,178$$

37. $\displaystyle\sum_{k=12}^{57}\frac{7-4k}{13}$

Write a few terms of the sum:

$$-\frac{41}{13}-\frac{45}{13}-\frac{49}{13}-\ .\ .\ .\ -\frac{221}{13}$$

Since k goes from 12 through 57, there are 46 terms. Thus, this is equivalent to a series coming from an arithmetic sequence with $a_1=-\frac{41}{13}$, $n=46$, and $a_{46}=-\frac{221}{13}$. Then

$$\begin{aligned}S_n&=\frac{n}{2}(a_1+a_n)\\S_{46}&=\frac{46}{2}\left(-\frac{41}{13}-\frac{221}{13}\right)\\&=23\left(-\frac{262}{13}\right)=-\frac{6026}{13}.\end{aligned}$$

38. First find $\displaystyle\sum_{k=101}^{200}(1.14k-2.8)$.

$112.34+113.48+\ .\ .\ .\ +225.2$

This is equivalent to a series coming from an arithmetic sequence with $a_1=112.34$, $n=100$, and $a_{100}=225.2$.

$$S_{100}=\frac{100}{2}(112.34+225.2)=16,877$$

Next find $\displaystyle\sum_{k=1}^{5}\left(\frac{k+4}{10}\right)$.

$$S_5=\frac{5}{2}\left(\frac{1}{2}+\frac{9}{10}\right)=3.5$$

Then $16,877-3.5=16,873.5$.

39. ***Familiarize***. We go from 50 poles in a row, down to six poles in the top row, so there must be 45 rows. We want the sum $50 + 49 + 48 + \ldots + 6$. Thus we want the sum of an arithmetic sequence. We will use the formula $S_n = \frac{n}{2}(a_1 + a_n)$.

Translate. We want to find the sum of the first 45 terms of an arithmetic sequence with $a_1 = 50$ and $a_{45} = 6$.

Carry out. Substituting into the formula, we have

$$S_{45} = \frac{45}{2}(50 + 6)$$
$$= \frac{45}{2} \cdot 56 = 1260$$

Check. We can do the calculation again, or we can do the entire addition:

$50 + 49 + 48 + \ldots + 6$.

State. There will be 1260 poles in the pile.

40. $a_{25} = 5000 + (25 - 1)(1125) = 5000 + 24 \cdot 1125 = 32,000$

$S_{25} = \frac{25}{2}(5000 + 32,000) = \frac{25}{2}(37,000) = \$462,500$

41. ***Familiarize***. We have a sequence 10, 20, 30, . . . It is an arithmetic sequence with $a_1 = 10$, $d = 10$, and $n = 31$.

Translate. We want to find $S_n = \frac{n}{2}(a_1 + a_n)$ where $a_n = a_1 + (n - 1)d$, $a_1 = 10$, $d = 10$, and $n = 31$.

Carry out. First we find a_{31}.

$a_{31} = 10 + (31 - 1)10 = 10 + 30 \cdot 10 = 310$

Then $S_{31} = \frac{31}{2}(10 + 310) = \frac{31}{2} \cdot 320 = 4960$.

Check. We can do the calculation again, or we can do the entire addition:

$10 + 20 + 30 + \ldots + 310$.

State. A total of 4960¢, or \$49.60 is saved.

42. ***Familiarize***. We have arithmetic sequence with $a_1 = 28$, $d = 4$, and $n = 20$.

Translate. We want to find $S_n = \frac{n}{2}(a_1 + a_n)$ where $a_n = a_1 + (n - 1)d$, $a_1 = 28$, $d = 4$, and $n = 20$.

Carry out. First we find a_{20}.

$a_{20} = 28 + (20 - 1)4 = 104$

Then $S_{20} = \frac{20}{2}(28 + 104) = 10 \cdot 132 = 1320$.

Check. We can do the calculations again, or we can do the entire addition:

$28 + 32 + 36 + \ldots 104$.

State. There are 1320 seats in the first balcony.

43. Yes; $d = 48 - 16 = 80 - 48 = 112 - 80 = 144 - 112 = 32$.

$a_{10} = 16 + (10 - 1)32 = 304$

$S_{10} = \frac{10}{2}(16 + 304) = 1600$ ft

44. Yes; $d = 0.6080 - 0.5908 = 0.6252 - 0.6080 = \ldots = 0.7112 - 0.6940 = 0.0172$

45. We first find how many plants will be in the last row.

Familiarize. The sequence is 35, 31, 27, ... It is an arithmetic sequence with $a_1 = 35$ and $d = -4$. Since each row must contain a positive number of plants, we must determine how many times we can add -4 to 35 and still have a positive result.

Translate. We find the largest integer x for which $35 + x(-4) > 0$. Then we evaluate the expression $35 - 4x$ for that value of x.

Carry out. We solve the inequality.

$$35 - 4x > 0$$
$$35 > 4x$$
$$\frac{35}{4} > x$$
$$8\frac{3}{4} > x$$

The integer we are looking for is 8. Thus $35 - 4x = 35 - 4(8) = 3$.

Check. If we add -4 to 35 eight times we get 3, a positive number, but if we add -4 to 35 more than eight times we get a negative number.

State. There will be 3 plants in the last row.

Next we find how many plants there are altogether.

Familiarize. We want to find the sum $35+31+27+\ldots+3$. We know $a_1 = 35$ $a_n = 3$, and, since we add -4 to 35 eight times, $n = 9$. (There are 8 terms after a_1, for a total of 9 terms.) We will use the formula $S_n = \frac{n}{2}(a_1 + a_n)$.

Translate. We want to find the sum of the first 9 terms of an arithmetic sequence in which $a_1 = 35$ and $a_9 = 3$.

Carry out. Substituting into the formula, we have

$$S_9 = \frac{9}{2}(35 + 3)$$
$$= \frac{9}{2} \cdot 38 = 171$$

Check. We can check the calculations by doing them again. We could also do the entire addition:

$35 + 31 + 27 + \ldots + 3$.

State. There are 171 plants altogether.

46. $a_8 = 10 + (8 - 1)(2) = 10 + 7 \cdot 2 = 24$ marchers

$S_8 = \frac{8}{2}(10 + 24) = 4 \cdot 34 = 136$ marchers

47. Yes; $d = 6 - 3 = 9 - 6 = 3n - 3(n - 1) = 3$.

48. $7x - 2y = 4, \quad (1)$

$x + 3y = 17 \quad (2)$

Multiply equation (1) by 3 and equation (2) by 2 and add.

$$\begin{array}{rl} 21x - 6y &= 12 \\ 2x + 6y &= 34 \\ \hline 23x \quad &= 46 \\ x &= 2 \end{array}$$

Back-substitute to find y.

$$\begin{aligned} 2+3y &= 17 \quad \text{Using equation (2)} \\ 3y &= 15 \\ y &= 5 \end{aligned}$$

The solution is $(2,5)$.

49. $$\begin{aligned} 2x + \ \ y + 3z &= 12 \\ x - 3y - 2z &= -1 \\ 5x + 2y - 4z &= -4 \end{aligned}$$

We will use Gauss-Jordan elimination with matrices. First we write the augmented matrix.

$$\left[\begin{array}{rrr|r} 2 & 1 & 3 & 12 \\ 1 & -3 & -2 & -1 \\ 5 & 2 & -4 & -4 \end{array}\right]$$

Next we interchange the first two rows.

$$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -1 \\ 2 & 1 & 3 & 12 \\ 5 & 2 & -4 & -4 \end{array}\right]$$

Now multiply the first row by -2 and add it to the second row. Also multiply the first row by -5 and add it to the third row.

$$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -1 \\ 0 & 7 & 7 & 14 \\ 0 & 17 & 6 & 1 \end{array}\right]$$

Multiply the second row by $\dfrac{1}{7}$.

$$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 17 & 6 & 1 \end{array}\right]$$

Multiply the second row by 3 and add it to the first row. Also multiply the second row by -17 and add it to the third row.

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -11 & -33 \end{array}\right]$$

Multiply the third row by $-\dfrac{1}{11}$.

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Multiply the third row by -1 and add it to the first row and also to the second row.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now we can read the solution from the matrix. It is $(2,-1,3)$.

50. $9x^2 + 16y^2 = 144$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Vertices: $(-4,0)$, $(4,0)$

$c^2 = a^2 - b^2 = 16 - 9 = 7$

$c = \sqrt{7}$

Foci: $(-\sqrt{7},0)$, $(\sqrt{7},0)$

51. The vertices are on the y-axis, so the transverse axis is vertical and $a=5$. The length of the minor axis is 4, so $b = 4/2 = 2$. The equation is

$$\frac{x^2}{4} + \frac{y^2}{25} = 1.$$

52. Let x = the first number in the sequence and let d = the common difference. Then the three numbers in the sequence are x, $x+d$, and $x+2d$. Solve:

$$x + x + 2d = 10,$$
$$x(x+d) = 15.$$

We get $x = 3$ and $d = 2$ so the numbers are 3, $3+2$, and $3+2\cdot 2$, or 3, 5, and 7.

53. $S_n = \dfrac{n}{2}(1+2n-1) = n^2$

54. $$\begin{aligned} a_1 &= \$8760 \\ a_2 &= \$8760 + (-\$798.23) = \$7961.77 \\ a_3 &= \$8760 + 2(-\$798.23) = \$7163.54 \\ a_4 &= \$8760 + 3(-\$798.23) = \$6365.31 \\ a_5 &= \$8760 + 4(-\$798.23) = \$5567.08 \\ a_6 &= \$8760 + 5(-\$798.23) = \$4768.85 \\ a_7 &= \$8760 + 6(-\$798.23) = \$3970.62 \\ a_8 &= \$8760 + 7(-\$798.23) = \$3172.39 \\ a_9 &= \$8760 + 8(-\$798.23) = \$2374.16 \\ a_{10} &= \$8760 + 9(-\$798.23) = \$1575.93 \\ S_{10} &= \frac{10}{2}(\$8760 + \$1575.93) = \$51,679.65 \end{aligned}$$

55. Let d = the common difference. Then $a_4 = a_2 + 2d$, or

$$\begin{aligned} 10p + q &= 40 - 3q + 2d \\ 10p + 4q - 40 &= 2d \\ 5p + 2q - 20 &= d. \end{aligned}$$

Also, $a_1 = a_2 - d$, so we have

$$\begin{aligned} a_1 &= 40 - 3q - (5p + 2q - 20) \\ &= 40 - 3q - 5p - 2q + 20 \\ &= 60 - 5p - 5q. \end{aligned}$$

56. $P(x) = x^4 + 4x^3 - 84x^2 - 176x + 640$ has at most 4 zeros because $P(x)$ is of degree 4. By the rational roots theorem, the possible zeros are

± 1, ± 2, ± 4, ± 5, ± 8, ± 10, ± 16, ± 20, ± 32, ± 40, ± 64, ± 80, ± 128, ± 160, ± 320, ± 640

Using synthetic division, we find that two zeros are -4 and 2.

Also by synthetic division we determine that ± 1 and -2 are not zeros. Therefore we determine that $d = 6$ in the arithmetic sequence.

Possible arithmetic sequences:

a) $-4,\ 2,\ 8,\ 14$

b) $-4,\ 2,\ 8$

c) $-16,\ -10,\ -4,\ 2$

The solution cannot be (a) because 14 is not a possible zero. Checking -16 by synthetic division we find that -16 is not a zero. Thus (*b*) is the only possible arithmetic sequence which contains all four zeros. Synthetic division confirms that -10 and 8 are also zeros. The zeros are -10, -4, 2, and 8.

57. 4, m_1, m_2, m_3, 12

We look for m_1, m_2, and m_3 such that 4, m_1, m_2, m_3, 12 is an arithmetic sequence. In this case, $a_1 = 4$, $n = 5$, and $a_5 = 12$. First we find d:

$$a_n = a_1 + (n-1)d$$
$$12 = 4 + (5-1)d$$
$$12 = 4 + 4d$$
$$8 = 4d$$
$$2 = d$$

Then we have

$$m_1 = a_1 + d = 4 + 2 = 6$$
$$m_2 = m_1 + d = 6 + 2 = 8$$
$$m_3 = m_2 + d = 8 + 2 = 10$$

58. -3, m_1, m_2, m_3, 5

We look for m_1, m_2, and m_3 such that -3, m_1, m_2, m_3, 5 is an arithmetic sequence. In this case, $a_1 = -3$, $n = 5$, and $a_5 = 5$. First we find d:

$$a_n = a_1 + (n-1)d$$
$$5 = -3 + (5-1)d$$
$$8 = 4d$$
$$2 = d$$

Then we have

$$m_1 = a_1 + d = -3 + 2 = -1,$$
$$m_2 = m_1 + d = -1 + 2 = 1,$$
$$m_3 = m_2 + d = 1 + 2 = 3.$$

59. 4, m_1, m_2, m_3, m_4, 13

We look for m_1, m_2, m_3, and m_4 such that 4, m_1, m_2, m_3, m_4, 13 is an arithmetic sequence. In this case $a_1 = 4$, $n = 6$, and $a_6 = 13$. First we find d.

$$a_n = a_1 + (n-1)d$$
$$13 = 4 + (6-1)d$$
$$9 = 5d$$
$$1\frac{4}{5} = d$$

Then we have

$$m_1 = a_1 + d = 4 + 1\frac{4}{5} = 5\frac{4}{5},$$
$$m_2 = m_1 + d = 5\frac{4}{5} + 1\frac{4}{5} = 6\frac{8}{5} = 7\frac{3}{5},$$
$$m_3 = m_2 + d = 7\frac{3}{5} + 1\frac{4}{5} = 8\frac{7}{5} = 9\frac{2}{5},$$
$$m_4 = m_3 + d = 9\frac{2}{5} + 1\frac{4}{5} = 10\frac{6}{5} = 11\frac{1}{5}.$$

60. 27, m_1, m_2, . . . ,m_9, m_{10}, 300

$$300 = 27 + (12-1)d$$
$$273 = 11d$$
$$\frac{273}{11} = d, \text{ or}$$
$$24\frac{9}{11} = d$$
$$m_1 = 27 + 24\frac{9}{11} = 51\frac{9}{11}$$
$$m_2 = 51\frac{9}{11} + 24\frac{9}{11} = 76\frac{7}{11}$$
$$m_3 = 76\frac{7}{11} + 24\frac{9}{11} = 101\frac{5}{11},$$
$$m_4 = 101\frac{5}{11} + 24\frac{9}{11} = 126\frac{3}{11},$$
$$m_5 = 126\frac{3}{11} + 24\frac{9}{11} = 151\frac{1}{11},$$
$$m_6 = 151\frac{1}{11} + 24\frac{9}{11} = 175\frac{10}{11},$$
$$m_7 = 175\frac{10}{11} + 24\frac{9}{11} = 200\frac{8}{11},$$
$$m_8 = 200\frac{8}{11} + 24\frac{9}{11} = 225\frac{6}{11},$$
$$m_9 = 225\frac{6}{11} + 24\frac{9}{11} = 250\frac{4}{11},$$
$$m_{10} = 250\frac{4}{11} + 24\frac{9}{11} = 275\frac{2}{11}$$

61. $1, 1+d, 1+2d, \ldots, 50$ has n terms and $S_n = 459$.

Find n:

$$459 = \frac{n}{2}(1+50)$$
$$18 = n$$

Find d:

$$50 = 1 + (18-1)d$$
$$\frac{49}{17} = d$$

The sequence has a total of 18 terms, so we insert 16 arithmetic means between 1 and 50 with $d = \frac{49}{17}$.

62. a) $a_t = \$5200 - t\left(\frac{\$5200 - \$1100}{8}\right)$

$a_t = \$5200 - \$512.50t$

b) $a_0 = \$5200 - \$512.50(0) = \$5200$

$a_1 = \$5200 - \$512.50(1) = \$4687.50$

$a_2 = \$5200 - \$512.50(2) = \$4175$

$$a_3 = \$5200 - \$512.50(3) = \$3662.50$$
$$a_4 = \$5200 - \$512.50(4) = \$3150$$
$$a_7 = \$5200 - \$512.50(7) = \$1612.50$$
$$a_8 = \$5200 - \$512.50(8) = \$1100$$

63.
$$\begin{array}{rl} m = p + d & \\ m = q - d & \\ \hline 2m = p + q & \text{Adding} \\ m = \dfrac{p+q}{2} & \end{array}$$

Exercise Set 11.3

1. 2, 4, 8, 16, . . .
$\frac{4}{2} = 2$, $\frac{8}{4} = 2$, $\frac{16}{8} = 2$
$r = 2$

2. $r = -\frac{6}{18} = -\frac{1}{3}$

3. 1, −1, 1, −1, . . .
$\frac{-1}{1} = -1$, $\frac{1}{-1} = -1$, $\frac{-1}{1} = -1$
$r = -1$

4. $r = \frac{-0.8}{8} = 0.1$

5. $\frac{2}{3}, -\frac{4}{3}, \frac{8}{3}, -\frac{16}{3}, \ldots$
$\dfrac{-\frac{4}{3}}{\frac{2}{3}} = -2$, $\dfrac{\frac{8}{3}}{-\frac{4}{3}} = -2$, $\dfrac{-\frac{16}{3}}{\frac{8}{3}} = -2$
$r = -2$

6. $r = \frac{15}{75} = \frac{1}{5}$

7. $\frac{0.6275}{6.275} = 0.1$, $\frac{0.06275}{0.6275} = 0.1$
$r = 0.1$

8. $r = \dfrac{\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x}$

9. $\dfrac{\frac{5a}{2}}{5} = \frac{a}{2}$, $\dfrac{\frac{5a^2}{4}}{\frac{5a}{2}} = \frac{a}{2}$, $\dfrac{\frac{5a^3}{8}}{\frac{5a}{4}} = \frac{a}{2}$
$r = \frac{a}{2}$

10. $r = \frac{\$858}{\$780} = 1.1$

11. 2, 4, 8, 16, . . .
$a_1 = 2$, $n = 7$, and $r = \frac{4}{2}$, or 2.
We use the formula $a_n = a_1 r^{n-1}$.
$$a_7 = 2(2)^{7-1} = 2 \cdot 2^6 = 2 \cdot 64 = 128$$

12. $a_9 = 2(-5)^{9-1} = 781,250$

13. 2, $2\sqrt{3}$, 6, . . .
$a_1 = 2,\ n = 9,$ and $r = \frac{2\sqrt{3}}{2}$, or $\sqrt{3}$
$a_n = a_1 r^{n-1}$
$a_9 = 2(\sqrt{3})^{9-1} = 2(\sqrt{3})^8 = 2 \cdot 81 = 162$

14. $a_{57} = 1(-1)^{57-1} = 1$

15. $\frac{7}{625}, -\frac{7}{25}, \ldots$
$a_1 = \frac{7}{625}$, $n = 23$, and $r = \dfrac{-\frac{7}{25}}{\frac{7}{625}} = -25$.
$a_n = a_1 r^{n-1}$
$$a_{23} = \frac{7}{625}(-25)^{23-1} = \frac{7}{625}(-25)^{22}$$
$$= \frac{7}{25^2} \cdot 25^2 \cdot 25^{20} = 7(25)^{20}, \text{ or } 7(5)^{40}$$

16. $a_5 = \$1000(1.06)^{5-1} \approx \$1262.48$

17. 1, 3, 9, . . .
$a_1 = 1$ and $r = \frac{3}{1}$, or 3
$a_n = a_1 r^{n-1}$
$a_n = 1(3)^{n-1} = 3^{n-1}$

18. $a_n = 25\left(\frac{1}{5}\right)^{n-1} = \frac{5^2}{5^{n-1}} = 5^{3-n}$

19. 1, −1, 1, −1, . . .
$a_1 = 1$ and $r = \frac{-1}{1} = -1$
$a_n = a_1 r^{n-1}$
$a_n = 1(-1)^{n-1} = (-1)^{n-1}$

20. $a_n = (-2)^n$

21. $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \ldots$
$a_1 = \frac{1}{x}$ and $r = \dfrac{\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x}$
$a_n = a_1 r^{n-1}$
$$a_n = \frac{1}{x}\left(\frac{1}{x}\right)^{n-1} = \frac{1}{x} \cdot \frac{1}{x^{n-1}} = \frac{1}{x^{1+n-1}} = \frac{1}{x^n}$$

22. $a_n = 5\left(\frac{a}{2}\right)^{n-1}$

23. $6 + 12 + 24 + \ldots$
$a_1 = 6,\ n = 7,$ and $r = \frac{12}{6}$, or 2
$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$
$$S_7 = \frac{6(1 - 2^7)}{1 - 2} = \frac{6(1 - 128)}{-1} = \frac{6(-127)}{-1} = 762$$

24. $S_{10} = \dfrac{16\left[1-\left(-\frac{1}{2}\right)^{10}\right]}{1-\left(-\frac{1}{2}\right)} = \dfrac{16\left(1-\frac{1}{1024}\right)}{\frac{3}{2}} =$

$\dfrac{16\left(\frac{1023}{1024}\right)}{\frac{3}{2}} = \dfrac{341}{32}$, or $10\dfrac{21}{32}$

25. $\dfrac{1}{18} - \dfrac{1}{6} + \dfrac{1}{2} - \ldots$

$a_1 = \dfrac{1}{18}$, $n = 9$, and $r = \dfrac{-\frac{1}{6}}{\frac{1}{18}} = -\dfrac{1}{6} \cdot \dfrac{18}{1} = -3$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_9 = \frac{\frac{1}{18}\left[1-(-3)^9\right]}{1-(-3)} = \frac{\frac{1}{18}(1+19,683)}{4}$$

$$\frac{\frac{1}{18}(19,684)}{4} = \frac{1}{18}(19,684)\left(\frac{1}{4}\right) = \frac{4921}{18}$$

26. $a_n = a_1 r^{n-1}$

$-\dfrac{1}{32} = (-8) \cdot \left(-\dfrac{1}{2}\right)^{n-1}$

$n = 9$

$$S_9 = \frac{-8\left[\left(-\frac{1}{2}\right)^9 - 1\right]}{-\frac{1}{2} - 1} = -\frac{171}{32}$$

27. Multiplying each term of the sequence by $-\sqrt{2}$ produces the next term, so it is true that the sequence is geometric.

28. The sequence with general term $3n$ does not have a common ratio, so the statement is false.

29. Since $\dfrac{2^{n+1}}{2^n} = 2$, the sequence has a common ratio so it is true that the sequence is geometric.

30. It is true that multiplying a term of a geometric sequence by the common ratio produces the next term of the sequence.

31. Since $|-0.75| < 1$, it is true that the series has a sum.

32. When $|r| \geq 1$, a geometric series does not have a limit, so the statement is false.

33. $4 + 2 + 1 + \ldots$

$|r| = \left|\dfrac{2}{4}\right| = \left|\dfrac{1}{2}\right| = \dfrac{1}{2}$, and since $|r| < 1$, the series does have a sum.

$$S_\infty = \frac{a_1}{1-r} = \frac{4}{1-\frac{1}{2}} = \frac{4}{\frac{1}{2}} = 4 \cdot \frac{2}{1} = 8$$

34. $|r| = \left|\dfrac{3}{7}\right| = \dfrac{3}{7} < 1$, so the series has a sum.

$$S_\infty = \frac{7}{1-\frac{3}{7}} = \frac{7}{\frac{4}{7}} = \frac{49}{4}$$

35. $25 + 20 + 16 + \ldots$

$|r| = \left|\dfrac{20}{25}\right| = \left|\dfrac{4}{5}\right| = \dfrac{4}{5}$, and since $|r| < 1$, the series does have a sum.

$$S_\infty = \frac{a_1}{1-r} = \frac{25}{1-\frac{4}{5}} = \frac{25}{\frac{1}{5}} = 25 \cdot \frac{5}{1} = 125$$

36. $|r| = \left|\dfrac{-10}{100}\right| = \dfrac{1}{10} < 1$, so the series has a sum.

$$S_\infty = \frac{100}{1-\left(-\frac{1}{10}\right)} = \frac{100}{\frac{11}{10}} = \frac{1000}{11}$$

37. $8 + 40 + 200 + \ldots$

$|r| = \left|\dfrac{40}{8}\right| = |5| = 5$, and since $|r| > 1$ the series does not have a sum.

38. $|r| = \left|\dfrac{3}{-6}\right| = \left|-\dfrac{1}{2}\right| = \dfrac{1}{2} < 1$, so the series has a sum.

$$S_\infty = \frac{-6}{1-\left(-\frac{1}{2}\right)} = \frac{-6}{\frac{3}{2}} = -6 \cdot \frac{2}{3} = -4$$

39. $0.6 + 0.06 + 0.006 + \ldots$

$|r| = \left|\dfrac{0.06}{0.6}\right| = |0.1| = 0.1$, and since $|r| < 1$, the series does have a sum.

$$S_\infty = \frac{a_1}{1-r} = \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{6}{9} = \frac{2}{3}$$

40. $\displaystyle\sum_{k=0}^{10} 3^k$

$a_1 = 1,\ |r| = 3,\ n = 11$

$$S_{11} = \frac{1(1-3^{11})}{1-3} = 88,573$$

41. $\displaystyle\sum_{k=1}^{11} 15\left(\frac{2}{3}\right)^k$

$a_1 = 15 \cdot \dfrac{2}{3}$ or 10; $|r| = \left|\dfrac{2}{3}\right| = \dfrac{2}{3},\ n = 11$

$$S_{11} = \frac{10\left[1-\left(\frac{2}{3}\right)^{11}\right]}{1-\frac{2}{3}} = \frac{10\left[1-\frac{2048}{177,147}\right]}{\frac{1}{3}}$$

$$= 10 \cdot \frac{175,099}{177,147} \cdot 3$$

$$= \frac{1,750,990}{59,049}, \text{ or } 29\frac{38,569}{59,049}$$

42. $\displaystyle\sum_{k=0}^{50} 200(1.08)^k$

$a_1 = 200,\ |r| = 1.08,\ n = 51$

$$S_{51} = \frac{200[1-(1.08)^{51}]}{1-1.08} \approx 124,134.354$$

43. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}$

$a_1 = 1,\ |r| = \left|\frac{1}{2}\right| = \frac{1}{2}$

$S_\infty = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

44. Since $|r| = |2| > 1$, the sum does not exist.

45. $\displaystyle\sum_{k=1}^{\infty} 12.5^k$

Since $|r| = 12.5 > 1$, the sum does not exist.

46. Since $|r| = 1.0625 > 1$, the sum does not exist.

47. $\displaystyle\sum_{k=1}^{\infty} \$500(1.11)^{-k}$

$a_1 = \$500(1.11)^{-1}$, or $\frac{\$500}{1.11}$; $|r| = |1.11^{-1}| = \frac{1}{1.11}$

$S_\infty = \frac{a_1}{1-r} = \frac{\frac{\$500}{1.11}}{1-\frac{1}{1.11}} = \frac{\frac{\$500}{1.11}}{\frac{0.11}{1.11}} \approx \$4545.\overline{45}$

48. $\displaystyle\sum_{k=1}^{\infty} \$1000(1.06)^{-k}$

$a_1 = \frac{\$1000}{1.06},\ |r| = \frac{1}{1.06}$

$S_\infty = \frac{\frac{\$1000}{1.06}}{1-\frac{1}{1.06}} = \frac{\frac{\$1000}{1.06}}{\frac{0.06}{1.06}} \approx \$16,666.\overline{66}$

49. $\displaystyle\sum_{k=1}^{\infty} 16(0.1)^{k-1}$

$a_1 = 16,\ |r| = |0.1| = 0.1$

$S_\infty = \frac{a_1}{1-r} = \frac{16}{1-0.1} = \frac{16}{0.9} = \frac{160}{9}$

50. $\displaystyle\sum_{k=1}^{\infty} \frac{8}{3}\left(\frac{1}{2}\right)^{k-1}$

$a_1 = \frac{8}{3},\ |r| = \frac{1}{2}$

$S_\infty = \frac{\frac{8}{3}}{1-\frac{1}{2}} = \frac{\frac{8}{3}}{\frac{1}{2}} = \frac{16}{3}$

51. $0.131313\ldots = 0.13 + 0.0013 + 0.000013 + \ldots$

This is an infinite geometric series with $a_1 = 0.13$.

$|r| = \left|\frac{0.0013}{0.13}\right| = |0.01| = 0.01 < 1$, so the series has a limit.

$S_\infty = \frac{a_1}{1-r} = \frac{0.13}{1-0.01} = \frac{0.13}{0.99} = \frac{13}{99}$

52. $0.2222 = 0.2 + 0.02 + 0.002 + 0.0002 + \ldots$

$|r| = \left|\frac{0.02}{0.2}\right| = |0.1| = 0.1$

$S_\infty = \frac{0.2}{1-0.1} = \frac{0.2}{0.9} = \frac{2}{9}$

53. We will find fraction notation for $0.999\overline{9}$ and then add 8.

$0.999\overline{9} = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots$

This is an infinite geometric series with $a_1 = 0.9$.

$|r| = \left|\frac{0.09}{0.9}\right| = |0.1| = 0.1 < 1$, so the series has a limit.

$S_\infty = \frac{a_1}{1-r} = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1$

Then $8.999\overline{9} = 8 + 1 = 9$.

54. $0.1\overline{6} = 0.16 + 0.0016 + 0.000016 + \ldots$

$|r| = \left|\frac{0.0016}{0.16}\right| = |0.01| = 0.01$

$S_\infty = \frac{0.16}{1-0.01} = \frac{0.16}{0.99} = \frac{16}{99}$

Then $6.1\overline{6} = 6 + \frac{16}{99} = \frac{610}{99}$.

55. $3.4125\overline{125} = 3.4 + 0.0125\overline{125}$

We will find fraction notation for $0.0125\overline{125}$ and then add 3.4, or $\frac{34}{10}$, or $\frac{17}{5}$.

$0.0125\overline{125} = 0.0125 + 0.0000125 + \ldots$

This is an infinite geometric series with $a_1 = 0.0125$.

$|r| = \left|\frac{0.0000125}{0.0125}\right| = |0.001| = 0.001 < 1$, so the series has a limit.

$S_\infty = \frac{a_1}{1-r} = \frac{0.0125}{1-0.001} = \frac{0.0125}{0.999} = \frac{125}{9990}$

Then $\frac{17}{5} + \frac{125}{9990} = \frac{33,966}{9990} + \frac{125}{9990} = \frac{34,091}{9990}$

56. $12.7809\overline{809} = 12.7 + 0.0809\overline{809}$

$0.0809\overline{809} = 0.0809 + 0.0000809 + \ldots$

$|r| = \left|\frac{0.0000809}{0.0809}\right| = |0.001| = 0.001$

$S_\infty = \frac{0.0809}{1-0.001} = \frac{0.0809}{0.999} = \frac{809}{9990}$

Then $12.7 + \frac{809}{9990} = \frac{127}{10} + \frac{809}{9990} = \frac{127,682}{9990} = \frac{63,841}{4995}$

57. ***Familiarize***. The total earnings are represented by the geometric series

$\$0.01 + \$0.01(2) + \$0.01(2)^2 + \ldots + \$0.01(2)^{27}$, where $a_1 = \$0.01$, $r = 2$, and $n = 28$.

Translate. Using the formula

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

we have

$$S_{28} = \frac{\$0.01(1-2^{28})}{1-2}.$$

Carry out. We carry out the computation and get \$2,684,354.55.

Check. Repeat the calculation.

State. You would earn \$2,684,354.55.

58. a) ***Familiarize***. The rebound distances form a geometric sequence:

$$\frac{1}{4}\times 16,\quad \left(\frac{1}{4}\right)^2\times 16,\quad \left(\frac{1}{4}\right)^3\times 16,\ \ldots,$$

or $4,\quad \frac{1}{4}\times 4,\quad \left(\frac{1}{4}\right)^2\times 4,\ \ldots$

The height of the 6th rebound is the 6th term of the sequence.

Translate. We will use the formula $a_n = a_1 r^{n-1}$, with $a_1 = 4$, $r = \frac{1}{4}$, and $n = 6$:

$$a_6 = 4\left(\frac{1}{4}\right)^{6-1}$$

Carry out. We calculate to obtain $a_6 = \frac{1}{256}$.

Check. We can do the calculation again.

State. It rebounds $\frac{1}{256}$ ft the 6th time.

b) $S_\infty = \dfrac{a}{1-r} = \dfrac{4}{1-\frac{1}{4}} = \dfrac{4}{\frac{3}{4}} = \dfrac{16}{3}$ ft, or $5\frac{1}{3}$ ft

59. a) ***Familiarize***. The rebound distances form a geometric sequence:

$$0.6\times 200,\ (0.6)^2\times 200,\ (0.6)^3\times 200,\ \ldots,$$

or $120,\ 0.6\times 120,\ (0.6)^2\times 120,\ \ldots$

The total rebound distance after 9 rebounds is the sum of the first 9 terms of this sequence.

Translate. We will use the formula

$S_n = \dfrac{a_1(1-r^n)}{1-r}$ with $a_1 = 120$, $r = 0.6$, and $n = 9$.

Carry out.

$$S_9 = \frac{120[1-(0.6)^9]}{1-0.6} \approx 297$$

Check. We repeat the calculation.

State. The bungee jumper has traveled about 297 ft upward after 9 rebounds.

b) $S_\infty = \dfrac{a_1}{1-r} = \dfrac{120}{1-0.6} = 300$ ft

60. a) $a_1 = 100,000$, $r = 1.03$. The population in 15 years will be the 16th term of the sequence
$100,000,\ (1.03)100,000,\ (1.03)^2 100,000,\ \ldots$

$$a_{16} = 100,000(1.03)^{16-1} \approx 155,797$$

b) Solve: $200,000 = 100,000(1.03)^{n-1}$

$n \approx 24$ yr

61. ***Familiarize***. The amount of the annuity is the geometric series

$\$1000 + \$1000(1.032) + \$1000(1.032)^2 + \ldots + \$1000(1.032)^{17}$, where $a_1 = \$1000$, $r = 1.032$, and $n = 18$.

Translate. Using the formula

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

we have

$$S_{18} = \frac{\$1000[1-(1.032)^{18}]}{1-1.032}.$$

Carry out. We carry out the computation and get $S_{18} \approx \$23,841.50$.

Check. Repeat the calculations.

State. The amount of the annuity is \$23,841.50.

62. a) We have a geometric sequence with $a_1 = P$ and $r = 1+i$. Then

$$S_N = V = \frac{P[1-(1+i)^N]}{1-(1+i)} = \frac{P[1-(1+i)^N]}{-i} = \frac{P[(1+i)^N - 1]}{i}$$

b) We have a geometric sequence with $a_1 = P$ and $r = 1 + \frac{i}{n}$.

The number of terms is nN. Then

$$S_{nN} = V = \frac{P\left[1-\left(1+\frac{i}{n}\right)^{nN}\right]}{1-\left(1+\frac{i}{n}\right)} = \frac{P\left[1-\left(1+\frac{i}{n}\right)^{nN}\right]}{-\frac{i}{n}} = \frac{P\left[\left(1+\frac{i}{n}\right)^{nN}-1\right]}{\frac{i}{n}}$$

63. ***Familiarize***. The amounts owed at the beginning of successive years form a geometric sequence:

$\$120,000,\ (1.12)\$120,000,\ (1.12)^2\$120,000,\ \ldots$

The amount to be repaid at the end of 13 years is the amount owed at the beginning of the 14th year.

Translate. Use the formula $a_n = a_1 r^{n-1}$ with $a_1 = 120,000$, $r = 1.12$, and $n = 14$:

$$a_{14} = 120,000(1.12)^{14-1}$$

Carry out. We perform the calculation, obtaining $a_{14} \approx \$523,619.17$.

Check. Repeat the calculation.

State. At the end of 13 years, \$523,619.17 will be repaid.

64. We have a sequence $0.01,\ 2(0.01),\ 2^2(0.01),\ 2^3(0.01),\ \ldots$. The thickness after 20 folds is given by the 21st term of the sequence.

$$a_{21} = 0.01(2)^{21-1} = 10,485.76 \text{ in.}$$

65. ***Familiarize***. The total effect on the economy is the sum of an infinite geometric series

$\$13,000,000,000 + \$13,000,000,000(0.85) +$
$\$13,000,000,000(0.85)^2 + \ldots$

with $a_1 = \$13,000,000,000$ and $r = 0.85$.

Translate. Using the formula

$$S_\infty = \frac{a_1}{1-r}$$

we have

$$S_\infty = \frac{\$13,000,000,000}{1-0.85}.$$

Carry out. Perform the calculation:

$S_\infty \approx \$86,666,666,667$.

Check. Repeat the calculation.

State. The total effect on the economy is \$86,666,666,667.

66. $S_\infty = \dfrac{3,000,000(0.3)}{1-0.3} \approx 1,285,714$

$\dfrac{1,285,714}{3,000,000} \approx 0.429$, so this is about 42.9% of the population.

67. $f(x) = x^2$, $g(x) = 4x + 5$

$(f \circ g)(x) = f(g(x)) = f(4x+5) = (4x+5)^2 =$
$16x^2 + 40x + 25$

$(g \circ f)(x) = g(f(x)) = g(x^2) = 4x^2 + 5$

68. $f(x) = x - 1$, $g(x) = x^2 + x + 3$

$(f \circ g)(x) = f(g(x)) = f(x^2+x+3) = x^2 + x + 3 - 1 =$
$x^2 + x + 2$

$(g \circ f)(x) = g(f(x)) = g(x-1) =$
$(x-1)^2 + (x-1) + 3 = x^2 - 2x + 1 + x - 1 + 3 = x^2 - x + 3$

69.
$$\begin{aligned} 5^x &= 35 \\ \ln 5^x &= \ln 35 \\ x \ln 5 &= \ln 35 \\ x &= \frac{\ln 35}{\ln 5} \\ x &\approx 2.209 \end{aligned}$$

70.
$$\begin{aligned} \log_2 x &= -4 \\ x &= 2^{-4} = \frac{1}{2^4} \\ x &= \frac{1}{16} \end{aligned}$$

71. See the answer section in the text.

72. The sequence is not geometric; $a_4/a_3 \neq a_3/a_2$.

73. a) If the sequence is arithmetic, then $a_2 - a_1 = a_3 - a_2$.

$$x + 7 - (x+3) = 4x - 2 - (x+7)$$
$$x = \frac{13}{3}$$

The three given terms are $\dfrac{13}{3} + 3 = \dfrac{22}{3}$, $\dfrac{13}{3} + 7 = \dfrac{34}{3}$, and $4 \cdot \dfrac{13}{3} - 2 = \dfrac{46}{3}$.

Then $d = \dfrac{12}{3}$, or 4, so the fourth term is $\dfrac{46}{3} + \dfrac{12}{3} = \dfrac{58}{3}$.

b) If the sequence is geometric, then $a_2/a_1 = a_3/a_2$.

$$\frac{x+7}{x+3} = \frac{4x-2}{x+7}$$
$$x = -\frac{11}{3} \text{ or } x = 5$$

For $x = -\dfrac{11}{3}$: The three given terms are $-\dfrac{11}{3} + 3 = -\dfrac{2}{3}$, $-\dfrac{11}{3} + 7 = \dfrac{10}{3}$, and $4\left(-\dfrac{11}{3}\right) - 2 = -\dfrac{50}{3}$.

Then $r = -5$, so the fourth term is $-\dfrac{50}{3}(-5) = \dfrac{250}{3}$.

For $x = 5$: The three given terms are $5 + 3 = 8$, $5 + 7 = 12$, and $4 \cdot 5 - 2 = 18$. Then $r = \dfrac{3}{2}$, so the fourth term is $18 \cdot \dfrac{3}{2} = 27$.

74. $S_n = \dfrac{1(1-x^n)}{1-x} = \dfrac{1-x^n}{1-x}$

75. $x^2 - x^3 + x^4 - x^5 + \ldots$

This is a geometric series with $a_1 = x^2$ and $r = -x$.

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{x^2(1-(-x)^n)}{1-(-x)} = \frac{x^2(1-(-x)^n)}{1+x}$$

76. $\dfrac{a_n + 1}{a_n} = r$, so $\dfrac{(a_n+1)^2}{(a_n)^2} = r^2$; Thus a_1^2, a_2^2, a_3^2, . . . is a geometric sequence with the common ratio r^2.

77. See the answer section in the text.

78. Let the arithmetic sequence have the common difference $d = a_{n+1} - a_n$. Then for the sequence $5^{a_1}, 5^{a_2}, 5^{a_3}, \ldots$, we have $\dfrac{5^{a_{n+1}}}{5^{a_n}} = 5^{a_{n+1}-a_n} = 5^d$. Thus, we have a geometric sequence with the common ratio 5^d.

79. ***Familiarize***. The length of a side of the first square is 16 cm. The length of a side of the next square is the length of the hypotenuse of a right triangle with legs 8 cm and 8 cm, or $8\sqrt{2}$ cm. The length of a side of the next square is the length of the hypotenuse of a right triangle with legs $4\sqrt{2}$ cm and $4\sqrt{2}$ cm, or 8 cm. The areas of the squares form a sequence:

$(16)^2$, $(8\sqrt{2})^2$, $(8)^2$, . . . , or

256, 128, 64, ...

This is a geometric sequence with $a_1 = 256$ and $r = \dfrac{1}{2}$.

Translate. We find the sum of the infinite geometric series $256 + 128 + 64 + \ldots$.

$$S_\infty = \frac{a_1}{1-r}$$
$$S_\infty = \frac{256}{1 - \frac{1}{2}}$$

Carry out. We calculate to obtain $S_\infty = 512$.

Check. We can do the calculation again.

State. The sum of the areas is 512 cm^2.

Exercise Set 11.4

1. $n^2 < n^3$

   $1^2 < 1^3$, $2^2 < 2^3$, $3^2 < 3^3$, $4^2 < 4^3$, $5^2 < 5^3$

   The first statement is false, and the others are true.

2. $1^2 - 1 + 41$ is prime, $2^2 - 2 + 41$ is prime, $3^3 - 3 + 41$ is prime, $4^2 - 4 + 41$ is prime, $5^2 - 5 + 41$ is prime. Each of these statements is true.

   The statement is false for $n = 41$; $41^2 - 41 + 41$ is not prime.

3. A polygon of n sides has $\dfrac{n(n-3)}{2}$ diagonals.

   A polygon of 3 sides has $\dfrac{3(3-3)}{2}$ diagonals.

   A polygon of 4 sides has $\dfrac{4(4-3)}{2}$ diagonals.

   A polygon of 5 sides has $\dfrac{5(5-3)}{2}$ diagonals.

   A polygon of 6 sides has $\dfrac{6(6-3)}{2}$ diagonals.

   A polygon of 7 sides has $\dfrac{7(7-3)}{2}$ diagonals.

   Each of these statements is true.

4. The sum of the angles of a polygon of 3 sides is $(3-2)\cdot 180°$.

   The sum of the angles of a polygon of 4 sides is $(4-2)\cdot 180°$.

   The sum of the angles of a polygon of 5 sides is $(5-2)\cdot 180°$.

   The sum of the angles of a polygon of 6 sides is $(6-2)\cdot 180°$.

   The sum of the angles of a polygon of 7 sides is $(7-2)\cdot 180°$.

   Each of these statements is true.

5. See the answer section in the text.

6. $S_n:\ 4+8+12+\ldots+4n = 2n(n+1)$

   $S_1:\ 4 = 2\cdot 1\cdot(1+1)$

   $S_k:\ 4+8+12+\ldots+4k = 2k(k+1)$

   $S_{k+1}:\ 4+8+12+\ldots+4k+4(k+1) = 2(k+1)(k+2)$

   1) *Basis step:* Since $2\cdot 1\cdot(1+1) = 2\cdot 2 = 4$, S_1 is true.

   2) *Induction step:* Let k be any natural number. Assume S_k. Deduce S_{k+1}. Starting with the left side of S_{k+1}, we have

   $$\begin{aligned} &\underbrace{4+8+12+\ldots+4k}\ +4(k+1) \\ &= \quad 2k(k+1) \quad +4(k+1) \qquad \text{By } S_k \\ &= (k+1)(2k+4) \\ &= 2(k+1)(k+2) \end{aligned}$$

7. See the answer section in the text.

8. $S_n:\ 3+6+9+\ldots+3n = \dfrac{3n(n+1)}{2}$

   $S_1:\ 3 = \dfrac{3\cdot 1(1+1)}{2}$

   $S_k:\ 3+6+9+\ldots+3k = \dfrac{3k(k+1)}{2}$

   $S_{k+1}:\ 3+6+9+\ldots+3k+3(k+1) = \dfrac{3(k+1)(k+2)}{2}$

   1) *Basis step:* Since $\dfrac{3\cdot 1(1+1)}{2} = \dfrac{3\cdot 2}{2} = 3$, S_1 is true.

   2) *Induction step:* Let k be any natural number. Assume S_k. Deduce S_{k+1}. Starting with the left side of S_{k+1}, we have

   $$\begin{aligned} &\underbrace{3+6+9+\ldots+3k}\ +3(k+1) \\ &= \quad \frac{3k(k+1)}{2} \quad +3(k+1) \qquad \text{By } S_k \\ &= \frac{3k(k+1)+6(k+1)}{2} \\ &= \frac{(k+1)(3k+6)}{2} \\ &= \frac{3(k+1)(k+2)}{2} \end{aligned}$$

9. See the answer section in the text.

10. $S_n:\ 2 \le 2^n$

    $S_1:\ 2 \le 2^1$

    $S_k:\ 2 \le 2^k$

    $S_{k+1}:\ 2 \le 2^{k+1}$

    1) *Basis step:* Since $2 \le 2$, S_1 is true.

    2) *Induction step:* Let k be any natural number. Assume S_k. Deduce S_{k+1}.

    $$\begin{aligned} 2 &\le 2^k && S_k \\ 2\cdot 2 &\le 2^k\cdot 2 && \text{Multiplying by 2} \\ 2 < 2\cdot 2 &\le 2^{k+1} && (2 < 2\cdot 2) \\ 2 &\le 2^{k+1} \end{aligned}$$

11. See the answer section in the text.

12. $S_n:\ 3^n < 3^{n+1}$

    $S_1:\ 3^1 < 3^{1+1}$

    $S_k:\ 3^k < 3^{k+1}$

    $S_{k+1}:\ 3^{k+1} < 3^{k+2}$

    1) *Basis step:* Since $3^1 < 3^{1+1}$, or $3 < 9$, S_1 is true.

    2) *Induction step:* Let k be any natural number. Assume S_k. Deduce S_{k+1}.

    $$\begin{aligned} 3^k &< 3^{k+1} && S_k \\ 3^k\cdot 3 &< 3^{k+1}\cdot 3 && \text{Multiplying by 3} \\ 3^{k+1} &< 3^{k+2} \end{aligned}$$

13. See the answer section in the text.

14. $S_n: \ \dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+...+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$

$S_1: \ \dfrac{1}{1\cdot 2}=\dfrac{1}{1+1}$

$S_k: \ \dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+...+\dfrac{1}{k(k+1)}=\dfrac{k}{k+1}$

$S_{k+1}: \ \dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+...+\dfrac{1}{k(k+1)}+\dfrac{1}{(k+1)(k+2)}$
$$=\frac{k+1}{k+2}$$

1) *Basis step:* Since $\dfrac{1}{1+1}=\dfrac{1}{2}=\dfrac{1}{1\cdot 2}$, S_1 is true.

2) *Induction step:* Let k be any natural number. Assume S_k. Deduce S_{k+1}.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad (S_k)$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=$$
$$\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$
$$\left(\text{Adding } \frac{1}{(k+1)(k+2)} \text{ on both sides}\right)$$
$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$
$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$
$$=\frac{(k+1))(k+1)}{(k+1)(k+2)}$$
$$=\frac{k+1}{k+2}$$

15. See the answer section in the text.

16. $S_n: \ x\le x^n$

$S_1: \ x\le x$

$S_k: \ x\le x^k$

$S_{k+1}: \ x\le x^{k+1}$

1) *Basis step:* Since $x=x$, S_1 is true.

2) *Induction step:* Let k be any natural number. Assume S_k. Deduce S_{k+1}.

$$\begin{aligned} x &\le x^k && S_k \\ x\cdot x &\le x^k\cdot x && \text{Multiplying by } x,\ x>1 \\ x\le x\cdot x &\le x^k\cdot x \\ x &\le x^{k+1} \end{aligned}$$

17. See the answer section in the text.

18. $S_n: \ 1^2+2^2+3^2+...+n^2=\dfrac{n(n+1)(2n+1)}{6}$

$S_1: \ 1^2=\dfrac{1(1+1)(2\cdot 1+1)}{6}$

$S_k: \ 1^2+2^2+3^2+...+k^2=\dfrac{k(k+1)(2k+1)}{6}$

$S_{k+1}: \ 1^2+2^2+...+k^2+(k+1)^2=$
$$\frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

1) *Basis step:* $1^2=\dfrac{1(1+1)(2\cdot 1+1)}{6}$ is true.

2) *Induction step:* Let k be any natural number. Assume S_k. Deduce S_{k+1}.

$$1^2+2^2+...+k^2=\frac{k(k+1)(2k+1)}{6}$$

$1^2+2^2+...+k^2+(k+1)^2$
$$=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$
$$=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$
$$=\frac{(k+1)(2k^2+7k+6)}{6}$$
$$=\frac{(k+1)(k+2)(2k+3)}{6}$$
$$=\frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

19. See the answer section in the text.

20. $S_n: \ 1^4+2^4+3^4+...+n^4=$
$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$S_1: \ 1^4=\dfrac{1(1+1)(2\cdot 1+1)(3\cdot 1^2+3\cdot 1-1)}{30}$

$S_k: \ 1^4+2^4+3^4+...+k^4=$
$$\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$$

$S_{k+1}: \ 1^4+2^4+...+k^4+(k+1)^4=$
$$\frac{(k+1)(k+1+1)(2(k+1)+1)(3(k+1)^2-3(k+1)-1)}{30}$$

1) *Basis step:*
$$1^4=\frac{1(1+1)(2\cdot 1+1)(3\cdot 1^2+3\cdot 1-1)}{30}$$
is true.

2) *Induction step:*
$$1^4+2^4+...+k^4=$$
$$\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$$

$1^4+2^4+...+k^4+(k+1)^4$
$$=\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+(k+1)^4$$
$$=\frac{k(k+1)(2k+1)(3k^2+3k-1)+30(k+1)^4}{30}$$
$$=\frac{(k+1)(6k^4+39k^3+91k^2+89k+30)}{30}$$
$$=\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}$$
$$=[(k+1)(k+1+1)(2(k+1)+1)(3(k+1)^2+$$
$$3(k+1)-1)]/30$$

21. See the answer section in the text.

22. $S_n: \quad 2+5+8+...+3n-1 = \dfrac{n(3n+1)}{2}$

$S_1: \quad 2 = \dfrac{1(3\cdot 1+1)}{2}$

$S_k: \quad 2+5+8+...+3k-1 = \dfrac{k(3k+1)}{2}$

$S_{k+1}: \quad 2+5+...+(3k-1)+(3(k+1)-1) = \dfrac{(k+1)(3(k+1)+1)}{2}$

1) *Basis step:* $2 = \dfrac{1(3\cdot 1+1)}{2}$ is true.

2) *Induction step:* Let k be any natural number. Assume S_k. Deduce S_{k+1}.

$$2+5+...+3k-1 = \frac{k(3k+1)}{2}$$

$$\begin{aligned}2+5+...+(3k-1)+(3(k+1)-1) &= \\ \frac{k(3k+1)}{2}&+(3(k+1)-1)\\ &= \frac{3k^2+k+6k+6-2}{2}\\ &= \frac{3k^2+7k+4}{2}\\ &= \frac{(k+1)(3k+4)}{2}\\ &= \frac{(k+1)(3(k+1)+1)}{2}\end{aligned}$$

23. See the answer section in the text.

24. $S_n \quad \left(1+\dfrac{1}{1}\right)\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{3}\right)\cdots\left(1+\dfrac{1}{n}\right) = n+1$

$S_1: \quad 1+\dfrac{1}{1} = 1+1$

$S_k \quad \left(1+\dfrac{1}{1}\right)\cdots\left(1+\dfrac{1}{k}\right) = k+1$

$S_{k+1} \quad \left(1+\dfrac{1}{1}\right)\cdots\left(1+\dfrac{1}{k}\right)\left(1+\dfrac{1}{k+1}\right) = (k+1)+1$

1) *Basis step:* $\left(1+\dfrac{1}{1}\right) = 1+1$ is true.

2) *Induction step:* Let k be any natural number. Assume S_k. Deduce S_{k+1}.

$$\left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{k}\right) = k+1$$

$$\left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right) = (k+1)\left(1+\frac{1}{k+1}\right)$$

Multiplying by $\left(1+\dfrac{1}{k+1}\right)$

$$\begin{aligned}&= (k+1)\left(\frac{k+1+1}{k+1}\right)\\ &= (k+1)+1\end{aligned}$$

25. See the answer section in the text.

26. $2x-3y = 1, \quad (1)$

$3x-4y = 3 \quad (2)$

Multiply equation (1) by 4 and multiply equation (2) by -3 and add.

$$\begin{array}{rcl} 8x - 12y &=& 4 \\ -9x + 12y &=& -9 \\ \hline -x \qquad &=& -5 \\ x &=& 5 \end{array}$$

Back-substitute to find y. We use equation (1).

$$\begin{aligned} 2\cdot 5 - 3y &= 1\\ 10-3y &= 1\\ -3y &= -9\\ y &= 3\end{aligned}$$

The solution is $(5,3)$.

27. $x + y + z = 3, \quad (1)$

$2x - 3y - 2z = 5, \quad (2)$

$3x + 2y + 2z = 8 \quad (3)$

We will use Gaussian elimination. First multiply equation (1) by -2 and add it to equation (2). Also multiply equation (1) by -3 and add it to equation (3).

$$\begin{aligned} x + y + z &= 3\\ -5y - 4z &= -1\\ -y - z &= -1\end{aligned}$$

Now multiply the last equation above by 5 to make the y-coefficient a multiple of the y-coefficient in the equation above it.

$$\begin{aligned} x + y + z &= 3 \quad (1)\\ -5y - 4z &= -1 \quad (4)\\ -5y - 5z &= -5 \quad (5)\end{aligned}$$

Multiply equation (4) by -1 and add it to equation (3).

$$\begin{aligned} x + y + z &= 3 \quad (1)\\ -5y - 4z &= -1 \quad (4)\\ -z &= -4 \quad (6)\end{aligned}$$

Now solve equation (6) for z.

$$\begin{aligned} -z &= -4\\ z &= 4\end{aligned}$$

Back-substitute 4 for z in equation (4) and solve for y.

$$\begin{aligned} -5y - 4\cdot 4 &= -1\\ -5y-16 &= -1\\ -5y &= 15\\ y &= -3\end{aligned}$$

Finally, back-substitute -3 for y and 4 for z in equation (1) and solve for x.

$$\begin{aligned} x-3+4 &= 3\\ x+1 &= 3\\ x &= 2\end{aligned}$$

The solution is $(2,-3,4)$.

28. Let h = the number of hardback books sold and p = the number of paperback books sold.

Solve: $h+p=80,$

$24.95h+9.95p=1546$

$h=50,\ p=30$

29. ***Familiarize***. Let x, y, and z represent the amounts invested at 1.5%, 2%, and 3%, respectively.

Translate. We know that simple interest for one year was \$104. This gives us one equation:

$0.015x+0.02y+0.03z=104$

The amount invested at 2% is twice the amount invested at 1.5%:

$y=2x$, or $-2x+y=0$

There is \$400 more invested at 3% than at 2%:

$z=y+400$, or $-y+z=400$

We have a system of equations:

$$\begin{aligned} 0.015x+0.02y+0.03z&=104,\\ -2x+y&=0\\ -y+z&=400 \end{aligned}$$

Carry out. Solving the system of equations, we get $(800,1600,2000)$.

Check. Simple interest for one year would be 0.015(\$800)+0.02(\$1600)+0.03(\$2000), or \$12+\$32+\$60, or \$104. The amount invested at 2%, \$1600, is twice \$800, the amount invested at 1.5%. The amount invested at 3%, \$2000, is \$400 more than \$1600, the amount invested at 2%. The answer checks.

State. Martin invested \$800 at 1.5%, \$1600 at 2%, and \$2000 at 3%.

30. $S_n:\quad a_1+a_1r+a_1r^2+\ldots+a_1r^{n-1}=\dfrac{a_1-a_1r^n}{1-r}$

$S_1:\quad a_1=\dfrac{a_1-a_1r}{1-r}$

$S_k:\quad a_1+a_1r+a_2r^2+\ldots+a_1r^{k-1}=\dfrac{a_1-a_1r^k}{1-r}$

$S_{k+1}:\ a_1+a_1r+\ldots+a_1r^{k-1}+a_1r^{(k+1)-1}=\dfrac{a_1-a_1r^{k+1}}{1-r}$

1) *Basis step:* $a_1=\dfrac{a_1(1-r)}{1-r}=\dfrac{a_1-a_1r}{1-r}$ is true.

2) *Induction step:* Let n be any natural number. Assume S_k. Deduce S_{k+1}.

$$a_1+a_1r+\ldots+a_1r^{k-1}=\frac{a_1-a_1r^k}{1-r}$$

$$a_1+a{+_1}\,r+\ldots+a_1r^{k-1}+a_1r^k=\frac{a_1-a_1r^k}{1-r}+a_1r^k$$

Adding a_1r^k

$$=\frac{a_1-a_1r^k+a_1r^k-a_1r^{k+1}}{1-r}$$

$$=\frac{a_1-a_1r^{k+1}}{1-r}$$

31. See the answer section in the text.

32. $S_n:\quad 2n+1<3^n$

$S_2:\quad 2\cdot2+1<3^2$

$S_k:\quad 2k+1<3^k$

$S_{k+1}:\quad 2(k+1)+1=3^{k+1}$

1) *Basis step:* $2\cdot2+1<3^2$ is true.

2) *Induction step:* Let k be any natural number greater than or equal to 2. Assume S_k. Deduce S_{k+1}.

$$\begin{aligned} 2k+1&<3^k\\ 3(2k+1)&<3\cdot3^k \quad \text{Multiplying by 3}\\ 6k+3&<3^{k+1}\\ 2k+3<6k+3&<3^{k+1} \quad (2k<6k)\\ 2(k+1)+1&<3^{k+1} \end{aligned}$$

33. See the answer section in the text.

34. $S_1:\ \overline{z^1}=\overline{z}^1$ If $z=a+bi$, $\overline{z}=a-bi$.

$S_k:\ \overline{z^k}=\overline{z}^k$

$\overline{z^k}\cdot\overline{z}=\overline{z}^k\cdot\overline{z}$ Multiplying both sides of S_k by $\overline{z}$

$\overline{z^k}\cdot\overline{z}=\overline{z}^{k+1}$

$\overline{z^{k+1}}=\overline{z}^{k+1}$

35. See the answer section in the text.

36. $S_2:\ \overline{z_1z_2}=\overline{z_1}\cdot\overline{z_2}$

$S_k:\ \overline{z_1z_2\cdots z_k}=\overline{z}_1\overline{z}_2\cdots\overline{z}_k$

Starting with the left side of S_{k+1}, we have

$$\begin{aligned} \overline{z_1z_2\cdots z_kz_{k+1}}&=\overline{z_1z_2\cdots z_k}\cdot\overline{z_{k+1}} \quad \text{By } S_2\\ &=\overline{z}_1\overline{z}_2\cdots\overline{z}_k\cdot\overline{z}_{k+1} \quad \text{By } S_k \end{aligned}$$

37. See the answer section in the text.

38. $S_1: 2$ is a factor of 1^2+1.

$S_k: 2$ is a factor of k^2+k.

$$\begin{aligned} (k+1)^2+(k+1)&=k^2+2k+1+k+1\\ &=k^2+k+2(k+1) \end{aligned}$$

By S_k, 2 is a factor of k^2+k; hence 2 is a factor of the right-hand side, so 2 is a factor of $(k+1)^2+(k+1)$.

39. See the answer section in the text.

40. a) The least number of moves for

1 disk(s) is $1=2^1-1$,

2 disk(s) is $3=2^2-1$,

3 disk(s) is $7=2^3-1$,

4 disk(s) is $15=2^4-1$; etc.

b) Let P_n be the least number of moves for n disks. We conjecture and must show:

$S_n:\quad P_n=2^n-1$.

1) *Basis step:* S_1 is true by substitution.

2) *Induction step:* Assume S_k for k disks: $P_k = 2^k - 1$. Show: $P_{k+1} = 2^{k+1} - 1$. Now suppose there are $k+1$ disks on one peg. Move k of them to another peg in $2^k - 1$ moves (by S_k) and move the remaining disk to the free peg (1 move). Then move the k disks onto it in (another) $2^k - 1$ moves. Thus the total moves P_{k+1} is $2(2^k - 1) + 1 = 2^{k+1} - 1$: $P_{k+1} = 2^{k+1} - 1$.

Chapter 11 Mid-Chapter Mixed Review

1. All of the terms of a sequence with general term $a_n = n$ are positive. Since the given sequence has negative terms, the given statement is false.

2. True; see page 921 in the text.

3. $a_2/a_1 = 7/3$; $a_3/a_2 = 3/-1 = -3$; since $7/3 \neq -3$, the sequence is not geometric. The given statement is false.

4. False; we must also show that S_1 is true.

5.
$$\begin{aligned} a_n &= 3n+5 \\ a_1 &= 3\cdot 1+5 = 8, \\ a_2 &= 3\cdot 2+5 = 11, \\ a_3 &= 3\cdot 3+5 = 14, \\ a_4 &= 3\cdot 4+5 = 17; \\ a_9 &= 3\cdot 9+5 = 32; \\ a_{14} &= 3\cdot 14+5 = 47 \end{aligned}$$

6.
$$\begin{aligned} a_n &= (-1)^{n+1}(n-1) \\ a_1 &= (-1)^{1+1}(1-1) = 0, \\ a_2 &= (-1)^{2+1}(2-1) = -1, \\ a_3 &= (-1)^{3+1}(3-1) = 2, \\ a_4 &= (-1)^{4+1}(4-1) = -3; \\ a_9 &= (-1)^{9+1}(9-1) = 8; \\ a_{14} &= (-1)^{14+1}(14-1) = -13 \end{aligned}$$

7. $3, 6, 9, 12, 15, \ldots$

These are multiples of 3, so the general term could be $3n$.

8. $a_n = (-1)^n n^2$

9. $S_4 = 1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} = 1\dfrac{7}{8}$, or $\dfrac{15}{8}$

10. $1(1+1) + 2(2+1) + 3(3+1) + 4(4+1) + 5(5+1) = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6 = 2+6+12+20+30 = 70$

11. $-4+8-12+16-20+\ldots$

This is an infinite sum of multiples of 4 with alternating signs. Sigma notation is

$$\sum_{k=1}^{\infty}(-1)^k 4k.$$

12.
$$\begin{aligned} a_1 &= 2, \\ a_2 &= 4\cdot 2-2 = 6, \\ a_3 &= 4\cdot 6-2 = 22, \\ a_4 &= 4\cdot 22-2 = 86 \end{aligned}$$

13. $7-12 = -5$; $2-7 = -5$; $-3-2 = -5$

The common difference is -5.

14.
$$\begin{aligned} d &= 6-4 = 2 \\ a_{10} &= 4+(10-1)2 = 4+9\cdot 2 = 22 \end{aligned}$$

15. In Exercise 14 we found that $d = 2$.
$$\begin{aligned} a_n &= a_1+(n-1)d \\ 44 &= 4+(n-1)2 \\ 44 &= 4+2n-2 \\ 44 &= 2+2n \\ 42 &= 2n \\ 21 &= n \end{aligned}$$

The 21st term is 44.

16.
$$\begin{aligned} d &= 11-6 = 5 \\ a_{16} &= 6+(16-1)5 = 6+15\cdot 5 = 81 \\ S_{16} &= \frac{16}{2}(6+81) = 8\cdot 87 = 696 \end{aligned}$$

17. $\dfrac{-8}{-16} = -\dfrac{1}{2}$; $\dfrac{4}{-8} = -\dfrac{1}{2}$; $\dfrac{-2}{4} = -\dfrac{1}{2}$

The common ratio is $-\dfrac{1}{2}$.

18. $r = \dfrac{\frac{1}{8}}{\frac{1}{16}} = \dfrac{1}{8}\cdot\dfrac{16}{1} = 2$

a) $a_8 = \dfrac{1}{16}\cdot 2^{8-1} = \dfrac{1}{16}\cdot 2^7 = 8$

b) $S_{10} = \dfrac{\frac{1}{16}(1-2^{10})}{1-2} = \dfrac{1}{16}\dfrac{(1-1024)}{-1} = -\dfrac{1}{16}(-1023) = \dfrac{1023}{16}$, or 63.9375

19. $|r| = \left|\dfrac{4}{-8}\right| = \left|-\dfrac{1}{2}\right| = \dfrac{1}{2} < 1$, so the series has a sum.

$$S_\infty = \frac{a_1}{1-r} = \frac{-8}{1-\left(-\frac{1}{2}\right)} = \frac{-8}{\frac{3}{2}} = -8\cdot\frac{2}{3} = -\frac{16}{3}$$

20. $|r| = |5| = 5 > 1$, so the series does not have a sum.

21. ***Familiarize***. The number of plants is represented by the arithmetic series $36+30+24+\ldots$ with $a_1 = 36$, $d = 30-36 = -6$, and $n = 6$.

Translate. We want to find $S_n = \dfrac{n}{2}(a_1+a_n)$ where $a_n = a_1+(n-1)d$.

Carry out.
$$\begin{aligned} a_6 &= 36+(6-1)(-6) = 36+5(-6) = 36-30 = 6 \\ S_6 &= \frac{6}{2}(36+6) = 3\cdot 42 = 126 \end{aligned}$$

Check. We can do the calculations again or we can do the entire addition $36+30+24+18+12+6$. The answer checks.

State. In all, there will be 126 plants.

22. We have a geometric sequence with $a_1 = \$1500$, $r = 104\%$, or 1.04, and $n = 4$.

$$S_4 = \frac{\$1500(1-1.04^4)}{1-1.04} \approx \$6369.70$$

23. See the answer section in the text.

24. The first formula can be derived from the second by substituting $a_1 + (n-1)d$ for a_n. When the first and last terms of the sum are known, the second formula is the better one to use. If the last term is not known, the first formula allows us to compute the sum in one step without first finding a_n.

25.
$$\begin{aligned} & 1+2+3+\ldots+100 \\ =\ & (1+100)+(2+99)+(3+98)+\ldots+ \\ & (50+51) \\ =\ & \underbrace{101+101+101+\ldots+101}_{\text{50 addends of 101}} \\ =\ & 50\cdot 101 \\ =\ & 5050 \end{aligned}$$

A formula for the first n natural numbers is $\frac{n}{2}(1+n)$.

26. Answers may vary. One possibility is given. Casey invests \$900 at 8% interest, compounded annually. How much will be in the account at the end of 40 years?

27. We can prove an infinite sequence of statements S_n by showing that a basis statement S_1 is true and then that for all natural numbers k, if S_k is true, then S_{k+1} is true.

Exercise Set 11.5

1. ${}_6P_6 = 6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$

2. ${}_4P_3 = 4\cdot 3\cdot 2 = 24$, or

$${}_4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = \frac{4\cdot 3\cdot 2\cdot 1}{1} = 24$$

3. Using formula (1), we have

${}_{10}P_7 = 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4 = 604,800.$

Using formula (2), we have

$${}_{10}P_7 = \frac{10!}{(10-7)!} = \frac{10!}{3!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3!}{3!} =$$
$604,800.$

4. ${}_{10}P_3 = 10\cdot 9\cdot 8 = 720$, or

$${}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10\cdot 9\cdot 8\cdot 7!}{7!} = 720$$

5. $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

6. $7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5040$

7. $0!$ is defined to be 1.

8. $1! = 1$

9. $\frac{9!}{5!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!} = 9\cdot 8\cdot 7\cdot 6 = 3024$

10. $\frac{9!}{4!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4!}{4!} = 9\cdot 8\cdot 7\cdot 6\cdot 5 = 15,120$

11. $(8-3)! = 5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

12. $(8-5)! = 3! = 3\cdot 2\cdot 1 = 6$

13. $\frac{10!}{7!3!} = \frac{10\cdot 9\cdot 8\cdot 7!}{7!3\cdot 2\cdot 1} = \frac{10\cdot 3\cdot 3\cdot 4\cdot 2}{3\cdot 2\cdot 1} =$
$10\cdot 3\cdot 4 = 120$

14. $\frac{7!}{(7-2)!} = \frac{7!}{5!} = \frac{7\cdot 6\cdot 5!}{5!} = 42$

15. Using formula (2), we have

$${}_8P_0 = \frac{8!}{(8-0)!} = \frac{8!}{8!} = 1.$$

16. ${}_{13}P_1 = 13$ (Using formula (1))

17. Using a calculator, we find
${}_{52}P_4 = 6,497,400$

18. ${}_{52}P_5 = 311,875,200$

19. Using formula (1), we have ${}_nP_3 = n(n-1)(n-2)$.

Using formula (2), we have

$${}_nP_3 = \frac{n!}{(n-3)!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!} =$$
$n(n-1)(n-2)$.

20. ${}_nP_2 = n(n-1)$

21. Using formula (1), we have ${}_nP_1 = n$.

Using formula (2), we have

$${}_nP_1 = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n.$$

22. ${}_nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$

23. ${}_6P_6 = 6! = 720$

24. ${}_4P_4 = 4! = 24$

25. ${}_9P_9 = 9! = 362,880$

26. ${}_8P_8 = 8! = 40,320$

27. ${}_9P_4 = 9\cdot 8\cdot 7\cdot 6 = 3024$

28. ${}_8P_5 = 8\cdot 7\cdot 6\cdot 5\cdot 4 = 6720$

29. Without repetition: ${}_5P_5 = 5! = 120$

With repetition: $5^5 = 3125$

30. ${}_7P_7 = 7! = 5040$

31. There are ${}_5P_5$ choices for the order of the rock numbers and ${}_4P_4$ choices for the order of the speeches, so we have ${}_5P_5 \cdot {}_4P_4 = 5!4! = 2880$.

32. $\frac{24!}{3!5!9!4!3!} = 16,491,024,950,400$

33. The first number can be any of the eight digits other than 0 and 1. The remaining 6 numbers can each be any of the ten digits 0 through 9. We have

$$8 \cdot 10^6 = 8,000,000$$

Accordingly, there can be 8,000,000 telephone numbers within a given area code before the area needs to be split with a new area code.

34. BUSINESS: 1 B, 1 U, 3 S's, 1 I, 1 N, 1 E, a total of 8.

$$= \frac{8!}{1! \cdot 1! \cdot 3! \cdot 1! \cdot 1! \cdot 1!}$$

$$= \frac{8!}{3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$$

BIOLOGY: 1 B, 1 I, 2 0's, 1 L, 1 G, 1 Y, a total of 7.

$$= \frac{7!}{1! \cdot 1! \cdot 2! \cdot 1! \cdot 1! \cdot 1!}$$

$$= \frac{7!}{2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$$

MATHEMATICS: 2 M's, 2 A's, 2 T's, 1 H, 1 E, 1 I, 1 C, 1 S, a total of 11.

$$= \frac{11!}{2! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 1! \cdot 1!}$$

$$= \frac{11!}{2! \cdot 2! \cdot 2!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2! \cdot 2! \cdot 2!}$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{2 \cdot 1 \cdot 2 \cdot 1}$$

$$= 4,989,600$$

35. $a^2b^3c^4 = a \cdot a \cdot b \cdot b \cdot b \cdot c \cdot c \cdot c \cdot c$

There are 2 a's, 3 b's, and 4 c's, for a total of 9. We have

$$\frac{9!}{2! \cdot 3! \cdot 4!}$$

$$= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 4!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{2 \cdot 3 \cdot 2} = 1260.$$

36. a) ${}_4P_4 = 4! = 24$

b) There are 4 choices for the first coin and 2 possibilities (head or tail) for each choice. This results in a total of 8 choices for the first selection.

Likewise there are 6 choices for the second selection, 4 for the third, and 2 for the fourth. Then the number of ways in which the coins can be lined up is $8 \cdot 6 \cdot 4 \cdot 2$, or 384.

37. a) ${}_6P_5 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$

b) $6^5 = 7776$

c) The first letter can only be D. The other four letters are chosen from A, B, C, E, F without repetition. We have

$$1 \cdot {}_5P_4 = 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 120.$$

d) The first letter can only be D. The second letter can only be E. The other three letters are chosen from A, B, C, F without repetition. We have

$$1 \cdot 1 \cdot {}_4P_3 = 1 \cdot 1 \cdot 4 \cdot 3 \cdot 2 = 24.$$

38. There are 80 choices for the number of the county, 26 choices for the letter of the alphabet, and 9999 choices for the number that follows the letter. By the fundamental counting principle we know there are $80 \cdot 26 \cdot 9999$, or 20,797,920 possible license plates.

39. a) Since repetition is allowed, each of the 5 digits can be chosen in 10 ways. The number of zip-codes possible is $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$, or 100,000.

b) Since there are 100,000 possible zip-codes, there could be 100,000 post offices.

40. $10^9 = 1,000,000,000$

41. a) Since repetition is allowed, each digit can be chosen in 10 ways. There can be $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$, or 1,000,000,000 social security numbers.

b) Since more than 303 million social security numbers are possible, each person can have a social security number.

42.

$$\begin{aligned} 4x - 9 &= 0 \\ 4x &= 9 \\ x &= \frac{9}{4}, \text{ or } 2.25 \end{aligned}$$

The solution is $\frac{9}{4}$, or 2.25.

43.

$$\begin{aligned} x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \\ x + 3 = 0 \quad &or \quad x - 2 = 0 \\ x = -3 \quad &or \quad x = 2 \end{aligned}$$

The solutions are -3 and 2.

44. $2x^2 - 3x - 1 = 0$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2}$$

$$= \frac{3 \pm \sqrt{17}}{4}$$

The solutions are $\frac{3+\sqrt{17}}{4}$ and $\frac{3-\sqrt{17}}{4}$, or $\frac{3 \pm \sqrt{17}}{4}$.

45. $f(x) = x^3 - 4x^2 - 7x + 10$

We use synthetic division to find one factor of the polynomial. We try $x - 1$.

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -7 & 10 \\ & & 1 & -3 & -10 \\ \hline & 1 & -3 & -10 & 0 \end{array}$$

$$\begin{aligned} x^3 - 4x^2 - 7x + 10 &= 0 \\ (x-1)(x^2 - 3x - 10) &= 0 \\ (x-1)(x-5)(x+2) &= 0 \end{aligned}$$

$$x - 1 = 0 \quad or \quad x - 5 = 0 \quad or \quad x + 2 = 0$$

$$x = 1 \quad or \quad x = 5 \quad or \quad x = -2$$

The solutions are -2, 1, and 5.

46.
$$
\begin{aligned}
{}_nP_5 &= 7 \cdot {}_nP_4\\
\frac{n!}{(n-5)!} &= 7 \cdot \frac{n!}{(n-4)!}\\
\frac{n!}{7(n-5)!} &= \frac{n!}{(n-4)!}\\
7(n-5)! &= (n-4)! \quad \text{The denominators must be the same.}\\
7(n-5)! &= (n-4)(n-5)!\\
7 &= n-4\\
11 &= n
\end{aligned}
$$

47.
$$
\begin{aligned}
{}_nP_4 &= 8 \cdot {}_{n-1}P_3\\
\frac{n!}{(n-4)!} &= 8 \cdot \frac{(n-1)!}{(n-1-3)!}\\
\frac{n!}{(n-4)!} &= 8 \cdot \frac{(n-1)!}{(n-4)!}\\
n! &= 8 \cdot (n-1)! \quad \text{Multiplying by } (n-4)!\\
n(n-1)! &= 8 \cdot (n-1)!\\
n &= 8 \quad \text{Dividing by } (n-1)!
\end{aligned}
$$

48.
$$
\begin{aligned}
{}_nP_5 &= 9 \cdot {}_{n-1}P_4\\
\frac{n!}{(n-5)!} &= 9 \cdot \frac{(n-1)!}{(n-1-4)!}\\
\frac{n!}{(n-5)!} &= 9 \cdot \frac{(n-1)!}{(n-5)!}\\
n! &= 9(n-1)!\\
n(n-1)! &= 9(n-1)!\\
n &= 9
\end{aligned}
$$

49.
$$
\begin{aligned}
{}_nP_4 &= 8 \cdot {}_nP_3\\
\frac{n!}{(n-4)!} &= 8 \cdot \frac{n!}{(n-3)!}\\
(n-3)! &= 8(n-4)! \quad \text{Multiplying by } \frac{(n-4)!(n-3)!}{n!}\\
(n-3)(n-4)! &= 8(n-4)!\\
n-3 &= 8 \quad \text{Dividing by } (n-4)!\\
n &= 11
\end{aligned}
$$

50. $n! = n(n-1)(n-2)(n-3)(n-4)\cdots 1 =$
$n(n-1)(n-2)[(n-3)(n-4)\cdots 1] =$
$n(n-1)(n-2)(n-3)!$

51. There is one losing team per game. In order to leave one tournament winner there must be $n-1$ losers produced in $n-1$ games.

52. 2 losses for each of $(n-1)$ losing teams means $2n-2$ losses. The tournament winner will have lost at most 1 game; thus at most there are $(2n-2)+1$ or $(2n-1)$ losses requiring $2n-1$ games.

Exercise Set 11.6

1.
$$
\begin{aligned}
{}_{13}C_2 &= \frac{13!}{2!(13-2)!}\\
&= \frac{13!}{2!11!} = \frac{13 \cdot 12 \cdot 11!}{2 \cdot 1 \cdot 11!}\\
&= \frac{13 \cdot 12}{2 \cdot 1} = \frac{13 \cdot 6 \cdot 2}{2 \cdot 1}\\
&= 78
\end{aligned}
$$

2.
$$
\begin{aligned}
{}_9C_6 &= \frac{9!}{6!(9-6)!}\\
&= \frac{9!}{6!3!} = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6! \cdot 3 \cdot 2 \cdot 1}\\
&= 84
\end{aligned}
$$

3.
$$
\begin{aligned}
\binom{13}{11} &= \frac{13!}{11!(13-11)!}\\
&= \frac{13!}{11!2!}\\
&= 78 \quad \text{(See Exercise 1.)}
\end{aligned}
$$

4.
$$
\begin{aligned}
\binom{9}{3} &= \frac{9!}{3!(9-3)!}\\
&= \frac{9!}{3!6!}\\
&= 84 \quad \text{(See Exercise 2.)}
\end{aligned}
$$

5.
$$
\begin{aligned}
\binom{7}{1} &= \frac{7!}{1!(7-1)!}\\
&= \frac{7!}{1!6!} = \frac{7 \cdot 6!}{1 \cdot 6!}\\
&= 7
\end{aligned}
$$

6.
$$
\begin{aligned}
\binom{8}{8} &= \frac{8!}{8!(8-8)!}\\
&= \frac{8!}{8!0!} = \frac{8!}{8! \cdot 1}\\
&= 1
\end{aligned}
$$

7.
$$
\begin{aligned}
\frac{{}_5P_3}{3!} &= \frac{5 \cdot 4 \cdot 3}{3!}\\
&= \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = \frac{5 \cdot 2 \cdot 2 \cdot 3}{3 \cdot 2 \cdot 1}\\
&= 5 \cdot 2 = 10
\end{aligned}
$$

8. $\frac{{}_{10}P_5}{5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 252$

9.
$$
\begin{aligned}
\binom{6}{0} &= \frac{6!}{0!(6-0)!}\\
&= \frac{6!}{0!6!} = \frac{6!}{6! \cdot 1}\\
&= 1
\end{aligned}
$$

10. $\binom{6}{1} = \frac{6}{1} = 6$

11. $\binom{6}{2} = \frac{6 \cdot 5}{2 \cdot 1} = 15$

12. $\binom{6}{3} = \frac{6!}{3!(6-3)!}$
$= \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3 \cdot 2 \cdot 1}$
$= 20$

13. $\binom{n}{r} = \binom{n}{n-r}$, so

$$\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}$$
$$= 2\left[\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3}\right]$$
$$= 2\left[\frac{7!}{7!0!} + \frac{7!}{6!1!} + \frac{7!}{5!2!} + \frac{7!}{4!3!}\right]$$
$$= 2(1 + 7 + 21 + 35) = 2 \cdot 64 = 128$$

14. $$\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$$
$$= 2\left[\binom{6}{0} + \binom{6}{1} + \binom{6}{2}\right] + \binom{6}{3}$$
$$= 2(1 + 6 + 15) + 20 = 64$$

15. We will use form (1).

$${}_{52}C_4 = \frac{52!}{4!(52-4)!}$$
$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 48!}$$
$$= \frac{52 \cdot 51 \cdot 50 \cdot 49}{4 \cdot 3 \cdot 2 \cdot 1}$$
$$= 270{,}725$$

16. We will use form (1).

$${}_{52}C_5 = \frac{52!}{5!(52-5)!}$$
$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 47!}$$
$$= 2{,}598{,}960$$

17. We will use form (2).

$$\binom{27}{11}$$
$$= \frac{27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdot 17}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$
$$= 13{,}037{,}895$$

18. We will use form (2).

$$\binom{37}{8}$$
$$= \frac{37 \cdot 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 \cdot 30}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$
$$= 38{,}608{,}020$$

19. $\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{1!(n-1)!} = n$

20. $\binom{n}{3} = \frac{n!}{3!(n-3)!} =$
$\frac{n(n-1)(n-2)(n-3)!}{3 \cdot 2 \cdot 1 \cdot (n-3)!} = \frac{n(n-1)(n-2)}{6}$

21. $\binom{m}{m} = \frac{m!}{m!(m-m)!} = \frac{m!}{m!0!} = 1$

22. $\binom{t}{4} = \frac{t!}{4!(t-4)!} = \frac{t(t-1)(t-2)(t-3)(t-4)!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot (t-4)!} =$
$\frac{t(t-1)(t-2)(t-3)}{12}$

23. $${}_{23}C_4 = \frac{23!}{4!(23-4)!}$$
$$= \frac{23!}{4!19!} = \frac{23 \cdot 22 \cdot 21 \cdot 20 \cdot 19!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 19!}$$
$$= \frac{23 \cdot 22 \cdot 21 \cdot 20}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{23 \cdot 2 \cdot 11 \cdot 3 \cdot 7 \cdot 4 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1}$$
$$= 8855$$

24. Playing all other teams once: ${}_9C_2 = 36$

Playing all other teams twice: $2 \cdot_9 C_2 = 72$

25. $${}_{13}C_{10} = \frac{13!}{10!(13-10)!}$$
$$= \frac{13!}{10!3!} = \frac{13 \cdot 12 \cdot 11 \cdot 10!}{10! \cdot 3 \cdot 2 \cdot 1}$$
$$= \frac{13 \cdot 12 \cdot 11}{3 \cdot 2 \cdot 1} = \frac{13 \cdot 3 \cdot 2 \cdot 2 \cdot 11}{3 \cdot 2 \cdot 1}$$
$$= 286$$

26. Using the fundamental counting principle, we have ${}_{58}C_6 \cdot_{42} C_4$.

27. ${}_{10}C_7 \cdot_5 C_3 = \binom{10}{7} \cdot \binom{5}{3}$ Using the fundamental counting principle

$$= \frac{10!}{7!(10-7)!} \cdot \frac{5!}{3!(5-3)!}$$
$$= \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7! \cdot 3!} \cdot \frac{5 \cdot 4 \cdot 3!}{3! \cdot 2!}$$
$$= \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \cdot \frac{5 \cdot 4}{2 \cdot 1} = 120 \cdot 10 = 1200$$

28. Since two points determine a line and no three of these 8 points are colinear, we need to find the number of combinations of 8 points taken 2 at a time, ${}_8C_2$.

$${}_8C_2 = \binom{8}{2} = \frac{8!}{2!(8-2)!}$$
$$= \frac{8 \cdot 7 \cdot 6!}{2 \cdot 1 \cdot 6!} = \frac{4 \cdot 2 \cdot 7}{2 \cdot 1}$$
$$= 28$$

Thus 28 lines are determined.

Since three noncolinear points determine a triangle, we need to find the number of combinations of 8 points taken 3 at a time, ${}_8C_3$.

$$_8C_3 = \binom{8}{3} = \frac{8!}{3!(8-3)!}$$
$$= \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2 \cdot 1 \cdot 5!} = \frac{8 \cdot 7 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1}$$
$$= 56$$

Thus 56 triangles are determined.

29. $_{52}C_5 = 2,598,960$

30. $_{52}C_{13} = 635,013,559,600$

31. a) $_{31}P_2 = 930$

b) $31 \cdot 31 = 961$

c) $_{31}C_2 = 465$

32.
$$3x - 7 = 5x + 10$$
$$-17 = 2x$$
$$-\frac{17}{2} = x$$

The solution is $-\frac{17}{2}$.

33.
$$2x^2 - x = 3$$
$$2x^2 - x - 3 = 0$$
$$(2x-3)(x+1) = 0$$
$$2x - 3 = 0 \quad or \quad x + 1 = 0$$
$$2x = 3 \quad or \quad x = -1$$
$$x = \frac{3}{2} \quad or \quad x = -1$$

The solutions are $\frac{3}{2}$ and -1.

34. $x^2 + 5x + 1 = 0$
$$x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$
$$= \frac{-5 \pm \sqrt{21}}{2}$$

The solutions are $\frac{-5+\sqrt{21}}{2}$ and $\frac{-5-\sqrt{21}}{2}$, or $\frac{-5 \pm \sqrt{21}}{2}$.

35.
$$x^3 + 3x^2 - 10x = 24$$
$$x^3 + 3x^2 - 10x - 24 = 0$$

We use synthetic division to find one factor of the polynomial on the left side of the equation. We try $x - 3$.

$$\begin{array}{r|rrrr} 3 & 1 & 3 & -10 & -24 \\ & & 3 & 18 & 24 \\ \hline & 1 & 6 & 8 & 0 \end{array}$$

Now we have:
$$(x-3)(x^2 + 6x + 8) = 0$$
$$(x-3)(x+2)(x+4) = 0$$
$$x - 3 = 0 \quad or \quad x + 2 = 0 \quad or \quad x + 4 = 0$$
$$x = 3 \quad or \quad x = -2 \quad or \quad x = -4$$

The solutions are -4, -2, and 3.

36. $_4C_3 \cdot {}_4C_2 = 24$

37. There are 13 diamonds, and we choose 5. We have $_{13}C_5 = 1287$.

38. $_nC_4$

39. Playing once: $_nC_2$

Playing twice: $2 \cdot {}_nC_2$

40.
$$\binom{n+1}{3} = 2 \cdot \binom{n}{2}$$
$$\frac{(n+1)!}{(n+1-3)!3!} = 2 \cdot \frac{n!}{(n-2)!2!}$$
$$\frac{(n+1)!}{(n-2)!3!} = 2 \cdot \frac{n!}{(n-2)!2!}$$
$$\frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!3 \cdot 2 \cdot 1} = 2 \cdot \frac{n(n-1)(n-2)!}{(n-2)! \cdot 2 \cdot 1}$$
$$\frac{(n+1)(n)(n-1)}{6} = n(n-1)$$
$$\frac{n^3 - n}{6} = n^2 - n$$
$$n^3 - n = 6n^2 - 6n$$
$$n^3 - 6n^2 + 5n = 0$$
$$n(n^2 - 6n + 5) = 0$$
$$n(n-5)(n-1) = 0$$
$$n = 0 \text{ or } n = 5 \text{ or } n = 1$$

Only 5 checks. The solution is 5.

41.
$$\binom{n}{n-2} = 6$$
$$\frac{n!}{(n-(n-2))!(n-2)!} = 6$$
$$\frac{n!}{2!(n-2)!} = 6$$
$$\frac{n(n-1)(n-2)!}{2 \cdot 1 \cdot (n-2)!} = 6$$
$$\frac{n(n-1)}{2} = 6$$
$$n(n-1) = 12$$
$$n^2 - n = 12$$
$$n^2 - n - 12 = 0$$
$$(n-4)(n+3) = 0$$
$$n = 4 \text{ or } n = -3$$

Only 4 checks. The solution is 4.

42. $$\binom{n}{3} = 2\cdot\binom{n-1}{2}$$
$$\frac{n!}{(n-3)!3!} = 2\cdot\frac{(n-1)!}{(n-1-2)!2!}$$
$$\frac{n!}{(n-3)!3!} = 2\cdot\frac{(n-1)!}{(n-3)!2!}$$
$$\frac{n!}{3!} = 2\cdot\frac{(n-1)!}{2!}$$
$$n! = 3!(n-1)!$$
$$\frac{n(n-1)!}{(n-1)!} = 6$$
$$n = 6$$

This number checks. The solution is 6.

43. $$\binom{n+2}{4} = 6\cdot\binom{n}{2}$$
$$\frac{(n+2)!}{(n+2-4)!4!} = 6\cdot\frac{n!}{(n-2)!2!}$$
$$\frac{(n+2)!}{(n-2)!4!} = 6\cdot\frac{n!}{(n-2)!2!}$$
$$\frac{(n+2)!}{4!} = 6\cdot\frac{n!}{2!} \quad \text{Multiplying by } (n–2)!$$
$$4!\cdot\frac{(n+2)!}{4!} = 4!\cdot 6\cdot\frac{n!}{2!}$$
$$(n+2)! = 72\cdot n!$$
$$(n+2)(n+1)n! = 72\cdot n!$$
$$(n+2)(n+1) = 72 \quad \text{Dividing by } n!$$
$$n^2+3n+2 = 72$$
$$n^2+3n-70 = 0$$
$$(n+10)(n-7) = 0$$
$$n = -10 \quad\text{or}\quad n = 7$$

Only 7 checks. The solution is 7.

44. $$\binom{n}{k-1} + \binom{n}{k}$$
$$= \frac{n!}{(k-1)!(n-k+1)!}\cdot\frac{k}{k} + \frac{n!}{k!(n-k)!}\cdot\frac{(n-k+1)}{(n-k+1)}$$
$$= \frac{n!(k+(n-k+1))}{k!(n-k+1)!}$$
$$= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$$

45. Line segments: ${}_nC_2 = \dfrac{n!}{2!(n-2)!} = \dfrac{n(n-1)(n-2)!}{2\cdot 1\cdot (n-2)!} = \dfrac{n(n-1)}{2}$

Diagonals: The n line segments that form the sides of the n-agon are not diagonals. Thus, the number of diagonals is ${}_nC_2 - n = \dfrac{n(n-1)}{2} - n = \dfrac{n^2-n-2n}{2} = \dfrac{n^2-3n}{2} = \dfrac{n(n-3)}{2}$, $n\geq 4$.

Let D_n be the number of diagonals on an n-agon. Prove the result above for diagonals using mathematical induction.

$S_n:\quad D_n = \dfrac{n(n-3)}{2}$, for $n = 4, 5, 6, \ldots$

$S_4:\quad D_4 = \dfrac{4\cdot 1}{2}$

$S_k:\quad D_k = \dfrac{k(k-3)}{2}$

$S_{k+1}:\quad D_{k+1} = \dfrac{(k+1)(k-2)}{2}$

1) *Basis step:* S_4 is true (a quadrilateral has 2 diagonals).

2) *Induction step:* Assume S_k. Observe that when an additional vertex V_{k+1} is added to the k-gon, we gain k segments, 2 of which are sides of the $(k+1)$-gon, and a former side $\overline{V_1V_k}$ becomes a diagonal. Thus the additional number of diagonals is $k-2+1$, or $k-1$. Then the new total of diagonals is $D_k + (k-1)$, or

$$\begin{aligned} D_{k+1} &= D_k + (k-1) \\ &= \frac{k(k-3)}{2} + (k-1) \quad (\text{by } S_k) \\ &= \frac{(k+1)(k-2)}{2} \end{aligned}$$

Exercise Set 11.7

1. Expand: $(x+5)^4$.

We have $a = x$, $b = 5$, and $n = 4$.

Pascal's triangle method: Use the fifth row of Pascal's triangle.

1 4 6 4 1

$$\begin{aligned} &(x+5)^4 \\ &= 1\cdot x^4 + 4\cdot x^3\cdot 5 + 6\cdot x^2\cdot 5^2 + \\ &\quad 4\cdot x\cdot 5^3 + 1\cdot 5^4 \\ &= x^4 + 20x^3 + 150x^2 + 500x + 625 \end{aligned}$$

Factorial notation method:

$$\begin{aligned} &(x+5)^4 \\ &= \binom{4}{0}x^4 + \binom{4}{1}x^3\cdot 5 + \binom{4}{2}x^2\cdot 5^2 + \\ &\quad \binom{4}{3}x\cdot 5^3 + \binom{4}{4}5^4 \\ &= \frac{4!}{0!4!}x^4 + \frac{4!}{1!3!}x^3\cdot 5 + \frac{4!}{2!2!}x^2\cdot 5^2 + \\ &\quad \frac{4!}{3!1!}x\cdot 5^3 + \frac{4!}{4!0!}5^4 \\ &= x^4 + 20x^3 + 150x^2 + 500x + 625 \end{aligned}$$

2. Expand: $(x-1)^4$.

Pascal's triangle method: Use the 5th row of Pascal's triangle.

1 4 6 4 1

$(x-1)^4$

$= 1\cdot x^4 + 4\cdot x^3(-1) + 6x^2(-1)^2 + 4x(-1)^3 +$
$1\cdot(-1)^4$

$= x^4 - 4x^3 + 6x^2 - 4x + 1$

Factorial notation method:

$(x-1)^4$

$= \binom{4}{0}x^4 + \binom{4}{1}x^3(-1) + \binom{4}{2}x^2(-1)^2 +$

$\binom{4}{3}x(-1)^3 + \binom{4}{4}(-1)^4$

$= x^4 - 4x^3 + 6x^2 - 4x + 1$

3. Expand: $(x-3)^5$.

We have $a = x$, $b = -3$, and $n = 5$.

Pascal's triangle method: Use the sixth row of Pascal's triangle.

1 5 10 10 5 1

$(x-3)^5$

$= 1\cdot x^5 + 5x^4(-3) + 10x^3(-3)^2 + 10x^2(-3)^3 +$
$5x(-3)^4 + 1\cdot(-3)^5$

$= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

Factorial notation method:

$(x-3)^5$

$= \binom{5}{0}x^5 + \binom{5}{1}x^4(-3) + \binom{5}{2}x^3(-3)^2 +$

$\binom{5}{3}x^2(-3)^3 + \binom{5}{4}x(-3)^4 + \binom{5}{5}(-3)^5$

$= \frac{5!}{0!5!}x^5 + \frac{5!}{1!4!}x^4(-3) + \frac{5!}{2!3!}x^3(9) +$

$\frac{5!}{3!2!}x^2(-27) + \frac{5!}{4!1!}x(81) + \frac{5!}{5!0!}(-243)$

$= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

4. Expand: $(x+2)^9$.

Pascal's triangle method: Use the 10th row of Pascal's triangle.

1 9 36 84 126 126 84 36 9 1

$(x+2)^9$

$= 1\cdot x^9 + 9x^8\cdot 2 + 36x^7\cdot 2^2 + 84x^6\cdot 2^3 +$
$126x^5\cdot 2^4 + 126x^4\cdot 2^5 + 84x^3\cdot 2^6 + 36x^2\cdot 2^7 +$
$9x\cdot 2^8 + 1\cdot 2^9$

$= x^9 + 18x^8 + 144x^7 + 672x^6 + 2016x^5 + 4032x^4 +$
$5376x^3 + 4608x^2 + 2304x + 512$

Factorial notation method:

$(x+2)^9$

$= \binom{9}{0}x^9 + \binom{9}{1}x^8\cdot 2 + \binom{9}{2}x^7\cdot 2^2 +$

$\binom{9}{3}x^6\cdot 2^3 + \binom{9}{4}x^5\cdot 2^4 + \binom{9}{5}x^4\cdot 2^5 +$

$\binom{9}{6}x^3\cdot 2^6 + \binom{9}{7}x^2\cdot 2^7 + \binom{9}{8}x\cdot 2^8 +$

$\binom{9}{9}2^9$

$= x^9 + 18x^8 + 144x^7 + 672x^6 + 2016x^5 + 4032x^4 +$
$5376x^3 + 4608x^2 + 2304x + 512$

5. Expand: $(x-y)^5$.

We have $a = x$, $b = -y$, and $n = 5$.

Pascal's triangle method: We use the sixth row of Pascal's triangle.

1 5 10 10 5 1

$(x-y)^5$

$= 1\cdot x^5 + 5x^4(-y) + 10x^3(-y)^2 + 10x^2(-y)^3 +$
$5x(-y)^4 + 1\cdot(-y)^5$

$= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

Factorial notation method:

$(x-y)^5$

$= \binom{5}{0}x^5 + \binom{5}{1}x^4(-y) + \binom{5}{2}x^3(-y)^2 +$

$\binom{5}{3}x^2(-y)^3 + \binom{5}{4}x(-y)^4 + \binom{5}{5}(-y)^5$

$= \frac{5!}{0!5!}x^5 + \frac{5!}{1!4!}x^4(-y) + \frac{5!}{2!3!}x^3(y^2) +$

$\frac{5!}{3!2!}x^2(-y^3) + \frac{5!}{4!1!}x(y^4) + \frac{5!}{5!0!}(-y^5)$

$= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

6. Expand: $(x+y)^8$.

Pascal's triangle method: Use the ninth row of Pascal's triangle.

1 8 28 56 70 56 28 8 1

$(x+y)^8$

$= x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 +$
$56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$

Factorial notation method:

$(x+y)^8$

$= \binom{8}{0}x^8 + \binom{8}{1}x^7y + \binom{8}{2}x^6y^2 +$

$\binom{8}{3}x^5y^3 + \binom{8}{4}x^4y^4 + \binom{8}{5}x^3y^5 +$

$\binom{8}{6}x^2y^6 + \binom{8}{7}xy^7 + \binom{8}{8}y^8$

$= x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 +$
$56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$

7. Expand: $(5x+4y)^6$.

We have $a=5x$, $b=4y$, and $n=6$.

Pascal's triangle method: Use the seventh row of Pascal's triangle.

1 6 15 20 15 6 1

$$\begin{aligned}&(5x+4y)^6\\ &= 1\cdot(5x)^6+6\cdot(5x)^5(4y)+15(5x)^4(4y)^2+\\ &\quad 20(5x)^3(4y)^3+15(5x)^2(4y)^4+6(5x)(4y)^5+\\ &\quad 1\cdot(4y)^6\\ &= 15{,}625x^6+75{,}000x^5y+150{,}000x^4y^2+\\ &\quad 160{,}000x^3y^3+96{,}000x^2y^4+30{,}720xy^5+4096y^6\end{aligned}$$

Factorial notation method:

$$\begin{aligned}&(5x+4y)^6\\ &= \binom{6}{0}(5x)^6+\binom{6}{1}(5x)^5(4y)+\\ &\quad \binom{6}{2}(5x)^4(4y)^2+\binom{6}{3}(5x)^3(4y)^3+\\ &\quad \binom{6}{4}(5x)^2(4y)^4+\binom{6}{5}(5x)(4y)^5+\binom{6}{6}(4y)^6\\ &= \frac{6!}{0!6!}(15{,}625x^6)+\frac{6!}{1!5!}(3125x^5)(4y)+\\ &\quad \frac{6!}{2!4!}(625x^4)(16y^2)+\frac{6!}{3!3!}(125x^3)(64y^3)+\\ &\quad \frac{6!}{4!2!}(25x^2)(256y^4)+\frac{6!}{5!1!}(5x)(1024y^5)+\\ &\quad \frac{6!}{6!0!}(4096y^6)\\ &= 15{,}625x^6+75{,}000x^5y+150{,}000x^4y^2+\\ &\quad 160{,}000x^3y^3+96{,}000x^2y^4+30{,}720xy^5+\\ &\quad 4096y^6\end{aligned}$$

8. Expand: $(2x-3y)^5$.

Pascal's triangle method: Use the sixth row of Pascal's triangle.

1 5 10 10 5 1

$$\begin{aligned}&(2x-3y)^5\\ &= 1\cdot(2x)^5+5(2x)^4(-3y)+10(2x)^3(-3y)^2+\\ &\quad 10(2x)^2(-3y)^3+5(2x)(-3y)^4+1\cdot(-3y)^5\\ &= 32x^5-240x^4y+720x^3y^2-1080x^2y^3+\\ &\quad 810xy^4-243y^5\end{aligned}$$

Factorial notation method:

$$\begin{aligned}&(2x-3y)^5\\ &= \binom{5}{0}(2x)^5+\binom{5}{1}(2x)^4(-3y)+\\ &\quad \binom{5}{2}(2x)^3(-3y)^2+\binom{5}{3}(2x)^2(-3y)^3+\\ &\quad \binom{5}{4}(2x)(-3y)^4+\binom{5}{5}(-3y)^5\\ &= 32x^5-240x^4y+720x^3y^2-1080x^2y^3+\\ &\quad 810xy^4-243y^5\end{aligned}$$

9. Expand: $\left(2t+\dfrac{1}{t}\right)^7$.

We have $a=2t$, $b=\dfrac{1}{t}$, and $n=7$.

Pascal's triangle method: Use the eighth row of Pascal's triangle.

1 7 21 35 35 21 7 1

$$\begin{aligned}&\left(2t+\frac{1}{t}\right)^7\\ &= 1\cdot(2t)^7+7(2t)^6\left(\frac{1}{t}\right)+21(2t)^5\left(\frac{1}{t}\right)^2+\\ &\quad 35(2t)^4\left(\frac{1}{t}\right)^3+35(2t)^3\left(\frac{1}{t}\right)^4+21(2t)^2\left(\frac{1}{t}\right)^5+\\ &\quad 7(2t)\left(\frac{1}{t}\right)^6+1\cdot\left(\frac{1}{t}\right)^7\\ &= 128t^7+7\cdot 64t^6\cdot\frac{1}{t}+21\cdot 32t^5\cdot\frac{1}{t^2}+\\ &\quad 35\cdot 16t^4\cdot\frac{1}{t^3}+35\cdot 8t^3\cdot\frac{1}{t^4}+21\cdot 4t^2\cdot\frac{1}{t^5}+\\ &\quad 7\cdot 2t\cdot\frac{1}{t^6}+\frac{1}{t^7}\\ &= 128t^7+448t^5+672t^3+560t+280t^{-1}+\\ &\quad 84t^{-3}+14t^{-5}+t^{-7}\end{aligned}$$

Factorial notation method:

$$\begin{aligned}&\left(2t+\frac{1}{t}\right)^7\\ &= \binom{7}{0}(2t)^7+\binom{7}{1}(2t)^6\left(\frac{1}{t}\right)+\\ &\quad \binom{7}{2}(2t)^5\left(\frac{1}{t}\right)^2+\binom{7}{3}(2t)^4\left(\frac{1}{t}\right)^3+\\ &\quad \binom{7}{4}(2t)^3\left(\frac{1}{t}\right)^4+\binom{7}{5}(2t)^2\left(\frac{1}{t}\right)^5+\\ &\quad \binom{7}{6}(2t)\left(\frac{1}{t}\right)^6+\binom{7}{7}\left(\frac{1}{t}\right)^7\\ &= \frac{7!}{0!7!}(128t^7)+\frac{7!}{1!6!}(64t^6)\left(\frac{1}{t}\right)+\frac{7!}{2!5!}(32t^5)\left(\frac{1}{t^2}\right)+\\ &\quad \frac{7!}{3!4!}(16t^4)\left(\frac{1}{t^3}\right)+\frac{7!}{4!3!}(8t^3)\left(\frac{1}{t^4}\right)+\\ &\quad \frac{7!}{5!2!}(4t^2)\left(\frac{1}{t^5}\right)+\frac{7!}{6!1!}(2t)\left(\frac{1}{t^6}\right)+\frac{7!}{7!0!}\left(\frac{1}{t^7}\right)\\ &= 128t^7+448t^5+672t^3+560t+280t^{-1}+\\ &\quad 84t^{-3}+14t^{-5}+t^{-7}\end{aligned}$$

10. Expand: $\left(3y-\dfrac{1}{y}\right)^4$.

Pascal's triangle method: Use the fifth row of Pascal's triangle.

1 4 6 4 1

$\left(3y - \frac{1}{y}\right)^4$

$= 1 \cdot (3y)^4 + 4(3y)^3\left(-\frac{1}{y}\right) + 6(3y)^2\left(-\frac{1}{y}\right)^2 +$

$4(3y)\left(-\frac{1}{y}\right)^3 + 1 \cdot \left(-\frac{1}{y}\right)^4$

$= 81y^4 - 108y^2 + 54 - 12y^{-2} + y^{-4}$

Factorial notation method:

$\left(3y - \frac{1}{y}\right)^4$

$= \binom{4}{0}(3y)^4 + \binom{4}{1}(3y)^3\left(-\frac{1}{y}\right) +$

$\binom{4}{2}(3y)^2\left(-\frac{1}{y}\right)^2 + \binom{4}{3}(3y)\left(-\frac{1}{y}\right)^3 +$

$\binom{4}{4}\left(-\frac{1}{y}\right)^4$

$= 81y^4 - 108y^2 + 54 - 12y^{-2} + y^{-4}$

11. Expand: $(x^2 - 1)^5$.

We have $a = x^2$, $b = -1$, and $n = 5$.

Pascal's triangle method: Use the sixth row of Pascal's triangle.

1 5 10 10 5 1

$(x^2 - 1)^5$

$= 1 \cdot (x^2)^5 + 5(x^2)^4(-1) + 10(x^2)^3(-1)^2 +$

$10(x^2)^2(-1)^3 + 5(x^2)(-1)^4 + 1 \cdot (-1)^5$

$= x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1$

Factorial notation method:

$(x^2 - 1)^5$

$= \binom{5}{0}(x^2)^5 + \binom{5}{1}(x^2)^4(-1) +$

$\binom{5}{2}(x^2)^3(-1)^2 + \binom{5}{3}(x^2)^2(-1)^3 +$

$\binom{5}{4}(x^2)(-1)^4 + \binom{5}{5}(-1)^5$

$= \frac{5!}{0!5!}(x^{10}) + \frac{5!}{1!4!}(x^8)(-1) + \frac{5!}{2!3!}(x^6)(1) +$

$\frac{5!}{3!2!}(x^4)(-1) + \frac{5!}{4!1!}(x^2)(1) + \frac{5!}{5!0!}(-1)$

$= x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1$

12. Expand: $(1 + 2q^3)^8$.

Pascal's triangle method: Use the ninth row of Pascal's triangle.

1 8 28 56 70 56 28 8 1

$(1 + 2q^3)^8$

$= 1 \cdot 1^8 + 8 \cdot 1^7(2q^3) + 28 \cdot 1^6(2q^3)^2 + 56 \cdot 1^5(2q^3)^3 +$

$70 \cdot 1^4(2q^3)^4 + 56 \cdot 1^3(2q^3)^5 + 28 \cdot 1^2(2q^3)^6 +$

$8 \cdot 1(2q^3)^7 + 1 \cdot (2q^3)^8$

$= 1 + 16q^3 + 112q^6 + 448q^9 + 1120q^{12} + 1792q^{15} +$

$1792q^{18} + 1024q^{21} + 256q^{24}$

Factorial notation method:

$(1 + 2q^3)^8$

$= \binom{8}{0}(1)^8 + \binom{8}{1}(1)^7(2q^3) + \binom{8}{2}(1)^6(2q^3)^2 +$

$\binom{8}{3}(1)^5(2q^3)^3 + \binom{8}{4}(1)^4(2q^3)^4 +$

$\binom{8}{5}(1)^3(2q^3)^5 + \binom{8}{6}(1)^2(2q^3)^6 +$

$\binom{8}{7}(1)(2q^3)^7 + \binom{8}{8}(2q^3)^8$

$= 1 + 16q^3 + 112q^6 + 448q^9 + 1120q^{12} + 1792q^{15} +$

$1792q^{18} + 1024q^{21} + 256q^{24}$

13. Expand: $(\sqrt{5} + t)^6$.

We have $a = \sqrt{5}$, $b = t$, and $n = 6$.

Pascal's triangle method: We use the seventh row of Pascal's triangle:

1 6 15 20 15 6 1

$(\sqrt{5} + t)^6 = 1 \cdot (\sqrt{5})^6 + 6(\sqrt{5})^5(t) +$

$15(\sqrt{5})^4(t^2) + 20(\sqrt{5})^3(t^3) +$

$15(\sqrt{5})^2(t^4) + 6\sqrt{5}t^5 + 1 \cdot t^6$

$= 125 + 150\sqrt{5}\,t + 375t^2 + 100\sqrt{5}\,t^3 +$

$75t^4 + 6\sqrt{5}\,t^5 + t^6$

Factorial notation method:

$(\sqrt{5} + t)^6 = \binom{6}{0}(\sqrt{5})^6 + \binom{6}{1}(\sqrt{5})^5(t) +$

$\binom{6}{2}(\sqrt{5})^4(t^2) + \binom{6}{3}(\sqrt{5})^3(t^3) +$

$\binom{6}{4}(\sqrt{5})^2(t^4) + \binom{6}{5}(\sqrt{5})(t^5) +$

$\binom{6}{6}(t^6)$

$= \frac{6!}{0!6!}(125) + \frac{6!}{1!5!}(25\sqrt{5})t + \frac{6!}{2!4!}(25)(t^2) +$

$\frac{6!}{3!3!}(5\sqrt{5})(t^3) + \frac{6!}{4!2!}(5)(t^4) +$

$\frac{6!}{5!1!}(\sqrt{5})(t^5) + \frac{6!}{6!0!}(t^6)$

$= 125 + 150\sqrt{5}\,t + 375t^2 + 100\sqrt{5}\,t^3 +$

$75t^4 + 6\sqrt{5}\,t^5 + t^6$

14. Expand: $(x - \sqrt{2})^6$.

Pascal's triangle method: Use the seventh row of Pascal's triangle.

1 6 15 20 15 6 1

$(x - \sqrt{2})^6$

$= 1 \cdot x^6 + 6x^5(-\sqrt{2}) + 15x^4(-\sqrt{2})^2 + 20x^3(-\sqrt{2})^3 +$

$15x^2(-\sqrt{2})^4 + 6x(-\sqrt{2})^5 + 1 \cdot (-\sqrt{2})^6$

$= x^6 - 6\sqrt{2}x^5 + 30x^4 - 40\sqrt{2}x^3 + 60x^2 - 24\sqrt{2}x + 8$

Factorial notation method:

$(x-\sqrt{2})^6$

$= \binom{6}{0}x^6 + \binom{6}{1}(x^5)(-\sqrt{2}) +$
$\binom{6}{2}(x^4)(-\sqrt{2})^2 + \binom{6}{3}(x^3)(-\sqrt{2})^3 +$
$\binom{6}{4}(x^2)(-\sqrt{2})^4 + \binom{6}{5}(x)(-\sqrt{2})^5 +$
$\binom{6}{6}(-\sqrt{2})^6$

$= x^6 - 6\sqrt{2}x^5 + 30x^4 - 40\sqrt{2}x^3 + 60x^2 - 24\sqrt{2}x + 8$

15. Expand: $\left(a - \frac{2}{a}\right)^9$.

We have $a = a$, $b = -\frac{2}{a}$, and $n = 9$.

Pascal's triangle method: Use the tenth row of Pascal's triangle.

1 9 36 84 126 126 84 36 9 1

$\left(a-\frac{2}{a}\right)^9 = 1\cdot a^9 + 9a^8\left(-\frac{2}{a}\right) + 36a^7\left(-\frac{2}{a}\right)^2 +$
$84a^6\left(-\frac{2}{a}\right)^3 + 126a^5\left(-\frac{2}{a}\right)^4 +$
$126a^4\left(-\frac{2}{a}\right)^5 + 84a^3\left(-\frac{2}{a}\right)^6 +$
$36a^2\left(-\frac{2}{a}\right)^7 + 9a\left(-\frac{2}{a}\right)^8 + 1\cdot\left(-\frac{2}{a}\right)^9$
$= a^9 - 18a^7 + 144a^5 - 672a^3 + 2016a -$
$4032a^{-1} + 5376a^{-3} - 4608a^{-5} +$
$2304a^{-7} - 512a^{-9}$

Factorial notation method:

$\left(a-\frac{2}{a}\right)^9$

$= \binom{9}{0}a^9 + \binom{9}{1}a^8\left(-\frac{2}{a}\right) + \binom{9}{2}a^7\left(-\frac{2}{a}\right)^2 +$
$\binom{9}{3}a^6\left(-\frac{2}{a}\right)^3 + \binom{9}{4}a^5\left(-\frac{2}{a}\right)^4 +$
$\binom{9}{5}a^4\left(-\frac{2}{a}\right)^5 + \binom{9}{6}a^3\left(-\frac{2}{a}\right)^6 +$
$\binom{9}{7}a^2\left(-\frac{2}{a}\right)^7 + \binom{9}{8}a\left(-\frac{2}{a}\right)^8 +$
$\binom{9}{9}\left(-\frac{2}{a}\right)^9$

$= \frac{9!}{9!0!}a^9 + \frac{9!}{8!1!}a^8\left(-\frac{2}{a}\right) + \frac{9!}{7!2!}a^7\left(\frac{4}{a^2}\right) +$
$\frac{9!}{6!3!}a^6\left(-\frac{8}{a^3}\right) + \frac{9!}{5!4!}a^5\left(\frac{16}{a^4}\right) +$
$\frac{9!}{4!5!}a^4\left(-\frac{32}{a^5}\right) + \frac{9!}{3!6!}a^3\left(\frac{64}{a^6}\right) +$
$\frac{9!}{2!7!}a^2\left(-\frac{128}{a^7}\right) + \frac{9!}{1!8!}a\left(\frac{256}{a^8}\right) +$
$\frac{9!}{0!9!}\left(-\frac{512}{a^9}\right)$

$= a^9 - 9(2a^7) + 36(4a^5) - 84(8a^3) + 126(16a) -$
$126(32a^{-1}) + 84(64a^{-3}) - 36(128a^{-5}) +$
$9(256a^{-7}) - 512a^{-9}$
$= a^9 - 18a^7 + 144a^5 - 672a^3 + 2016a - 4032a^{-1} +$
$5376a^{-3} - 4608a^{-5} + 2304a^{-7} - 512a^{-9}$

16. Expand: $(1+3)^n$

Use the factorial notation method.

$(1+3)^n$

$= \binom{n}{0}(1)^n + \binom{n}{1}(1)^{n-1}3 + \binom{n}{2}(1)^{n-2}3^2 +$
$\binom{n}{3}(1)^{n-3}3^3 + \cdots + \binom{n}{n-2}(1)^2 3^{n-2} +$
$\binom{n}{n-1}(1)3^{n-1} + \binom{n}{n}3^n$
$= 1 + 3n + \binom{n}{2}3^2 + \binom{n}{3}3^3 + \cdots +$
$\binom{n}{n-2}3^{n-2} + 3^{n-1}n + 3^n$

17. $(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6$

First, expand $(\sqrt{2}+1)^6$.

$(\sqrt{2}+1)^6 = \binom{6}{0}(\sqrt{2})^6 + \binom{6}{1}(\sqrt{2})^5(1) +$
$\binom{6}{2}(\sqrt{2})^4(1)^2 + \binom{6}{3}(\sqrt{2})^3(1)^3 +$
$\binom{6}{4}(\sqrt{2})^2(1)^4 + \binom{6}{5}(\sqrt{2})(1)^5 +$
$\binom{6}{6}(1)^6$
$= \frac{6!}{6!0!}\cdot 8 + \frac{6!}{5!1!}\cdot 4\sqrt{2} + \frac{6!}{4!2!}\cdot 4 +$
$\frac{6!}{3!3!}\cdot 2\sqrt{2} + \frac{6!}{2!4!}\cdot 2 + \frac{6!}{1!5!}\cdot\sqrt{2} + \frac{6!}{0!6!}$
$= 8 + 24\sqrt{2} + 60 + 40\sqrt{2} + 30 + 6\sqrt{2} + 1$
$= 99 + 70\sqrt{2}$

Next, expand $(\sqrt{2}-1)^6$.

$(\sqrt{2}-1)^6$

$= \binom{6}{0}(\sqrt{2})^6 + \binom{6}{1}(\sqrt{2})^5(-1) +$
$\binom{6}{2}(\sqrt{2})^4(-1)^2 + \binom{6}{3}(\sqrt{2})^3(-1)^3 +$
$\binom{6}{4}(\sqrt{2})^2(-1)^4 + \binom{6}{5}(\sqrt{2})(-1)^5 +$
$\binom{6}{6}(-1)^6$
$= \frac{6!}{6!0!}\cdot 8 - \frac{6!}{5!1!}\cdot 4\sqrt{2} + \frac{6!}{4!2!}\cdot 4 - \frac{6!}{3!3!}\cdot 2\sqrt{2} +$
$\frac{6!}{2!4!}\cdot 2 - \frac{6!}{1!5!}\cdot\sqrt{2} + \frac{6!}{0!6!}$
$= 8 - 24\sqrt{2} + 60 - 40\sqrt{2} + 30 - 6\sqrt{2} + 1$
$= 99 - 70\sqrt{2}$

$$(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$$
$$=(99+70\sqrt{2})-(99-70\sqrt{2})$$
$$=99+70\sqrt{2}-99+70\sqrt{2}$$
$$=140\sqrt{2}$$

18. $(1-\sqrt{2})^4=1\cdot 1^4+4\cdot 1^3(-\sqrt{2})+6\cdot 1^2(-\sqrt{2})^2+$
$4\cdot 1\cdot(-\sqrt{2})^3+1\cdot(-\sqrt{2})^4$
$=1-4\sqrt{2}+12-8\sqrt{2}+4$
$=17-12\sqrt{2}$

$(1+\sqrt{2})^4=1+4\sqrt{2}+12+8\sqrt{2}+4$ Using the result above
$=17+12\sqrt{2}$

$(1-\sqrt{2})^4+(1+\sqrt{2})^4=17-12\sqrt{2}+17+12\sqrt{2}=34$

19. Expand: $(x^{-2}+x^2)^4$.

We have $a=x^{-2}$, $b=x^2$, and $n=4$.

Pascal's triangle method: Use the fifth row of Pascal's triangle.

1 4 6 4 1.

$$(x^{-2}+x^2)^4$$
$$=1\cdot(x^{-2})^4+4(x^{-2})^3(x^2)+6(x^{-2})^2(x^2)^2+4(x^{-2})(x^2)^3+1\cdot(x^2)^4$$
$$=x^{-8}+4x^{-4}+6+4x^4+x^8$$

Factorial notation method:

$$(x^{-2}+x^2)^4$$
$$=\binom{4}{0}(x^{-2})^4+\binom{4}{1}(x^{-2})^3(x^2)+\binom{4}{2}(x^{-2})^2(x^2)^2+\binom{4}{3}(x^{-2})(x^2)^3+\binom{4}{4}(x^2)^4$$
$$=\frac{4!}{4!0!}(x^{-8})+\frac{4!}{3!1!}(x^{-6})(x^2)+\frac{4!}{2!2!}(x^{-4})(x^4)+\frac{4!}{1!3!}(x^{-2})(x^6)+\frac{4!}{0!4!}(x^8)$$
$$=x^{-8}+4x^{-4}+6+4x^4+x^8$$

20. Expand: $\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)^6$.

Pascal's triangle method: We use the seventh row of Pascal's triangle:

1 6 15 20 15 6 1

$$\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)^6$$
$$=1\cdot\left(\frac{1}{\sqrt{x}}\right)^6+6\left(\frac{1}{\sqrt{x}}\right)^5(-\sqrt{x})+15\left(\frac{1}{\sqrt{x}}\right)^4(-\sqrt{x})^2+20\left(\frac{1}{\sqrt{x}}\right)^3(-\sqrt{x})^3+15\left(\frac{1}{\sqrt{x}}\right)^2(-\sqrt{x})^4+6\left(\frac{1}{\sqrt{x}}\right)(-\sqrt{x})^5+1\cdot(\sqrt{x})^6$$
$$=x^{-3}-6x^{-2}+15x^{-1}-20+15x-6x^2+x^3$$

Factorial notation method:

$$\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)^6$$
$$=\binom{6}{0}\left(\frac{1}{\sqrt{x}}\right)^6+\binom{6}{1}\left(\frac{1}{\sqrt{x}}\right)^5(-\sqrt{x})+\binom{6}{2}\left(\frac{1}{\sqrt{x}}\right)^4(-\sqrt{x})^2+\binom{6}{3}\left(\frac{1}{\sqrt{x}}\right)^3(-\sqrt{x})^3+\binom{6}{4}\left(\frac{1}{\sqrt{x}}\right)^2(-\sqrt{x})^4+\binom{6}{5}\left(\frac{1}{\sqrt{x}}\right)(-\sqrt{x})^5+\binom{6}{6}(-\sqrt{x})^6$$
$$=x^{-3}-6x^{-2}+15x^{-1}-20+15x-6x^2+x^3$$

21. Find the 3rd term of $(a+b)^7$.

First, we note that $3=2+1$, $a=a$, $b=b$, and $n=7$. Then the 3rd term of the expansion of $(a+b)^7$ is

$\binom{7}{2}a^{7-2}b^2$, or $\frac{7!}{2!5!}a^5b^2$, or $21a^5b^2$.

22. $\binom{8}{5}x^3y^5=56x^3y^5$

23. Find the 6th term of $(x-y)^{10}$.

First, we note that $6=5+1$, $a=x$, $b=-y$, and $n=10$. Then the 6th term of the expansion of $(x-y)^{10}$ is

$\binom{10}{5}x^5(-y)^5$, or $-252x^5y^5$.

24. $\binom{9}{4}p^5(-2q)^4=2016p^5q^4$

25. Find the 12th term of $(a-2)^{14}$.

First, we note that $12=11+1$, $a=a$, $b=-2$, and $n=14$. Then the 12th term of the expansion of $(a-2)^{14}$ is

$$\binom{14}{11}a^{14-11}\cdot(-2)^{11}=\frac{14!}{3!11!}a^3(-2048)$$
$$=364a^3(-2048)$$
$$=-745,472a^3$$

26. $\binom{12}{10}x^{12-10}(-3)^{10}=3,897,234x^2$

27. Find the 5th term of $(2x^3-\sqrt{y})^8$.

First, we note that $5=4+1$, $a=2x^3$, $b=-\sqrt{y}$, and $n=8$. Then the 5th term of the expansion of $(2x^3-\sqrt{y})^8$ is

$$\binom{8}{4}(2x^3)^{8-4}(-\sqrt{y})^4$$
$$=\frac{8!}{4!4!}(2x^3)^4(-\sqrt{y})^4$$
$$=70(16x^{12})(y^2)$$
$$=1120x^{12}y^2$$

28. $\binom{7}{3}\left(\frac{1}{b^2}\right)^{7-3}\left(\frac{b}{3}\right)^3=\frac{35}{27}b^{-5}$

29. The expansion of $(2u-3v^2)^{10}$ has 11 terms so the 6th term is the middle term. Note that $6=5+1$, $a=2u$, $b=-3v^2$, and $n=10$. Then the 6th term of the expansion of $(2u-3v^2)^{10}$ is

$$\binom{10}{5}(2u)^{10-5}(-3v^2)^5$$
$$=\frac{10!}{5!5!}(2u)^5(-3v^2)^5$$
$$=252(32u^5)(-243v^{10})$$
$$=-1,959,552u^5v^{10}$$

30. 3rd term: $\binom{5}{2}(\sqrt{x})^{5-2}(\sqrt{3})^2=30x\sqrt{x}$

4th term: $\binom{5}{3}(\sqrt{x})^{5-3}(\sqrt{3})^3=30x\sqrt{3}$

31. The number of subsets is 2^7, or 128

32. 2^6, or 64

33. The number of subsets is 2^{24}, or 16,777,216.

34. 2^{26}, or 67,108,864

35. The term of highest degree of $(x^5+3)^4$ is the first term, or

$$\binom{4}{0}(x^5)^{4-0}3^0=\frac{4!}{4!0!}x^{20}=x^{20}.$$

Therefore, the degree of $(x^5+3)^4$ is 20.

36. The term of highest degree of $(2-5x^3)^7$ is the last term, or

$$\binom{7}{7}(-5x^3)^7=-78,125x^{21}.$$

Therefore, the degree of $(2-5x^3)^7$ is 21.

37. We use factorial notation. Note that $a=3$, $b=i$, and $n=5$.

$$(3+i)^5$$
$$=\binom{5}{0}(3^5)+\binom{5}{1}(3^4)(i)+\binom{5}{2}(3^3)(i^2)+$$
$$\binom{5}{3}(3^2)(i^3)+\binom{5}{4}(3)(i^4)+\binom{5}{5}(i^5)$$
$$=\frac{5!}{0!5!}(243)+\frac{5!}{1!4!}(81)(i)+\frac{5!}{2!3!}(27)(-1)+$$
$$\frac{5!}{3!2!}(9)(-i)+\frac{5!}{4!1!}(3)(1)+\frac{5!}{5!0!}(i)$$
$$=243+405i-270-90i+15+i$$
$$=-12+316i$$

38.
$$(1+i)^6$$
$$=\binom{6}{0}1^6+\binom{6}{1}1^5\cdot i+\binom{6}{2}1^4\cdot i^2+$$
$$\binom{6}{3}1^3\cdot i^3+\binom{6}{4}1^2\cdot i^4+\binom{6}{5}1\cdot i^5+$$
$$\binom{6}{6}i^6$$
$$=1+6i-15-20i+15+6i-1$$
$$=-8i$$

39. We use factorial notation. Note that $a=\sqrt{2}$, $b=-i$, and $n=4$.

$$(\sqrt{2}-i)^4=\binom{4}{0}(\sqrt{2})^4+\binom{4}{1}(\sqrt{2})^3(-i)+$$
$$\binom{4}{2}(\sqrt{2})^2(-i)^2+\binom{4}{3}(\sqrt{2})(-i)^3+$$
$$\binom{4}{4}(-i)^4$$
$$=\frac{4!}{0!4!}(4)+\frac{4!}{1!3!}(2\sqrt{2})(-i)+$$
$$\frac{4!}{2!2!}(2)(-1)+\frac{4!}{3!1!}(\sqrt{2})(i)+$$
$$\frac{4!}{4!0!}(1)$$
$$=4-8\sqrt{2}i-12+4\sqrt{2}i+1$$
$$=-7-4\sqrt{2}i$$

40.
$$\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^{11}$$
$$=\binom{11}{0}\left(\frac{\sqrt{3}}{2}\right)^{11}+\binom{11}{1}\left(\frac{\sqrt{3}}{2}\right)^{10}\left(-\frac{1}{2}i\right)+$$
$$\binom{11}{2}\left(\frac{\sqrt{3}}{2}\right)^{9}\left(-\frac{1}{2}i\right)^{2}+$$
$$\binom{11}{3}\left(\frac{\sqrt{3}}{2}\right)^{8}\left(-\frac{1}{2}i\right)^{3}+$$
$$\binom{11}{4}\left(\frac{\sqrt{3}}{2}\right)^{7}\left(-\frac{1}{2}i\right)^{4}+$$
$$\binom{11}{5}\left(\frac{\sqrt{3}}{2}\right)^{6}\left(-\frac{1}{2}i\right)^{5}+$$
$$\binom{11}{6}\left(\frac{\sqrt{3}}{2}\right)^{5}\left(-\frac{1}{2}i\right)^{6}+$$
$$\binom{11}{7}\left(\frac{\sqrt{3}}{2}\right)^{4}\left(-\frac{1}{2}i\right)^{7}+$$
$$\binom{11}{8}\left(\frac{\sqrt{3}}{2}\right)^{3}\left(-\frac{1}{2}i\right)^{8}+$$
$$\binom{11}{9}\left(\frac{\sqrt{3}}{2}\right)^{2}\left(-\frac{1}{2}i\right)^{9}+$$
$$\binom{11}{10}\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}i\right)^{10}+\binom{11}{11}\left(-\frac{1}{2}i\right)^{11}$$
$$=\frac{243\sqrt{3}}{2048}-\frac{2673}{2048}i-\frac{4455\sqrt{3}}{2048}+\frac{13,365}{2048}i+$$
$$\frac{8910\sqrt{3}}{2048}-\frac{12,474}{2048}i-\frac{4158\sqrt{3}}{2048}+\frac{2970}{2048}i+$$
$$\frac{495\sqrt{3}}{2048}-\frac{165}{2048}i-\frac{11\sqrt{3}}{2048}+\frac{1}{2048}i$$
$$=\frac{1024\sqrt{3}}{2048}+\frac{1024}{2048}i$$
$$=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

41.
$$\begin{aligned}(a-b)^n &= \binom{n}{0}a^n(-b)^0+\binom{n}{1}a^{n-1}(-b)^1+\\ &\quad\binom{n}{2}a^{n-2}(-b)^2+\cdots+\\ &\quad\binom{n}{n-1}a^1(-b)^{n-1}+\binom{n}{n}a^0(-b)^n\\ &= \binom{n}{0}(-1)^0a^nb^0+\binom{n}{1}(-1)^1a^{n-1}b^1+\\ &\quad\binom{n}{2}(-1)^2a^{n-2}b^2+\cdots+\\ &\quad\binom{n}{n-1}(-1)^{n-1}a^1b^{n-1}+\\ &\quad\binom{n}{n}(-1)^na^0b^n\\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^ka^{n-k}b^k\end{aligned}$$

42.
$$\begin{aligned}&\frac{(x+h)^{13}-x^{13}}{h}\\ &=(x^{13}+13x^{12}h+78x^{11}h^2+286x^{10}h^3+\\ &\quad 715x^9h^4+1287x^8h^5+1716x^7h^6+1716x^6h^7+\\ &\quad 1287x^5h^8+715x^4h^9+286x^3h^{10}+78x^2h^{11}+\\ &\quad 13xh^{12}+h^{13}-x^{13})/h\\ &=13x^{12}+78x^{11}h+286x^{10}h^2+715x^9h^3+\\ &\quad 1287x^8h^4+1716x^7h^5+1716x^6h^6+1287x^5h^7+\\ &\quad 715x^4h^8+286x^3h^9+78x^2h^{10}+13xh^{11}+h^{12}\end{aligned}$$

43.
$$\begin{aligned}&\frac{(x+h)^n-x^n}{h}\\ &=\frac{\binom{n}{0}x^n+\binom{n}{1}x^{n-1}h+\cdots+\binom{n}{n}h^n-x^n}{h}\\ &=\binom{n}{1}x^{n-1}+\binom{n}{2}x^{n-2}h+\cdots+\binom{n}{n}h^{n-1}\\ &=\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1}\end{aligned}$$

44. $(f+g)(x)=f(x)+g(x)=(x^2+1)+(2x-3)=$
x^2+2x-2

45. $(fg)(x)=f(x)g(x)=(x^2+1)(2x-3)=$
$2x^3-3x^2+2x-3$

46. $(f\circ g)(x)=f(g(x))=f(2x-3)=(2x-3)^2+1=4x^2-$
$12x+9+1=4x^2-12x+10$

47. $(g\circ f)(x)=g(f(x))=g(x^2+1)=2(x^2+1)-3=$
$2x^2+2-3=2x^2-1$

48. $\displaystyle\sum_{k=0}^{8}\binom{8}{k}x^{8-k}3^k=0$

The left side of the equation is sigma notation for $(x+3)^8$, so we have:

$$\begin{aligned}(x+3)^8&=0\\ x+3&=0 \quad \text{Taking the 8th root on both sides}\\ x&=-3\end{aligned}$$

49. $\displaystyle\sum_{k=0}^{4}\binom{4}{k}(-1)^kx^{4-k}6^k=\sum_{k=0}^{4}\binom{4}{k}x^{4-k}(-6)^k$, so the left side of the equation is sigma notation for $(x-6)^4$. We have:

$$\begin{aligned}(x-6)^4&=81\\ x-6&=\pm 3 \quad \text{Taking the 4th root on both sides}\\ x-6=3 \quad &\text{or} \quad x-6=-3\\ x=9 \quad &\text{or} \quad x=3\end{aligned}$$

The solutions are 9 and 3.

If we also observe that $(3i)^4=81$, we also find the imaginary solutions $6\pm 3i$.

50. The $(k+1)$st term of $\left(\dfrac{3x^2}{2}-\dfrac{1}{3x}\right)^{12}$ is $\dbinom{12}{k}\left(\dfrac{3x^2}{2}\right)^{12-k}\left(-\dfrac{1}{3x}\right)^k$. In the term which does not contain x, the exponent of x in the numerator is equal to the exponent of x in the denominator.

$$\begin{aligned}2(12-k)&=k\\ 24-2k&=k\\ 24&=3k\\ 8&=k\end{aligned}$$

Find the $(8+1)$st, or 9th term:

$$\binom{12}{8}\left(\frac{3x^2}{2}\right)^4\left(-\frac{1}{3x}\right)^8=\frac{12!}{4!8!}\left(\frac{3^4x^8}{2^4}\right)\left(\frac{1}{3^8x^8}\right)=\frac{55}{144}$$

51. The expansion of $(x^2-6y^{3/2})^6$ has 7 terms, so the 4th term is the middle term.

$$\binom{6}{3}(x^2)^3(-6y^{3/2})^3=\frac{6!}{3!3!}(x^6)(-216y^{9/2})=$$
$$-4320x^6y^{9/2}$$

52.
$$\frac{\binom{5}{3}(p^2)^2\left(-\frac{1}{2}p\sqrt[3]{q}\right)^3}{\binom{5}{2}(p^2)^3\left(-\frac{1}{2}p\sqrt[3]{q}\right)^2}=\frac{-\frac{1}{8}p^7q}{\frac{1}{4}p^8\sqrt[3]{q^2}}=-\frac{\sqrt[3]{q}}{2p}$$

53. The $(k+1)$st term of $\left(\sqrt[3]{x}-\dfrac{1}{\sqrt{x}}\right)^7$ is

$\dbinom{7}{k}(\sqrt[3]{x})^{7-k}\left(-\dfrac{1}{\sqrt{x}}\right)^k$. The term containing $\dfrac{1}{x^{1/6}}$ is the term in which the sum of the exponents is $-1/6$. That is,

$$\begin{aligned}\left(\frac{1}{3}\right)(7-k)+\left(-\frac{1}{2}\right)(k)&=-\frac{1}{6}\\ \frac{7}{3}-\frac{k}{3}-\frac{k}{2}&=-\frac{1}{6}\\ -\frac{5k}{6}&=-\frac{15}{6}\\ k&=3\end{aligned}$$

Find the $(3+1)$st, or 4th term.

$\binom{7}{3}(\sqrt[3]{x})^4\left(-\frac{1}{\sqrt{x}}\right)^3 = \frac{7!}{4!3!}(x^{4/3})(-x^{-3/2}) =$

$-35x^{-1/6}$, or $-\frac{35}{x^{1/6}}$.

54. The total number of subsets of a set of 7 bills is 2^7. This includes the empty set. Thus, 2^7-1, or 127, different sums of money can be formed.

55. ${}_{100}C_0 + {}_{100}C_1 + \cdots + {}_{100}C_{100}$ is the total number of subsets of a set with 100 members, or 2^{100}.

56. ${}_nC_0 + {}_nC_1 + ... + {}_nC_n$ is the total number of subsets of a set with n members, or 2^n.

57. $\sum_{k=0}^{23}\binom{23}{k}(\log_a x)^{23-k}(\log_a t)^k =$

$(\log_a x + \log_a t)^{23} = [\log_a(xt)]^{23}$

58. $\sum_{k=0}^{15}\binom{15}{k} i^{30-2k}$

$= i^{30} + 15i^{28} + 105i^{26} + 455i^{24} + 1365i^{22} +$
$3003i^{20} + 5005i^{18} + 6435i^{16} + 6435i^{14} +$
$5005i^{12} + 3003i^{10} + 1365i^8 + 455i^6 + 105i^4 +$
$15i^2 + 1$

$= -1 + 15 - 105 + 455 - 1365 + 3003 - 5005 +$
$6435 - 6435 + 5005 - 3003 + 1365 - 455 +$
$105 - 15 + 1$

$= 0$

59. See the answer section in the text.

Exercise Set 11.8

1. a) We use Principle P.

For 1: $P = \frac{18}{100}$, or 0.18

For 2: $P = \frac{24}{100}$, or 0.24

For 3: $P = \frac{23}{100}$, or 0.23

For 4: $P = \frac{23}{100}$, or 0.23

For 5: $P = \frac{12}{100}$, or 0.12

b) Opinions may vary, but it seems that people tend not to select the first or last numbers.

2. The total number of gumdrops is $7+8+9+4+5+6$, or 39.

Lemon: $\frac{8}{39}$

Lime: $\frac{5}{39}$

Orange: $\frac{9}{39} = \frac{3}{13}$

Grape: $\frac{6}{39} = \frac{2}{13}$

Strawberry: $\frac{7}{39}$

Licorice: $\frac{0}{39} = 0$

3. The company can expect 78% of the 15,000 pieces of advertising to be opened and read. We have:

$78\%(15,000) = 0.78(15,000) = 11,700$ pieces.

4. a) B: $136/9136 \approx 1.5\%$

C: $273/9136 \approx 3.0\%$

D: $286/9136 \approx 3.1\%$

E: $1229/9136 \approx 13.5\%$

F: $173/9136 \approx 1.9\%$

G: $190/9136 \approx 2.1\%$

H: $399/9136 \approx 4.4\%$

I: $539/9136 \approx 5.9\%$

J: $21/9136 \approx 0.2\%$

K: $57/9136 \approx 0.6\%$

L: $417/9136 \approx 4.6\%$

M: $231/9136 \approx 2.5\%$

N: $597/9136 \approx 6.5\%$

O: $705/9136 \approx 7.7\%$

P: $238/9136 \approx 2.6\%$

Q: $4/9136 \approx 0.04\%$

R: $609/9136 \approx 6.7\%$

S: $745/9136 \approx 8.2\%$

T: $789/9136 \approx 8.6\%$

U: $240/9136 \approx 2.6\%$

V: $113/9136 \approx 1.2\%$

W: $127/9136 \approx 1.4\%$

X: $20/9136 \approx 0.2\%$

Y: $124/9136 \approx 1.4\%$

b) $\frac{853+1229+539+705+240}{9136} = \frac{3566}{9136} \approx 39\%$

c) In part (b) we found that there are 3566 vowels. Thus, there are $9136-3566$, or 5570 consonants.

$$P = \frac{5570}{9136} \approx 61\%$$

(We could also find this by subtracting the probability of a vowel occurring from 100%: $100\% - 39\% = 61\%$)

5. a) Since there are 14 equally likely ways of selecting a marble from a bag containing 4 red marbles and 10 green marbles, we have, by Principle P,

$P(\text{selecting a red marble}) = \frac{4}{14} = \frac{2}{7}$.

b) Since there are 14 equally likely ways of selecting a marble from a bag containing 4 red marbles and 10 green marbles, we have, by Principle P,

$P(\text{selecting a green marble}) = \frac{10}{14} = \frac{5}{7}$.

c) Since there are 14 equally likely ways of selecting a marble from a bag containing 4 red marbles and 10 green marbles, we have, by Principle P,

$P(\text{selecting a purple marble}) = \frac{0}{14} = 0.$

d) Since there are 14 equally likely ways of selecting a marble from a bag containing 4 red marbles and 10 green marbles, we have, by Principle P,

$P(\text{selecting a red or a green marble}) = \frac{4+10}{14} = 1.$

6. Total number of coins: $5+3+7=15$

a) $\frac{3}{15}$, or $\frac{1}{5}$

b) $\frac{7}{15}$

c) $\frac{0}{15}$, or 0

d) $\frac{15}{15}$, or 1

7. There are 6 possible outcomes. There are 3 numbers less than 4, so the probability is $\frac{3}{6}$, or $\frac{1}{2}$.

8. $\frac{2}{6}$, or $\frac{1}{3}$

9. a) There are 4 queens, so the probability is $\frac{4}{52}$, or $\frac{1}{13}$.

b) There are 4 aces and 4 tens, so the probability is $\frac{4+4}{52}$, or $\frac{8}{52}$, or $\frac{2}{13}$.

c) There are 13 hearts, so the probability is $\frac{13}{52}$, or $\frac{1}{4}$.

d) There are two black 6's, so the probability is $\frac{2}{52}$, or $\frac{1}{26}$.

10. a) $\frac{4}{52}$, or $\frac{1}{13}$

b) $\frac{8}{52}$, or $\frac{2}{13}$

c) $\frac{2}{52}$, or $\frac{1}{26}$

d) $\frac{26}{52}$, or $\frac{1}{2}$

11. The number of ways of drawing 3 cards from a deck of 52 is ${}_{52}C_3$. The number of ways of drawing 3 aces is ${}_4C_3$. The probability is

$$\frac{{}_4C_3}{{}_{52}C_3} = \frac{4}{22,100} = \frac{1}{5525}.$$

12. $\frac{{}_{26}C_4}{{}_{52}C_4} = \frac{14,950}{270,725} = \frac{46}{833}$

13. The total number of people on the sales force is $10+10$, or 20. The number of ways to choose 4 people from a group of 20 is ${}_{20}C_4$. The number of ways of selecting 2 people from a group of 10 is ${}_{10}C_2$. This is done for both the men and the women.

$P(\text{choosing 2 men and 2 women}) = \frac{{}_{10}C_2 \cdot {}_{10}C_2}{{}_{20}C_4} = \frac{45 \cdot 45}{4845} = \frac{135}{323}$

14. The total number of coins is $7+5+10$, or 22 and the total number of coins to be drawn is $4+3+1$, or 8. The number of ways of selecting 8 coins from a group of 22 is ${}_{22}C_8$. Four dimes can be selected in ${}_7C_4$ ways, 3 nickels in ${}_5C_3$ ways, and 1 quarter in ${}_{10}C_1$ ways.

$P(\text{selecting 4 dimes, 3 nickels, and 1 quarter}) = \frac{{}_7C_4 \cdot {}_5C_3 \cdot {}_{10}C_1}{{}_{22}C_8}$, or $\frac{350}{31,977}$

15. The number of ways of selecting 5 cards from a deck of 52 cards is ${}_{52}C_5$. Three sevens can be selected in ${}_4C_3$ ways and 2 kings in ${}_4C_2$ ways.

$P(\text{drawing 3 sevens and 2 kings}) = \frac{{}_4C_3 \cdot {}_4C_2}{{}_{52}C_5}$, or $\frac{1}{108,290}$.

16. Since there are only 4 aces in the deck, $P(\text{5 aces}) = 0$.

17. The number of ways of selecting 5 cards from a deck of 52 cards is ${}_{52}C_5$. Since 13 of the cards are spades, then 5 spades can be drawn in ${}_{13}C_5$ ways

$P(\text{drawing 5 spades}) = \frac{{}_{13}C_5}{{}_{52}C_5} = \frac{1287}{2,598,960} = \frac{33}{66,640}$

18. $\frac{{}_4C_4 \cdot {}_4C_1}{{}_{52}C_5} = \frac{1}{649,740}$

19. a) HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

b) Three of the 8 outcomes have exactly one head. Thus, $P(\text{exactly one head}) = \frac{3}{8}$.

c) Seven of the 8 outcomes have exactly 0, 1, or 2 heads. Thus, $P(\text{at most two heads}) = \frac{7}{8}$.

d) Seven of the 8 outcomes have 1, 2, or 3 heads. Thus, $P(\text{at least one head}) = \frac{7}{8}$.

e) Three of the 8 outcomes have exactly two tails. Thus, $P(\text{exactly two tails}) = \frac{3}{8}$.

20. $\frac{18}{38}$, or $\frac{9}{19}$

21. The roulette wheel contains 38 equally likely slots. Eighteen of the 38 slots are colored black. Thus, by Principle P,

$P(\text{the ball falls in a black slot}) = \frac{18}{38} = \frac{9}{19}$.

22. $\frac{1}{38}$

23. The roulette wheel contains 38 equally likely slots. Only 1 slot is numbered 0. Then, by Principle P,

$P(\text{the ball falls in the 0 slot}) = \frac{1}{38}$.

24. $\dfrac{2}{38} = \dfrac{1}{19}$

25. The roulette wheel contains 38 equally likely slots. Thirty-six of the slots are colored red or black. Then, by Principle P,

P(the ball falls in a red or a black slot) $= \dfrac{36}{38} = \dfrac{18}{19}$.

26. $\dfrac{1}{38}$

27. The roulette wheel contains 38 equally likely slots. Eighteen of the slots are odd-numbered. Then, by Principle P,

P(the ball falls in a an odd-numbered slot) $=$

$\dfrac{18}{38} = \dfrac{9}{19}$.

28. The dartboard can be thought of as having 18 areas of equal size. Of these, 6 are red, 4 are green, 3 are blue, and 5 are yellow.

$P(\text{red}) = \dfrac{6}{18} = \dfrac{1}{3}$

$P(\text{green}) = \dfrac{4}{18} = \dfrac{2}{9}$

$P(\text{blue}) = \dfrac{3}{18} = \dfrac{1}{6}$

$P(\text{yellow}) = \dfrac{5}{18}$

29. zero

30. one-to-one

31. function; domain; range; domain; range

32. zero

33. combination

34. inverse variation

35. factor

36. geometric sequence

37. a) There are $\binom{13}{2}$ ways to select 2 denominations from the 13 denominations. Then in each denomination there are $\binom{4}{2}$ ways to choose 2 of the 4 cards. Finally there are $\binom{44}{1}$ ways to choose the fifth card from the 11 remaining denominations ($4 \cdot 11$, or 44 cards). Thus the number of two pairs hands is $\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot \binom{44}{1}$, or 123,552.

b) $\dfrac{123,552}{{}_{52}C_5} = \dfrac{123,552}{2,598,960} \approx 0.0475$

38. a) There are 13 ways to select a denomination. Then from that denomination there are ${}_4C_3$ ways to pick 3 of the 4 cards in that denomination. Now there are 12 ways to select any one of the remaining 12 denominations and ${}_4C_2$ ways to pick 2 cards from the 4 cards in that denomination. Thus the number of full houses is $(13 \cdot {}_4C_3) \cdot (12 \cdot {}_4C_2)$ or 3744.

b) $\dfrac{3744}{{}_{52}C_5} = \dfrac{3744}{2,598,960} \approx 0.00144$

39. a) There are 13 ways to select a denomination and then $\binom{4}{3}$ ways to choose 3 of the 4 cards in that denomination. Now there are $\binom{48}{2}$ ways to choose 2 cards from the 12 remaining denominations ($4 \cdot 12$, or 48 cards). But these combinations include the 3744 hands in a full house like Q-Q-Q-4-4 (Exercise 38), so these must be subtracted. Thus the number of three of a kind hands is $13 \cdot \binom{4}{3} \cdot \binom{48}{2} - 3744$, or 54,912.

b) $\dfrac{54,912}{{}_{52}C_5} = \dfrac{54,912}{2,598,960} \approx 0.0211$

40. a) There are 13 ways to choose a denomination. Then there are 48 ways to choose one of the 48 cards remaining after 4 cards of the same denomination are chosen. Thus there are $13 \cdot 48$, or 624, four of a kind hands.

b) $\dfrac{624}{{}_{52}C_5} = \dfrac{624}{2,598,960} \approx 2.4 \times 10^{-4}$

Chapter 11 Review Exercises

1. The statement is true. See page 912 in the text.

2. An infinite geometric series has a sum if $|r| < 1$, so the statement is false.

3. The statement is true. See page 947 in the text.

4. The total number of subsets of a set with n elements is 2^n, so the statement is false.

5.
$$a_n = (-1)^n\left(\frac{n^2}{n^4+1}\right)$$
$$a_1 = (-1)^1\left(\frac{1^2}{1^4+1}\right) = -\frac{1}{2}$$
$$a_2 = (-1)^2\left(\frac{2^2}{2^4+1}\right) = \frac{4}{17}$$
$$a_3 = (-1)^3\left(\frac{3^2}{3^4+1}\right) = -\frac{9}{82}$$
$$a_4 = (-1)^4\left(\frac{4^2}{4^4+1}\right) = \frac{16}{257}$$
$$a_{11} = (-1)^{11}\left(\frac{11^2}{11^4+1}\right) = -\frac{121}{14,642}$$
$$a_{23} = (-1)^{23}\left(\frac{23^2}{23^4+1}\right) = -\frac{529}{279,842}$$

6. $(-1)^{n+1}(n^2+1)$

7. $$\sum_{k=1}^{4} \frac{(-1)^{k+1}3^k}{3^k-1}$$
$$= \frac{(-1)^{1+1}3^1}{3^1-1} + \frac{(-1)^{2+1}3^2}{3^2-1} + \frac{(-1)^{3+1}3^3}{3^3-1} + \frac{(-1)^{4+1}3^4}{3^4-1}$$
$$= \frac{3}{2} - \frac{9}{8} + \frac{27}{26} - \frac{81}{80}$$
$$= \frac{417}{1040}$$

8. $\sum_{k=1}^{7}(k^2-1)$

9. $$a_n = a_1 + (n-1)d$$
$$a_1 = \frac{3}{4}, d = \frac{13}{12} - \frac{3}{4} = \frac{1}{3}, \text{ and } n = 10$$
$$a_{10} = \frac{3}{4} + (10-1)\frac{1}{3} = \frac{3}{4} + 3 = \frac{15}{4}$$

10. $$a_n = a_1 + (n-1)d$$
$$a_1 = a - b, d = a - (a-b) = b$$
$$a_6 = a - b + (6-1)b = a - b + 5b = a + 4b$$

11. $$a_n = a_1 + (n-1)d$$
$$a_{18} = 4 + (18-1)3 = 4 + 51 = 55$$
$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{18} = \frac{18}{2}(4 + 55) = 531$$

12. $1 + 2 + 3 + \ . \ . \ . \ + 199 + 200$
$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{200} = \frac{200}{2}(1 + 200) = 20,100$$

13. $a_1 = 5,\ a_{17} = 53$
$$53 = 5 + (17-1)d;\ d = 3$$
$$a_3 = 5 + (3-1)3 = 11$$

14. $d = 3,\ a_{10} = 23$
$$23 = a_1 + (10-1)3$$
$$23 = a_1 + 27$$
$$-4 = a_1$$

15. $a_1 = -2,\ r = 2,\ a_n = -64$
$$a_n = a_1 r^{n-1}$$
$$-64 = -2 \cdot 2^{n-1}$$
$$-64 = -2^n;\ n = 6$$
$$S_n = \frac{a_1(1-r^n)}{1-r}$$
$$S_n = \frac{-2(1-2^6)}{1-2} = 2(1-64) = -126$$

16. $r = \frac{1}{2},\ n = 5,\ S_n = \frac{31}{2}$
$$S_n = \frac{a_1(1-r^n)}{1-r}$$
$$\frac{31}{2} = \frac{a_1\left(1 - \left(\frac{1}{2}\right)^5\right)}{1 - \frac{1}{2}}$$
$$\frac{31}{2} = \frac{a_1\left(1 - \frac{1}{32}\right)}{\frac{1}{2}}$$
$$\frac{31}{2} = \frac{\frac{31}{32}a_1}{\frac{1}{2}}$$
$$\frac{31}{2} = \frac{31}{32}a_1 \cdot \frac{2}{1}$$
$$\frac{31}{2} = \frac{31}{16}a_1$$
$$8 = a_1$$
$$a_n = a_1 r^{n-1}$$
$$a_5 = 8\left(\frac{1}{2}\right)^{5-1} = 8\left(\frac{1}{2}\right)^4 = 8 \cdot \frac{1}{16} = \frac{1}{2}$$

17. Since $|r| = \left|\frac{27.5}{2.5}\right| = |1.1| = 1.1 > 1$, the sum does not exist.

18. Since $|r| = \left|\frac{0.0027}{0.27}\right| = |0.01| = 0.01 < 1$, the series has a sum.
$$S_\infty = \frac{0.27}{1-0.01} = \frac{0.27}{0.99} = \frac{27}{99} = \frac{3}{11}$$

19. Since $|r| = \left|\frac{-\frac{1}{6}}{\frac{1}{2}}\right| = \left|-\frac{1}{3}\right| = \frac{1}{3} < 1$, the series has a sum.
$$S_\infty = \frac{\frac{1}{2}}{1 - \left(-\frac{1}{3}\right)} = \frac{\frac{1}{2}}{\frac{4}{3}} = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

20. $2.\overline{43} = 2 + 0.43 + 0.0043 + 0.000043 + \ . \ . \ . \ .$

We will find fraction notation for $0.\overline{43}$ and then add 2.

$|r| = \left|\frac{0.0043}{0.43}\right| = |0.01| = 0.01 < 1$, so the series has a limit.
$$S_\infty = \frac{0.43}{1-0.01} = \frac{0.43}{0.99} = \frac{43}{99}$$
Then $2 + \frac{43}{99} = \frac{198}{99} + \frac{43}{99} = \frac{241}{99}$

21. $5,\ m_1,\ m_2,\ m_3,\ m_4,\ 9$

We look for m_1, m_2, m_3, and m_4, such that 5, m_1, m_2, m_3, m_4, 9 is an arithmetic sequence. In this case, $a_1 = 5$, $n = 6$, and $a_6 = 9$. First we find d:

$a_n = a_1 + (n-1)d$
$9 = 5 + (6-1)d$
$4 = 5d$
$\frac{4}{5} = d$

Then we have:

$m_1 = a_1 + d = 5 + \frac{4}{5} = 5\frac{4}{5}$
$m_2 = m_1 + d = 5\frac{4}{5} + \frac{4}{5} = 6\frac{3}{5}$
$m_3 = m_2 + d = 6\frac{3}{5} + \frac{4}{5} = 7\frac{2}{5}$
$m_4 = m_3 + d = 7\frac{2}{5} + \frac{4}{5} = 8\frac{1}{5}$

22. ***Familiarize***. The distances the ball drops are given by a geometric sequence:

$30,\ \frac{3}{4}\cdot 30,\ \left(\frac{3}{4}\right)^2\cdot 30,\ \left(\frac{3}{4}\right)^3\cdot 30,\ \left(\frac{3}{4}\right)^4\cdot 30,\ \left(\frac{3}{4}\right)^5\cdot 30.$

Then the total distance the ball drops is the sum of these 6 terms. The total rebound distance is this sum less 30 ft, the distance the ball drops initially.

Translate. To find the total distance the ball drops we will use the formula

$S_n = \frac{a_1(1-r^n)}{1-r}$ with $a_1 = 30$, $r = \frac{3}{4}$, and $n = 6$.

Carry out.

$$S_6 = \frac{30\left(1-\left(\frac{3}{4}\right)^6\right)}{1-\frac{3}{4}} = \frac{50,505}{512}$$

Then the total rebound distance is $\frac{50,505}{12} - 30$, or $\frac{35,145}{512}$, and the total distance the ball has traveled is $\frac{50,505}{512} + \frac{34,145}{512}$, or $\frac{42,825}{256}$, or about 167.3 ft.

Check. We can perform the calculations again.

State. The ball will have traveled about 167.3 ft when it hits the pavement for the sixth time.

23. $S_n = \frac{a_1(1-r^n)}{1-r}$

$S_{18} = \frac{2000\left[1-(1.028)^{18}\right]}{1-1.028} \approx \$45,993.04$

24. a), b) ***Familiarize***. We have an arithmetic sequence with $a_1 = 10$, $d = 2$, and $n = 365$.

Translate. We want to find $a_n = a_1 + (n-1)d$ and $S_n = \frac{n}{2}(a_1 + a_n)$ where $a_1 = 10$, $d = 2$, and $n = 365$.

Carry out. First we find a_{365}.

$a_{365} = 10 + (365-1)(2) = 738¢$, or \$7.38

Then $S_{365} = \frac{365}{2}(10 + 738) = 136,510¢$ or \$1365.10.

Check. We can repeat the calculations.

State. a) You will receive \$7.38 on the 365th day.

b) The sum of all the gifts is \$1365.10.

25. $a_1 = 24,000,000,000$; $r = 0.73$

$S_\infty = \frac{24,000,000,000}{1-0.73} \approx \$88,888,888,889$

26. $S_n : 1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$

$S_1 : 1 = \frac{1(3-1)}{2}$

$S_k : 1 + 4 + 7 + \ldots + (3k-2) = \frac{k(3k-1)}{2}$

$S_{k+1} : 1 + 4 + 7 + \ldots + (3k-2) + [3(k+1)-2]$
$= 1 + 4 + 7 + \ldots + (3k-2) + (3k+1)$
$= \frac{(k+1)(3k+2)}{2}$

1. *Basis step*: $\frac{1(3-1)}{2} = \frac{2}{2} = 1$ is true.
2. *Induction step*: Assume S_k. Add $3k+1$ to both sides.

$1 + 4 + 7 + \ldots + (3k-2) + (3k+1)$
$= \frac{k(3k-1)}{2} + (3k+1)$
$= \frac{k(3k-1)}{2} + \frac{2(3k+1)}{2}$
$= \frac{3k^2 - k + 6k + 2}{2}$
$= \frac{3k^2 + 5k + 2}{2}$
$= \frac{(k+1)(3k+2)}{2}$

27. $S_n : 1 + 3 + 3^2 + \ldots + 3^{n-1} = \frac{3^n - 1}{2}$

$S_1 : 1 = \frac{3^1 - 1}{2}$

$S_k : 1 + 3 + 3^2 + \ldots + 3^{k-1} = \frac{3^k - 1}{2}$

$S_{k+1} : 1 + 3 + 3^2 + \ldots + 3^{(k+1)-1} = \frac{3^{k+1} - 1}{2}$

1. *Basis step*: $\frac{3^1 - 1}{2} = \frac{2}{2} = 1$ is true.
2. *Induction step*: Assume S_k. Add 3^k to both sides.

$1 + 3 + \ldots + 3^{k-1} + 3^k$
$= \frac{3^k - 1}{2} + 3^k = \frac{3^k - 1}{2} + 3^k \cdot \frac{2}{2}$
$= \frac{3 \cdot 3^k - 1}{2} = \frac{3^{k+1} - 1}{2}$

28. $S_n : \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n}\right) = \frac{1}{n}$

$S_2 : \left(1-\frac{1}{2}\right) = \frac{1}{2}$

$S_k : \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{k}\right) = \frac{1}{k}$

$S_{k+1} : \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{k}\right)\left(1-\frac{1}{k+1}\right) = \frac{1}{k+1}$

1. *Basis step*: S_2 is true by substitution.

2. *Induction step*: Assume S_k. Deduce S_{k+1}. Starting with the left side of S_{k+1}, we have

$$\underbrace{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)}\left(1-\frac{1}{k+1}\right)$$
$$= \frac{1}{k} \cdot \left(1-\frac{1}{k+1}\right) \quad \text{By } S_k$$
$$= \frac{1}{k}\cdot\left(\frac{k+1-1}{k+1}\right)$$
$$= \frac{1}{k}\cdot\frac{k}{k+1}$$
$$= \frac{1}{k+1}. \quad \text{Simplifying}$$

29. $6! = 720$

30. $9 \cdot 8 \cdot 7 \cdot 6 = 3024$

31. $\binom{15}{8} = \dfrac{15!}{8!(15-8)!} = 6435$

32. $24 \cdot 23 \cdot 22 = 12,144$

33. $\dfrac{9!}{1!4!2!2!} = 3780$

34. $3 \cdot 4 \cdot 3 = 36$

35. a) ${}_6P_5 = \dfrac{6!}{(6-5)!} = 720$

b) $6^5 = 7776$

c) ${}_5P_4 = \dfrac{5!}{(5-4)!} = 120$

d) ${}_3P_2 = \dfrac{3!}{(3-2)!} = 6$

36. $2^8 = 256$

37. $(m+n)^7$

Pascal's triangle method: Use the 8th row of Pascal's triangle.

1 7 21 35 35 21 7 1

$$(m+n)^7 = m^7 + 7m^6n + 21m^5n^2 + 35m^4n^3 + 35m^3n^4 + 21m^2n^5 + 7mn^6 + n^7$$

Factorial notation method:

$$(m+n)^7 = \binom{7}{0}m^7+\binom{7}{1}m^6n+\binom{7}{2}m^5n^2+\binom{7}{3}m^4n^3+\binom{7}{4}m^3n^4+\binom{7}{5}m^2n^5+\binom{7}{6}mn^6+\binom{7}{7}n^7$$
$$= m^7 + 7m^6n + 21m^5n^2 + 35m^4n^3 + 35m^3n^4 + 21m^2n^5 + 7mn^6 + n^7$$

38. Expand: $(x-\sqrt{2})^5$

Pascal's triangle method: Use the 6th row.

1 5 10 10 5 1

$$(x-\sqrt{2})^5 = x^5 + 5x^4(-\sqrt{2}) + 10x^3(-\sqrt{2})^2 + 10x^2(-\sqrt{2})^3 + 5x(-\sqrt{2})^4 + (-\sqrt{2})^5$$
$$= x^5 - 5\sqrt{2}x^4 + 20x^3 - 20\sqrt{2}x^2 + 20x - 4\sqrt{2}$$

Factorial notation method:

$$(x-\sqrt{2})^5 = \binom{5}{0}x^5+\binom{5}{1}x^4(-\sqrt{2})+\binom{5}{2}x^3(-\sqrt{2})^2+\binom{5}{3}x^2(-\sqrt{2})^3+\binom{5}{4}x(-\sqrt{2})^4+\binom{5}{5}(-\sqrt{2})^5$$
$$= x^5 - 5\sqrt{2}x^4 + 20x^3 - 20\sqrt{2}x^2 + 20x - 4\sqrt{2}$$

39. Expand: $(x^2-3y)^4$

Pascal's triangle method: Use the 5th row.

1 4 6 4 1

$$(x^2-3y)^4 = (x^2)^4 + 4(x^2)^3(-3y) + 6(x^2)^2(-3y)^2 + 4(x^2)(-3y)^3 + (-3y)^4$$
$$= x^8 - 12x^6y + 54x^4y^2 - 108x^2y^3 + 81y^4$$

Factorial notation method:

$$(x^2-3y)^4 = \binom{4}{0}(x^2)^4 + \binom{4}{1}(x^2)^3(-3y) + \binom{4}{2}(x^2)^2(-3y)^2 + \binom{4}{3}(x^2)(-3y)^3 + \binom{4}{4}(-3y)^4$$
$$= x^8 - 12x^6y + 54x^4y^2 - 108x^2y^3 + 81y^4$$

40. Expand: $\left(a+\dfrac{1}{a}\right)^8$

Pascal's triangle method: Use the 9th row.

1 8 28 56 70 56 28 8 1

$$\left(a+\frac{1}{a}\right)^8 = a^8 + 8a^7\left(\frac{1}{a}\right) + 28a^6\left(\frac{1}{a}\right)^2 + 56a^5\left(\frac{1}{a}\right)^3 + 70a^4\left(\frac{1}{a}\right)^4 + 56a^3\left(\frac{1}{a}\right)^5 + 28a^2\left(\frac{1}{a}\right)^6 + 8a\left(\frac{1}{a}\right)^7 + \left(\frac{1}{a}\right)^8$$
$$= a^8 + 8a^6 + 28a^4 + 56a^2 + 70 + 56a^{-2} + 28a^{-4} + 8a^{-6} + a^{-8}$$

Factorial notation method:

$$\left(a+\frac{1}{a}\right)^8 = \binom{8}{0}a^8 + \binom{8}{1}a^7\left(\frac{1}{a}\right) + \binom{8}{2}a^6\left(\frac{1}{a}\right)^2 + \binom{8}{3}a^5\left(\frac{1}{a}\right)^3 + \binom{8}{4}a^4\left(\frac{1}{a}\right)^4 + \binom{8}{5}a^3\left(\frac{1}{a}\right)^5 + \binom{8}{6}a^2\left(\frac{1}{a}\right)^6 + \binom{8}{7}a\left(\frac{1}{a}\right)^7 + \binom{8}{8}\left(\frac{1}{a}\right)^8$$
$$= a^8 + 8a^6 + 28a^4 + 56a^2 + 70 + 56a^{-2} + 28a^{-4} + 8a^{-6} + a^{-8}$$

41. Expand: $(1+5i)^6$

Pascal's triangle method: Use the 7th row.

1 6 15 20 15 6 1

$$\begin{aligned}(1+5i)^6 &= 1^6+6(1)^5(5i)+15(1)^4(5i)^2+20(1)^3(5i)^3+\\ &\quad 15(1)^2(5i)^4+6(1)(5i)^5+(5i)^6\\ &= 1+30i-375-2500i+9375+\\ &\quad 18,750i-15,625\\ &= -6624+16,280i\end{aligned}$$

Factorial notation method:

$$\begin{aligned}(1+5i)^6 &= \binom{6}{0}1^6+\binom{6}{1}(1)^5(5i)+\binom{6}{2}(1)^4(5i)^2+\\ &\quad \binom{6}{3}(1)^3(5i)^3+\binom{6}{4}(1)^2(5i)^4+\\ &\quad \binom{6}{5}(1)(5i)^5+\binom{6}{6}(5i)^6\\ &= 1+30i-375-2500i+9375+\\ &\quad 18,750i-15,625\\ &= -6624+16,280i\end{aligned}$$

42. Find 4th term of $(a+x)^{12}$.

$$\binom{12}{3}a^9x^3 = 220a^9x^3$$

43. Find 12th term of $(2a-b)^{18}$.

$$\binom{18}{11}(2a)^7(-b)^{11} = -\binom{18}{11}128a^7b^{11}$$

44. Of 36 possible combinations, we can get a 10 with $(4,6)$, $(6,4)$ or $(5,5)$.

Probability $= \dfrac{3}{36} = \dfrac{1}{12}$

Since we cannot get a 10 on one die, the probability of getting a 10 on one die is 0.

45. Of 52 cards, 13 are clubs.

Probability $= \dfrac{13}{52} = \dfrac{1}{4}$.

46. $\dfrac{{}_4C_2 \cdot {}_4C_1}{{}_{52}C_3} = \dfrac{6\cdot 4}{22,100} = \dfrac{6}{5525}$

47. A: $\dfrac{86}{86+97+23} = \dfrac{86}{206} \approx 0.42$

B: $\dfrac{97}{86+97+23} = \dfrac{97}{206} \approx 0.47$

C: $\dfrac{23}{86+97+23} = \dfrac{23}{206} \approx 0.11$

48. 12, 10, 8, 6, . . .

$d = 10-12 = -2$

$S_{25} = 12+(25-1)(-2) = 12+24(-2) = 12-48 = -36$

Answer B is correct.

49. There are 3 pairs that total 4: 1 and 3, 2 and 2, 3 and 1. There are $6\cdot 6$, or 36, possible outcomes. Thus, we have $\dfrac{3}{36}$, or $\dfrac{1}{12}$. Answer A is correct.

50. $a_n = n-1$

Only integers $n \geq 1$ are inputs.

$a_1 = 1-1 = 0$, $a_2 = 2-1 = 1$, $a_3 = 3-1 = 2$,
$a_4 = 4-1 = 3$, $a_5 = 5-1 = 4$, $a_6 = 6-1 = 5, \ldots$

The graph of the sequence contains the points $(1,0)$, $(2,1)$, $(3,2)$, $(4,3)$, $(5,4)$, and $(6,5)$. Thus, D is the correct answer.

51. S_1 fails for both (a) and (b).

52. $\dfrac{a_{k+1}}{a_k} = r_1$, $\dfrac{b_{k+1}}{b_k} = r_2$, so $\dfrac{a_{k+1}b_{k+1}}{a_kb_k} = r_1r_2$, a constant.

53. a) If all of the terms of $a_1, a_2, \ldots, a_n$ are positive or if they area all negative, then $b_1, b_2, \ldots, b_n$ is an arithmetic sequence whose common difference is $|d|$, where d is the common difference of $a_1, a_2, \ldots, a_n$.

b) Yes; if d is the common difference of $a_1, a_2, \ldots, a_n$, then it is also the common difference of $b_1, b_2, \ldots, b_n$ and consequently $b_1, b_2, \ldots, b_n$ is an arithmetic sequence.

c) Yes; if d is the common difference of $a_1, a_2, \ldots, a_n$, then each term of $b_1, b_2, \ldots, b_n$ is obtained by adding $7d$ to the previous term and $b_1, b_2, \ldots, b_n$ is an arithmetic sequence.

d) No (unless a_n is constant)

e) No (unless a_n is constant)

f) No (unless a_n is constant)

54. $f(x) = x^4-4x^3-4x^2+16x = x(x^3-4x^2-4x+16)$

We know that 0 is a zero.

Consider $x^3-4x^2-4x+16$.

Possibilities for p/q: $\pm1, \pm2, \pm4, \pm8, \pm16$

$$\begin{array}{r|rrrr} -2 & 1 & -4 & -4 & 16\\ & & -2 & 12 & -16\\ \hline & 1 & -6 & 8 & 0\end{array}$$

$$\begin{aligned}f(x) &= x(x+2)(x^2-6x+8)\\ &= x(x+2)(x-2)(x-4)\end{aligned}$$

The zeros are -2, 0, 2, and 4.

55. $r = -\dfrac{1}{3}$, $S_\infty = \dfrac{3}{8}$

$$S_\infty = \frac{a_1}{1-r}$$

$$\frac{3}{8} = \frac{a_1}{1-\left(-\frac{1}{3}\right)}$$

$$a_1 = \frac{1}{2}$$

$$a_2 = \frac{1}{2}\left(-\frac{1}{3}\right) = -\frac{1}{6}$$

$$a_3 = -\frac{1}{6}\left(-\frac{1}{3}\right) = \frac{1}{18}$$

56.
$$\sum_{k=0}^{10}(-1)^k\binom{10}{k}(\log x)^{10-k}(\log y)^k$$
$$= (\log x - \log y)^{10}$$
$$= \left(\log \frac{x}{y}\right)^{10}$$

57.
$$\binom{n}{6} = 3\cdot\binom{n-1}{5}$$
$$\frac{n!}{6!(n-6)!} = 3\cdot\frac{(n-1)!}{5!(n-1-5)!}$$
$$6!(n-6)!\cdot\frac{n!}{6!(n-6)!} = 6!(n-6)!\cdot\frac{3(n-1)!}{5!(n-6)!}$$
$$n! = 18(n-1)!$$
$$\frac{n!}{(n-1)!} = \frac{18(n-1)!}{(n-1)!}$$
$$n = 18$$

58.
$$\binom{n}{n-1} = 36$$
$$\frac{n!}{(n-1)![n-(n-1)]!} = 36$$
$$\frac{n(n-1)!}{(n-1)!1!} = 36$$
$$n = 36$$

59.
$$\sum_{k=0}^{5}\binom{5}{k}9^{5-k}a^k = 0$$
$$(9+a)^5 = 0$$
$$9+a = 0$$
$$a = -9$$

60. Put the following in the form of a paragraph.

First find the number of seconds in a year (365 days):

$365 \cancel{\text{days}}\cdot\frac{24\ \cancel{\text{hr}}}{1\ \cancel{\text{day}}}\cdot\frac{60\ \cancel{\text{min}}}{1\ \cancel{\text{hr}}}\cdot\frac{60\text{ sec}}{1\ \cancel{\text{min}}} =$ 31,536,000 sec.

The number of arrangements possible is 15!.

The time is $\dfrac{15!}{31,536,000} \approx 41,466$ yr.

61. For each circular arrangement of the numbers on a clock face there are 12 distinguishable ordered arrangements on a line. The number of arrangements of 12 objects on a line is ${}_{12}P_{12}$, or 12!. Thus, the number of circular permutations is $\dfrac{{}_{12}P_{12}}{12} = \dfrac{12!}{12} = 11! = 39,916,800$.

In general, for each circular arrangement of n objects, there are n distinguishable ordered arrangements on a line. The total number of arrangements of n objects on a line is ${}_nP_n$, or $n!$. Thus, the number of circular permutations is $\dfrac{n!}{n} = \dfrac{n(n-1)!}{n} = (n-1)!$.

62. Choosing k objects from a set of n objects is equivalent to not choosing the other $n-k$ objects.

63. Order is considered in a combination lock.

64. In expanding $(a+b)^n$, it would probably be better to use Pascal's triangle when n is relatively small. When n is large, and many rows of Pascal's triangle must be computed to get to the $(n+1)$st row, it would probably be better to use factorial notation. In addition, factorial notation allows us to write a particular term of the expansion more efficiently than Pascal's triangle.

Chapter 11 Test

1.
$$a_n = (-1)^n(2n+1)$$
$$a_{21} = (-1)^{21}[2(21)+1]$$
$$= -43$$

2.
$$a_n = \frac{n+1}{n+2}$$
$$a_1 = \frac{1+1}{1+2} = \frac{2}{3}$$
$$a_2 = \frac{2+1}{2+2} = \frac{3}{4}$$
$$a_3 = \frac{3+1}{3+2} = \frac{4}{5}$$
$$a_4 = \frac{4+1}{4+2} = \frac{5}{6}$$
$$a_5 = \frac{5+1}{5+2} = \frac{6}{7}$$

3.
$$\sum_{k=1}^{4}(k^2+1) = (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)$$
$$= 2+5+10+7$$
$$= 34$$

4. $\displaystyle\sum_{k=1}^{6} 4k$

5. $\displaystyle\sum_{k=1}^{\infty} 2^k$

6.
$$a_{n+1} = 2+\frac{1}{a_n}$$
$$a_1 = 2+\frac{1}{1} = 2+1 = 3$$
$$a_2 = 2+\frac{1}{3} = 2\frac{1}{3}$$
$$a_3 = 2+\frac{1}{\frac{7}{3}} = 2+\frac{3}{7} = 2\frac{3}{7}$$
$$a_4 = 2+\frac{1}{\frac{17}{7}} = 2+\frac{7}{17} = 2\frac{7}{17}$$

7.
$$d = 5-2 = 3$$
$$a_n = a_1+(n-1)d$$
$$a_{15} = 2+(15-1)3 = 44$$

8. $a_1 = 8$, $a_{21} = 108$, $n = 21$

$$a_n = a_1 + (n-1)d$$
$$108 = 8 + (21-1)d$$
$$100 = 20d$$
$$5 = d$$

Use $a_n = a_1 + (n-1)d$ again to find a_7.

$a_7 = 8 + (7-1)(5) = 8 + 30 = 38$

9. $a_1 = 17$, $d = 13 - 17 = -4$, $n = 20$

First find a_{20}:

$$a_n = a_1 + (n-1)d$$
$$a_{20} = 17 + (20-1)(-4) = 17 - 76 = -59$$

Now find S_{20}:

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{20} = \frac{20}{2}(17 - 59) = 10(-42) = -420$$

10. $\displaystyle\sum_{k=1}^{25} (2k+1)$

$$a_1 = 2 \cdot 1 + 1 = 3$$
$$a_{25} = 2 \cdot 25 + 1 = 51$$
$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{25} = \frac{25}{2}(3 + 51) = \frac{25}{2} \cdot 54 = 675$$

11. $a_1 = 10,\ r = \dfrac{-5}{10} = -\dfrac{1}{2}$

$$a_n = a_1 r^{n-1}$$
$$a_{11} = 10\left(-\frac{1}{2}\right)^{11-1} = \frac{5}{512}$$

12. $r = 0.2$, $S_4 = 1248$

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$
$$1248 = \frac{a_1(1 - 0.2^4)}{1 - 0.2}$$
$$1248 = \frac{0.9984a_1}{0.8}$$
$$a_1 = 1000$$

13. $\displaystyle\sum_{k=1}^{8} 2^k$

$$a_1 = 2^1 = 2,\ r = 2,\ n = 8$$
$$S_8 = \frac{2(1 - 2^8)}{1 - 2} = 510$$

14. $a_1 = 18$, $r = \dfrac{6}{18} = \dfrac{1}{3}$

Since $|r| = \dfrac{1}{3} < 1$, the series has a sum.

$$S_\infty = \frac{18}{1 - \frac{1}{3}} = \frac{18}{\frac{2}{3}} = 18 \cdot \frac{3}{2} = 27$$

15. $0.\overline{56} = 0.56 + 0.0056 + 0.000056 + \ldots$

$|r| = \left|\dfrac{0.0056}{0.56}\right| = |0.01| = 0.01 < 1$, so the series has a sum.

$$S_\infty = \frac{0.56}{1 - 0.01} = \frac{0.56}{0.99} = \frac{56}{99}$$

16. $a_1 = \$10,000$

$$a_2 = \$10,000 \cdot 0.80 = \$8000$$
$$a_3 = \$8000 \cdot 0.80 = \$6400$$
$$a_4 = \$6400 \cdot 0.80 = \$5120$$
$$a_5 = \$5120 \cdot 0.80 = \$4096$$
$$a_6 = \$4096 \cdot 0.80 = \$3276.80$$

17. We have an arithmetic sequence \$8.50, \$8.75, \$9.00, \$9.25, and so on with $d = \$0.25$. Each year there are 12/3, or 4 raises, so after 4 years the sequence will have the original hourly wage plus the $4 \cdot 4$, or 16, raises for a total of 17 terms. We use the formula $a_n = a_1 + (n-1)d$ with $a_1 = \$8.50$, $d = \$0.25$, and $n = 17$.

$a_{17} = \$8.50 + (17-1)(\$0.25) = \$8.50 + 16(\$0.25) = \$8.50 + \$4.00 = \$12.50$

At the end of 4 years Tamika's hourly wage will be \$12.50.

18. We use the formula $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$ with $a_1 = \$2500$, $r = 1.056$, and $n = 18$.

$$S_{18} = \frac{2500[1 - (1.056)^{18}]}{1 - 1.056} = \$74,399.77$$

19.

$$S_n:\quad 2 + 5 + 8 + \ldots + (3n-1) = \frac{n(3n+1)}{2}$$
$$S_1:\quad 2 = \frac{1(3 \cdot 1 + 1)}{2}$$
$$S_k:\quad 2 + 5 + 8 + \ldots + (3k-1) = \frac{k(3k+1)}{2}$$
$$S_{k+1}:\quad 2 + 5 + 8 + \ldots + (3k-1) + [3(k+1) - 1] = \frac{(k+1)[3(k+1)+1]}{2}$$

1) *Basis step:* $\dfrac{1(3 \cdot 1 + 1)}{2} = \dfrac{1 \cdot 4}{2} = 2$, so S_1 is true.

2) *Induction step:*

$$\begin{aligned}
&\underbrace{2 + 5 + 8 + \ldots + (3k-1)} + [3(k+1) - 1] \\
&= \frac{k(3k+1)}{2} + [3k + 3 - 1] \quad \text{By } S_k \\
&= \frac{3k^2}{2} + \frac{k}{2} + 3k + 2 \\
&= \frac{3k^2}{2} + \frac{7k}{2} + 2 \\
&= \frac{3k^2 + 7k + 4}{2} \\
&= \frac{(k+1)(3k+4)}{2} \\
&= \frac{(k+1)[3(k+1)+1]}{2}
\end{aligned}$$

20. ${}_{15}P_6 = \dfrac{15!}{(15-6)!} = 3,603,600$

21. ${}_{21}C_{10} = \dfrac{21!}{10!(21-10)!} = 352,716$

22. $$\begin{aligned}\binom{n}{4} &= \frac{n!}{4!(n-4)!}\\ &= \frac{n(n-1)(n-2)(n-3)(n-4)!}{4!(n-4)!}\\ &= \frac{n(n-1)(n-2)(n-3)}{24}\end{aligned}$$

23. ${}_6P_4 = \dfrac{6!}{(6-4)!} = 360$

24. a) $6^4 = 1296$

b) ${}_5P_3 = \dfrac{5!}{(5-3)!} = 60$

25. ${}_{28}C_4 = \dfrac{28!}{4!(28-4)!} = 20,475$

26. ${}_{12}C_8 \cdot {}_8C_4 = \dfrac{12!}{8!(12-8)!} \cdot \dfrac{8!}{4!(8-4)!} = 34,650$

27. Expand: $(x+1)^5$.

Pascal's triangle method: Use the 6th row.

1 5 10 10 5 1

$$\begin{aligned}(x+1)^5 &= x^5+5x^4\cdot 1+10x^3\cdot 1^2+10x^2\cdot 1^3+5x\cdot 1^4+1^5\\ &= x^5+5x^4+10x^3+10x^2+5x+1\end{aligned}$$

Factorial notation method:

$$\begin{aligned}(x+1)^5 &= \binom{5}{0}x^5 + \binom{5}{1}x^4\cdot 1 + \binom{5}{2}x^3\cdot 1^2 +\\ &\quad \binom{5}{3}x^2\cdot 1^3 + \binom{5}{4}x\cdot 1^4 + \binom{5}{5}1^5\\ &= x^5+5x^4+10x^3+10x^2+5x+1\end{aligned}$$

28. Find 5th term of $(x-y)^7$.

$\binom{7}{4}x^3(-y)^4 = 35x^3y^4$

29. $2^9 = 512$

30. $\dfrac{8}{6+8} = \dfrac{8}{14} = \dfrac{4}{7}$

31. $\dfrac{{}_6C_1 \cdot {}_5C_2 \cdot {}_4C_5}{{}_{15}C_6} = \dfrac{6\cdot 10\cdot 4}{5005} = \dfrac{48}{1001}$

32. $a_n = 2_n - 2$

Only integers $n \geq 1$ are inputs.

$a_1 = 2\cdot 1 - 2 = 0$, $a_2 = 2\cdot 2 - 2 = 2$, $a_3 = 2\cdot 3 - 2 = 4$, $a_4 = 2\cdot 4 - 2 = 6$

Some points on the graph are $(1,0)$, $(2,2)$, $(3,4)$, and $(4,6)$. Thus the correct answer is B.

33. $$\begin{aligned}{}_nP_7 &= 9\cdot {}_nP_6\\ \frac{n!}{(n-7)!} &= 9\cdot\frac{n!}{(n-6)!}\\ \frac{n!}{(n-7)!}\cdot\frac{(n-6)!}{n!} &= 9\cdot\frac{n!}{(n-6)!}\cdot\frac{(n-6)!}{n!}\\ \frac{(n-6)(n-7)!}{(n-7)!} &= 9\\ n-6 &= 9\\ n &= 15\end{aligned}$$