How Do Functions Change? Let Me Count the Ways...

Three Case Studies Concerning the Nature of Students' Understanding of Graphs of Functions

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C & I 77225
December 11, 2001

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Rationale

Guershon Harel and Ed Dubinsky argue (1992) that

> the concept of function is the single most important concept from kindergarten to graduate school and is critical throughout the full range of education. Arithmetic in early grades, algebra in middle and high school, and transformational geometry in high school are all coming to be based on the idea of function. (p. vii)

Mathematicians and students of mathematics use several different representations of functions, of which tables, graphs, and analytic formulas are the most common. The understanding that all represent the same general concept, as well as understanding the limitations of each, is fundamental to understanding the function concept (Sierpinska, 1992). Sierpinska claims that developing the ability to interpret a graph or table is not trivial for students and it is in the course of learning this ability that "the fundamental acts of understanding find favorable conditions to occur" (p.49).

This paper is the result of the author's curiosity about students' understanding of the concept of rate of change, particularly as it relates to graphs of functions. Would students recognize a constant rate of change if given a graph or a table of values? How easily do they make the connection from table to graph -- in general and with respect to rates of change? Would they use their knowledge of constant rates of change to match a table with its graph or to interpret a given graph? What strategies do they use and what are their personal constructions as they attempt to interpret a given graph? Do special circumstances or characteristics of a graph make a student more likely to interpret it correctly?

Bell and Janvier (1981) identified several aspects of understanding found to be significant in their work of student's conceptions of graphs. Among these are:

> 1. the recognition of *global features* by a progression from point reading to interval and to gradient;
> 2. *measuring* intervals or gradients and *comparing* intervals or gradients;
> 3. various *distractors*, in particular pictorial distractors, when the shape of the graph is confused with that of the hill being climbed or the race track being traversed; and
> *situational* distractors, when experience of the situation interferes with attention to to the meanings of the abstract features of the graphs. (p. 37)

These aspects of students' understanding of graphs are investigated in the three case studies that follow. The case studies involve college freshmen enrolled in college algebra, the entry-level mathematics course for non-majors, at a large Midwestern public university. Two students are male and one is female. The analysis begins with LD, a female.

Case Study I: LD

Episode 1.

INT: [TASK] Look at the data in these tables and without plotting points, choose the best sketch from this set of graphs that would match each set of data (Bush & Greer, 1999). (See Episode 1 in Appendix .)

LD: Um. the cooking times for turkey, I notice that for every two times that the weight goes up, you have to cook it 1/2 hour more. Let's see, it would be a graph going up this way, it's an even chart, so...
INT: It's an even chart, what do you mean?
LD: Like, for every two it has to be a half-hour. So...I mean, it has the same variables...each...
INT: It has the same variables each time? I'm not sure I know what you mean, it has the same variables each time.
LD: Um, well it's like the same time, a half hour, a half hour for every two pounds, additional from the 6 pounds.

LD's first comment and her reference to an "even chart," though not a conventional use of vocabulary, suggests that she noticed that the rate of change was constant in the table. She immediately looked at choices A and B, the linear graphs. The above conversation does not indicate, however, if she was focusing on the rate of change of the independent variable or the dependent variable. Her strategy was more evident below:

LD: For each month that the baby grows older (she's looking at Table #3), he gets longer, let's see, the first one is 5, the second one is 7,..., increases by 6, by 4, 4, and 4. Okay, I was just determining between each month that the baby grows, how long that he grows between each month. And so ...I discovered that the age would be the x-axis and the length would be the y-axis, and let's see, it goes up and then it kind of ... it goes back down.

Note that LD is looking for differences of the dependent variable only as she attempts to find a graph that matches the data. She apparently realizes that the rate of change concerns the dependent variable and has little difficulty identifying which variable is dependent upon the other. Her relative ease in doing so concurs with research which also found that students have little difficulty doing so when *time* is the independent variable. LD is not rigid in this thinking, however, as evidenced by the fact that time was her *dependent* variable in the "Cooking Times for Turkey" scenario. When asked how she decided upon the dependent and independent variables, LD chose the independent one as the one which "...almost increased the same or evenly throughout rather than the dependent which might have not changed the same." To LD, the independent variable generally has a uniform rate of increase, while the dependent one may not.

LD also made an appropriate connection between the rate of change and shape of the graph:

LD: I was determining how much the baby grew in length in between each month that he got older. Between the second and 3rd month he grew 5 cm and between the 3rd and the 4th he grew ...
INT: Can you make any conjectures or any conclusions about the rate of change in this particular one? Does this rate of change differ in any way from the rate of change that you looked at for the cooking turkey time?
LD: It didn't increase in a steady increment. Um... it varied for each month. It didn't stay constant, or

stay 4, or 5...
INT: And how does that affect the way the graph looks?
LD: It wouldn't be a straight line.

When analyzing the data in Table 4, "After Three Pints of Beer..", LD similarly makes the connection between a constant rate of change and the linear nature of the graph:

LD: And, let's see here...I'm going to have to go with E. The graph of E as closest to the way, um that this graph is, because it goes down at a continual rate, um and since between the increments of the time and the alcohol in the blood, it's even as well. So...
INT: By even you mean...
LD: Even, going down 15 mg each hour.

Further, when analyzing the "Life Expectancy" data in Table 6, LD made a similar connection:

LD: ...I know for sure it's not A, it's not B, it's not C, it's not D, it's not E...
INT: This one starts going down, why wouldn't it be E? E is decreasing...?
LD: Um.. because E it goes down at a steady rate, whereas the time would have to be the same number between each year, where it starts at 5, then it skips to 10, so there's 5 years in between , but then it goes to 10 years in between, so and on the number of survivors is it the same for each of the years, so...um.. So I wouldn't pick E. It wouldn't be F because it doesn't level off to the same number of survivors for a certain number of years. Um.. let's see here.... It could be H...Okay. It wouldn't be I, it wouldn't be J,.. I'm going to have to say I have it narrowed down between H and L.

To LD, a constant rate of change in the dependent variable results in a graph that is a straight line. If the rate of change is not constant, the graph is not a straight line. The independent variable generally increases in a constant fashion, but does not determine the shape of the graph. Interestingly, when given the opposite task of looking at a graph and determining an appropriate table of values (see Episode 1B in Appendix), she was a tad less consistent. Though she correctly identified when the graph of a parabola (graph *p*) was increasing or decreasing and correctly noted its symmetry, she did NOT square the independent variable values even though she recognized the graph as a parabola. Note her table of values for graph *p*.

(p)

x indep	1	2	3	4	5	6	7	8	9
y dep	2	4	5	6	7	6	5	4	2

Monk (1992) differentiates between a *pointwise analysis* of a graph and an *across- time* analysis. A pointwise question asks for values of a function for a specific input value. An *across-time* analysis involves asking students to describe a *pattern of change* in the value of a function that results from a *pattern of change* in the input values. LD successively did an across-time analysis given the tables of values, analyzing the patterns of change in the output. She was less able to do this when given the graph first, however, as evidenced by her inability to recognize the squaring pattern of the parabola. Perhaps her difficulty relates more to the non-linear nature of the graph than her ability to analyze *across-time*, since she did note the increasing and decreasing nature of the graph, as well as its symmetry correctly.

Monk maintains that students find *across time* questions more difficult than *pointwise* questions.

Episode 2.

LD's confidence and consistency in recognizing a constant rate of change from a table were not so evident when asked to analyze graphs qualitatively, i.e. *across time*. Her difficulties here support Monk's (1992) claim that *across-time* analysis is difficult for students.

INT: [TASK] The rough sketch in the figure below describes what happens when three athletes, A,B, and C enter a 400-meter hurdles race. Imagine that you are the race commentator. Describe what is happening as carefully as you can. You do not need to measure anything accurately (Bush & Greer, 1999). (See Episode 2 in Appendix.)

LD: .... took runner C the longest to run the 400 meters, it looks like he started out the fastest and kind of slowed down. it took him longer to finish the 400.
INT: How can you tell it took him longer to finish the 400?
LD: Um... probably at about 20 seconds, he kind of planes off. and then , um... is at a steady rate for a little while *(pointing to the horizontal section of graph of runner C)* and doesn't keep going.
INT: So this here *(pointing to graph)* means he's going at a steady rate?
LD: Right
INT: Where it's horizontal?
LD: Right

Monk (1992) maintains that one source of difficulty students have with *across-time* analysis is their incomplete understanding of relevant concepts. Bell and Janvier (1981) called the same difficulty one of a "situational distractor" (p. 37). Here, the related concepts of speed, distance, and time come into play. LD appears to be able to be unable to completely integrate the three. She can use the concepts for some purposes (e.g. identifying the winner by point reading) but is unable to correctly differentiate change in position with change in speed. She appears to have what Monk calls a "blurred concept" of these ideas, "sometimes fused, conflated, or exchanged" (p. 176). She has exchanged the concept of change in position with that of a change in speed.

Episode 3.

LD's tendency toward a point-wise analysis and her ease in doing so is further evident in the following conversation:

INT: [TASK] This graph shows how long it takes students to leave the building during a fire drill. How many students were in the building before the fire drill (Leinhardt, Zaslavsky, & Stein)? (See Episode 3 in Appendix.)

LD: 400 at the beginning. " How many students were in the building after 10 seconds?" *(reads question)* I go to 10 seconds on the x axis follow it out to the line, there's 350. "30 seconds?" *(reads question)*. Follow it out again, there's 250. And 40 seconds, there's 200. " How many seconds had passed when there were only 50 students left in the building?" (*reads question).* I'm finding 50 students on the y-axis and I'm following it over until it meets the point, on the x and it was 70 seconds. "How many students did it take for all of the students to leave the building" (*reads question)* It took 80 seconds by the time it reached 0.

Bell and Janvier (1981) similarly found that reading and plotting points was successfully done by 90-95% of pupils in their study of some 1400 pupils in British secondary schools.

Episode 4.

The following analysis by LD again shows her relative ease in interpreting graphs when the independent variable is time. Since only two variables are involved, (time and plant height) as opposed to the three mentioned in Episode #2 (speed, distance, and time), LD is apparently able to escape Monk's (1992) blurring of concepts and Bell and Janvier's (1981) situational distractors. She successfully performs *across-time* analysis. Time is a concept, of course, very familiar to all, and plant height is a concept familiar to LD, perhaps more so than that of distance, rate, and time, sometimes thought to be of more interest to males.

INT: [TASK] Sue plans to study the effect of growing sunflowers in different size pots. The graphs show four possibilities, four possible outcomes of her experiment. Which graph is best described by each of the following statements (Leinhardt, Zaslavsky, & Stein)? (See Episode 4 in Appendix.)

LD: Okay... "as the pot size increases, the plant height decreases." *(reading question).* Okay, so...pot size is on the x, plant is on the h, so this increases, decreases. *(pauses..takes quite a bit of time).* I think it would be B. It would depend on the height of the plant that's starting. Um.... Yes, If it's starting at zero for the height of the plant and it's increasing, then on the x as the size of the pots are increasing, then the height of it goes down.
INT: OK, so your choice is...?
LD: B.

When asked in later episodes to sketch a graph of rates of change vs. time, (temperature of ice water over time and number of passengers in a commuter train over time), LD had little difficulty doing so accurately. When *time* is the independent variable, changes occurring in the dependent apparently cause LD little difficulty, particularly when she fully understands the concept represented by the dependent variable.

Episode 5.

LD's difficulty with time, distance, and speed is evident, again however, in the following episode:

INT: [TASK] Which of the graphs below could represent a journey, a trip? Describe what happens in each case (Leinhardt, Zaslavsky, & Stein). (See Episode 5 in Appendix.)

LD: Um it looks like in C, starting time is 0 and distance is 0 and they start going a faster pace so they increase their distance more at a faster time and it looks like they slow down their time and their distance (*pointing decreasing portion of graph)* and then they're increasing their time again because they're going a little bit faster with their distance again .
INT: Let's get back to C again. Could you just explain what's happening in that graph?

LD: And then once they get to this point, it looks like, um as the time goes on they slow down so that the distance, um . . .
INT: How can you tell they slow down?
LD: Um because it drops from that point, to that point, and then it increases again,
INT: ...how do you compare how he's going?
LD: He's going at a faster rate.
INT: Because it's going up?
LD: Yes

LD again is confusing the change in distance with change is speed. I maintain that part of the problem is her lack of familiarity with the distance, rate, and time scenario.

Episode 6.

Further evidence that LD uses *pointwise* analysis easily follows. In the following scenario three variables are changing (gender, time, and weight) but two different graphs are given, thus separating the differences in gender. Note also that the concepts (gender, time, and weight) are not only very familiar ones, but also those that certainly would not lend themselves easily to Monk's (1992) "exchanging" of concepts. The questions begin by requiring simple point-reading, then three types of interval reading: an *x*-interval corresponding to a *y*-interval, an *x*-interval defined by a certain condition i.e. G>B, and finally an interval of greatest rate of increase. LD handled each with relative aplomb:

INT: [TASK] We're looking now at the graph of weight in kg of boys vs girls over time. And if you would read the questions out loud (Bell & Janvier, 1981). (See Episode 6 in Appendix.)
LD: "The average weight of boys at age 9 is..." So On the x axis is the years and the y is the weight. Age 9 on the x. And then, trace up, find the straight line for which boys are. And go over to the y-axis and it's about 25 kg.
LD: "Girls at age 17". I'm going to find 17 on the x-axis, and trace that up to the dotted line which is the girls, and trace that over to the y, and that's about 58.

Note that when asked to read intervals in the discussion below, LD traces back to each axis rather than looking at the *global* characteristics of the graph. This finding is similar to that found by Bell & Janvier (1981): "More sophisticated methods, such as reading the difference directly form the scale on the axis, without subtraction, or reading it even more directly from the grid in the body of the graph, were rarely used" (p. 37).

LD: "From what age do the girls on average weigh more than 20 kg?" So, ...find that...It's about age 8. (FINDS 20 ON Y- AXIS, DRAWS LINE TO DOTTED LINE, THEN DOWN TO X-AXIS).

Below, she initially exhibits confusion of rate of growth with height, an error found frequently in the literature (Clement, 1989), though with questioning, she is able to correct her error:

LD: "When, at what ages do the girls weigh more than the boys?" Let's see here. Girls weigh more than the boys from about age 12- 15 because this is increased more on the chart.
INT: Because it increased more on the chart? (APPARENT CONFUSION -- RATE VS HEIGHT)

LD: Between this age range, the dotted line, which is the girls, is more than the boys.
INT: Is more in what respect?
LD: More in weight.
INT: More in weight? What numbers are you comparing? You said two things, that's why I'm asking ...is it increasing more, or weigh more?
LD: Oh, They weigh more. They weigh more here, between the age of 12 and 15.
INT: How can you tell?
LD: Because um, the dotted line for the girls is higher on the graph in the weight range.

She had little difficulty, however, answering the following *across-time*, qualitative question and did not, contrary to Bell & Janvier's findings (1981), refer back to the axes here. Rather, she refers to the characteristic of the graph, itself, correctly relating rate of change to the steepness of the graph. Again, I maintain that her familiarity with the scenario aids in her interpretation.

LD: " At what age do the girls put on weight the most rapidly?" *(reads question)*
LD: Let me see, I would think that they'd put on the weight most rapidly when it, um...when the line of the girls is going straight up more than leaning off. Um... so therefore, right here the ages 12-15 again is when it increases the most.

Episode 7.

In the following scenario, LD avoids making the height vs. rate error, but reverts to the less sophisticated method of referring back to the axes:

INT [TASK]: This is a position vs. time graph. At the instant when t = 2 seconds, is the speed of object A greater than, less than, or equal to the speed of object B? (Clement, 1989).
LD: *(repeats question)* so... um...*(long pause)* Let's see, the whole position thing is throwing me off.
INT: That's okay. Just think about it.
LD. Let's see, I notice that B starts at 0, and A starts a little bit higher as the position, so um... the speed of the object,... um, ... I think it's going to be... less than....I think?
INT: And why are you thinking that?
LD: Because since B starts at 0, and when it gets at 2, since A has already started further along, then by the time it's at 2 it's taken longer so I think A is less than B. (ANALYZES POINT-WISE, NOT SLOPE), if that makes sense.
INT: Does it make sense to you?
LD: I think so.

Another error mentioned frequently in the research is the tendency of students to treat a graph as a literal picture of the problem situation. Clement (1989), Bell & Janvier (1981), Monk (1992) and Sierpinski (1992) all noted this as a common problem among students. Clement differentiates between a *global correspondence* error and a *local correspondence* error (p. 83). In the latter, a visual feature of the problem scene (e.g. the same location of the cars) is matched to a specific feature of the graph (i.e. the point of intersection). This latter type is evident in LD's interpretation below. She is able to interpret the changes in velocity correctly, but insists on interpreting a specific point as a *location* vs. time rather than the *velocity* vs. time. Her interpretation illustrates an apparent reversal of her preference to correctly use a point-wise analysis. Here she interprets across-time correctly but has difficulty with a point-wise interpretation:

INT [TASK]: This grid shows speed or velocity vs. time for two different cars. Describe the relative position of the cars at t = 1 hour. Now they're both starting at the same spot, this is a velocity vs. time graph. (Clement, 1989) (See Episode 7B in Appendix.)

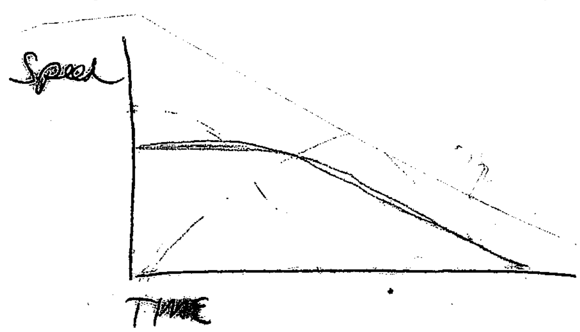
LD: Ok, at one hour, um... let's see... the bottom line, the velocity started out a little bit slower, but then increased more towards the one hour period, to get to that point. And the top line, seems like they started at a higher velocity, remained at that higher velocity, which is kind of, it's gradual until they got to that point, so they did meet at the end, like right about here, this one started increasing velocity more to get to that one other point.
INT: So at this point *(points to the point of intersection)*, what's happening with the two cars?
LD: They're meeting.
INT: They're meeting?
LD: Yes.

LD continues to see graphs as pictures in the following two episodes. In both, she is making Clement's *global correspondence* error; i.e. she is confusing the graph, as a whole with the problem situation. An incomplete understanding of velocity may be distracting her. It is significant that she was correctly able to interpret rate of change in Episode 6, when considering rates of growth of children.

Below, she made the error when asked both to *draw* a graph of a situation and to *interpret* a given graph:

<u>Episode 8.</u>

INT [TASK]: Draw the shape of the graph of speed vs time for a bike rider coasting over the edge of a hill. This is speed *(pointing to y-axis),* this is time *(pointing to x-axis)* (Bell & Janiver, 1981).
LD *(repeats question).* So, if he's coasting, ...hm... I'm not really sure...
INT: Just imagine what's going on, if you're riding a bicycle up a hill and then go over the top.
LD: Over the top, then the speed, going over the top would be like that,... maybe... but once you go down the hill, you start going faster... and maybe it would be more less of the time, ... *(long pause)* um...but...
INT: Whatever you think....



LD:.... would be more like that one right there.... The speed, let's see, I guess that would be the speed is decreasing as time's progressing, but if you're going down the hill, you're going faster, so maybe it should ...hm...(laughs) ...hm...(REALIZES SPEED INCREASING, BUT STILL DRAWS HILL)
INT: It's harder to draw from scratch, it just is...
LD: hm...*(long pause).* I'm second guessing myself.
INT: That's Okay, that's called thinking.

LD: um... *(rereads/repeats question).* Um.... going over the edge. if he's coasting, you're kind of going at the same speed, going over the edge of the hill, as time's increasing and once you go down the hill its like that. Yes.
Like that. (CONFUSION, COASTING = SAME SPEED)
INT: Are you happy with that?
LD: Yes.

Interestingly, LD notices the contradiction inherent in her sketch, but still is unable to correctly draw the graph.

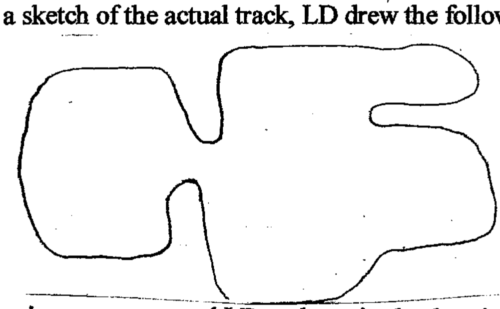
Episode 9.

INT [TASK]: The following graph represents the speed of a racing car as a function of its distance along the track. This time, this is the distance along the track. This is [on] a racetrack of some type, this is the speed at which it's going up along the y-axis here. The speed is a function of its distance along the track. I want you to study it for a bit. Can you tell me, from the graph, how many bends there are along the track on which the car was driving (Bell & Janvier, 1981)? (See Episode 9 in Appendix.)
LD: Okay, let's see here,... looks like speed is starting off at 160, here's one bend, here's another, so two, three, four five, six, seven, eight, nine. Nine.
INT: So you think there's nine bends in the track?
LD: Nine bends

Even when encouraged to rethink her interpretation, LD insists (incorrectly) that there are nine bends in the road, i.e. she sees the actual number of bends in the graph as the number of bends in the actual track:

INT: Look at the graph again, study it -- there's no hurry, think about what you've answered, see if you can sketch a graph of the actual track in the space below.
INT: This is not necessarily a picture, remember.
INT: How many bends do you have in there?
LD: (counts.) 1, 2,3,4,5,7,8,9
INT: Let's go back a minute, let's look at this graph again. Remember that this graph is not a picture. This is graph of distance that's traveled and the speed. Okay. Convince yourself again how many bends there are on the graph. This is the distance here *(points to x-axis).* How many bends?
LD: Hm.. I still think there's nine.

When asked to make a sketch of the actual track, LD drew the following, attempting to sketch nine curves.



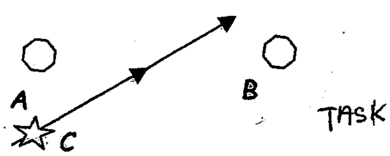
It wasn't until the interviewer encouraged LD to do an in-depth point-wise analysis that was she able to correctly count only three bends in the road. It took several minutes of questioning to do so. She strongly held the "graph as picture" misconception. Only her tried and true strategy of *point-wise analysis* eventually convinced her otherwise.

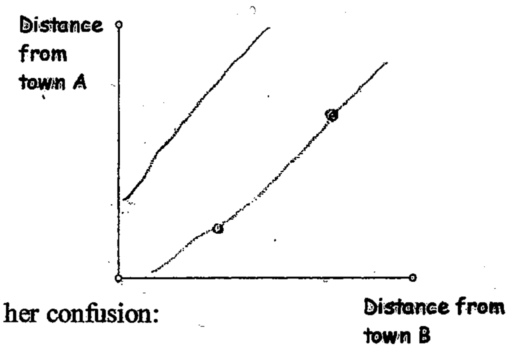
One last episode illustrates LD's difficulty with *situational distractors*. As mentioned previously, LD has had relatively little difficulty sketching a graph when time was the independent variable. I was curious if indeed LD would have difficulty making a sketch of a common, everyday situation when *neither* variable was *time*. That she had a great deal of difficulty is evidenced below. She clings to her "graph as picture" interpretation as she attempts the following, albeit difficult, task:

Episode 10.

INT [TASK]: This is a car, at C, this little hexagon is town A, this hexagon is town B, and the car is traveling along the road in the direction of the arrow. It's following along the road in a straight path at a constant speed. What I'd like you to do is draw a graph here. Along one axis is distance from town A and the other axis is distance from town B (Leinhardt, Zaslavsky, & Stein, 1990). (See Episode 10 in Appendix.)

LD made the following sketch:





Her belief that *time* is the independent variable leads to her confusion:

LD: hmm…OK. Car C traveled at a constant speed along the road. Complete the sketch of the graph. So the car is just going straight in-between from town A and town B….so.. hmm..OK. This is the time, no I guess this isthe time, hmm.. oops

LD: distance from town B, with the …I'm not really sure what, almost think that graph would almost be in the same direction that the arrows are going.
INT: the graph would be in the same direction?
LD: right. I really don't know if there's anything else I need to add. I don't think so.
INT: So you have two that are kind of parallel. Do you want to think about that a little bit? Point to a particular point on the graph and see if it would fit into what you're thinking about it.
LD: hm…*(thinking a long time)*. I think it would be more like this one. *(points to lower graph)*

Clearly, LD saw her sketch as the path of the car and the endpoints on the axis as the actual locations of the towns. When I encouraged LD to do a point-wise analysis, a strategy that has helped her in the past, she was still unable to get past her "graph as picture" interpretation:

INT: If you put your point here, what does this point on the graph mean?
LD: The distance that it's traveled away, more towards B, away from A.
INT: At that particular point, how would you interpret that point on the graph: in terms of the distance from town A and the distance from town B?
LD: Um……I'd say it's closer to town B, *(at lower point)*, rather than A, um….
INT: So does that reveal sort of what's happening on the graph?
LD: Yes, uh huh…The further along you're moving you're moving more away from town A and moving closer to town B.

Case Study 2: AB

Episode 1.

INT: [TASK] Look at the data in these tables and without plotting points, choose the best sketch from this set of graphs that would match each set of data (Bush & Greer, 1999). (See Episode 1 in Appendix .)

From the table of values AB did recognize one of the global features of the data: the fact that it was decreasing. He also had little trouble, as did LD, identifying the independent variable:

AB: OK. Um… on the x-axis I usually put the time, because that's how I was taught is what increases by equal increments usually goes on the bottom and since it's usually time I place that on the bottom on the x. Um… let's see 5, 10, 15… It has a negative correlation. As the time increases, the temperature decreases and the graph slopes down and to the right.

Though he noticed that the decrease in temperature was not uniform, he chose a linear model for the graph:

INT: Is there any pattern in the changing that you're noticing?
AB: The temperature doesn't seem to be uniform or anything, the time is just measured in increments increasing by 5 minutes, it doesn't, let's see, that's five, … it doesn't look like in temperature there's any, umm…
INT: So you do see a pattern?
AB: Yeah, I do see it there, Just working the other way, I didn't. I was working left to right instead of right to left.
AB: For the first one, the cooling coffee one, I went with E because it shows, decreases….
INT: And why did you pick E for that one?
AB: Because on the graph that I just did, it shows that it decreases over time and that's about the same picture.

As did LD, AB correctly focused on the rate of change of the dependent variable, though was more rigid in his insistence that *time* is *always* the *independent* variable, even when the situation suggests otherwise, as in the data of table #2. Later, AB comments that "…because of what I've done with all my classes in like, physics and my algebra II trig and my algebra classes, I just remember, like, how my graphs were supposed to look and all that kind of stuff. It's just stuff that I know, I can't describe it." His comment suggests a rote, rather than real understanding of the concept of *independent variable.*

AB: Now on time, for the cooking turkey one, it looks like the time increase and the weight increases, um…it's either A or B because both graphs increase, uum..
INT: When you say increasing, what do you mean?
AB: It's uhm on the x axis is time. So as the time increases the weight would increase and have a positive correlation since they both increase. um…

To AB, "correlation" meant slope, or rate of change. Again he noticed an important global feature of the data and thus of its corresponding graph. Interestingly however, even when prompted, AB did not notice any differences or similarities in the *type* of rate of change and has a tendency to think in linear terms.

INT: Is there anything that's similar or different in the ways that your y values are changing between the

three graphs or tables that you've looked at so far?
AB: It's changing?

INT: IS there a relationship between the two variables?
AB: A relationship between the two variables? Well yeah, as the time increases, the temperature decreases. That's the relationship. and then the weight and the time increases for number 2, they both increase,...

When looking at table #5, whose dependent values are clearly NOT increasing linearly, AB initially chooses a linear graph:

AB: and in number 5...
INT: What's happening there?
AB: The years are of course increasing and so is the number of species. It starts out at 0, 1880, so that would probably be A because both the time and the number of species increases and since it starts out at the origin... Hold on again... Maybe it is B. (N.B. ONLY LINEAR THINKING! THIS IS NOT LINEAR!).

A *pointwise* analysis of the data and graph caused a perturbation in AB's thinking, however, and he was able to eventually make a better choice of graph to match the data.

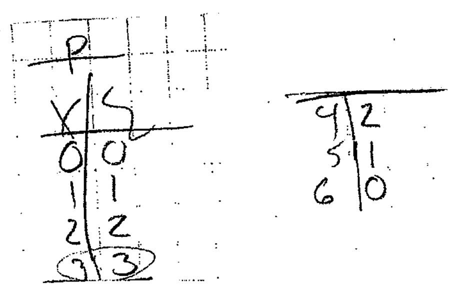
AB: ...Time starts in 1880 and number of species is 0, so it would be. Starting on the x-axis. Yeah I think this one is...I think it might actually be K.
INT: What makes you think that?
AB: Um Time line start down. The uh graph will start down on the x-axis and then increase slightly and then it takes a big jump between 1900 and 1910 uh and it goes from, let's see, a big jump right about here huge increase. Then from 1920 which would be like right at the top of the graph at K it flattens and remains constant, uh, like 30 ...species from 1920 - 1940 and the graph shows that it flattens out right there. So I'd have to go with K for #5. .... I keep thinking....
INT: You keep thinking.....about what? (PERTURBATION!!!)
AB: Back on like #3 and #4, because of what I said that it starts out at zero. Ah... I think it will be okay. Do you want me to label the axes?
INT: Why did you think some of these others weren't right?
AB: Paranoia. Because I had, I just, uh... For some reason I just thought it didn't ...like where the graph actually touches the x-axis didn't make sense too much.
INT: Was there something in the data ?
AB: Yeah, I don' t think I looked at it took well.
INT: I notice you were going to pick A or B for this one and then the fact that it leveled off seemed to be what attracted you to this one. But actually the early graph here in A is different than the early graph here in K and I'm wondering how you reconciled those differences.
AB: This curve here shows that there's a big jump, which means that it's not like uniform...
INT: What's not uniform?
AB: It increases, like by, like the species here from 1880-1890 increase by one and then 1890-1900 increases by 4, then...it's not like a , it doesn't increase by, say like 5 species then 10 then 15, like this one (#1) and I'm thinking that maybe...

In his attempt to *compare* the rates of change, however he exhibited some initial confusion with the concept of *uniform change,* thinking that the rate of change may be uniform if the *second differences*, rather than the *first differences* remain constant. This confusion may be the reason he chose a linear graph initially for table #1. With questioning, however, he is able to reconcile his differences and change his choice for table #1:

INT: Was this rate of change constant in #1? How does the rate of change...

AB: The temperature?
INT: Yes.
AB: It decreased by one less than the one before it.
INT: How does that differ from the way you analyzed that one there (*table #2?)*
AB: It increases by 2 every time in the second one. So that means it's like a steady increase. And since this one is...like at 25 min if that was n, the temperature, well the temperature at 25 minutes is 49. This would be like an n-5, if this was n, then this becomes n-6, then this is like that quantity, n-5. I can think it, but can't say it. It's confusing....
INT: Is that any different than what's happening in #2?
AB: Yeah. Because this increases by 1/2 every time. SO this is half and hour, this is half an hour, this is half an hour. And this is 7 degrees, 6 degrees...
INT: Will that affect the way it looks on the graph then?
AB: I think so. I just don't know... I think it might be like a curve. like #1. Because I know that #2 if the time increases by 1/2 hour then the weight increases by 2, the line is going to be straight.
INT: How do you know that?
AB: Cuz the ,um, that (sighs), ...there's no real change in any.. uniform change between the time and the weight and the weight being 2 pounds every time and the time being half hour every time. There wouldn't be any bend in the graph to bring it. It's kind of hard to say...
INT: Then for table #1, which graph would you chose?
AB: This one. I think it might actually be F, maybe G. It does decrease and there's a bend in G and the temperature won't reach zero like the graph in E is showing. So it would have to eventually just bottom out.

When asked to create a table of values to match the given graphs (see TASK IB in the Appendix), he, like LD before him, noted some global characteristics correctly, but had difficulty if the rate of change was not linear. His tendency to linear thinking reveals itself in his table for graph *p:*



Episode 2.

Any difficulties AB had with uniform rate of change while converting from table to graph or vice versa appeared negligible when asked simply to *interpret a given graph,* rather than *construct* one. Leinhardt, Zaslavsky & Stein (1990) differentiate between these two tasks and note that "releasing the student from constructing a graph from scratch with paper and pencil alters the task domain enormously" (p. 12). Note AB's relative ease in interpreting the graph in the task below, even in terms of constant vs. non-uniform rates of change:

INT: [TASK] The rough sketch in the figure below describes what happens when three athletes, A,B, and C enter a 400-meter hurdles race. Imagine that you are the race commentator. Describe what is happening as carefully as you can. You do not need to measure anything accurately (Bush & Greer, 1999). (See Episode 2 in Appendix.)

AB: Ok, um...,it looks like racer A, the first runner, looks like they keep a pretty constant speed over time related to the distance.

INT: How can you tell that?
AB: It just increases, it looks like they slow down a little bit, but it's basically a straight line. There's a little bit of a bend, um...
INT: So what's increasing? I'm not sure...
AB: OF course, time goes on and distance goes on, but it looks like they keep about the same, I would say about the same speed and then they just slow down at the end., um... and then racer B does about the same. They keep a constant speed, maybe perhaps at the end, they sprint, to the finish line, because the line goes up toward the end. Um... and then runner C is like, ... is again running the fastest at the beginning and then they meet, they're keeping the same distance, as time goes on they're, looks perhaps maybe they hurt themselves, because they're staying at whatever this distance is, right here *(pointing to the horizontal line).* As time goes on, so they fell maybe and get right back up and continue on with the race, and run at a constant speed.
INT: How can you tell it's a constant speed?
AB: Because it doesn't uh, it's a straight line.

AB does not appear to share LD's confusion with the related concepts of distance, rate and time. It is interesting to note, however, that as LD frequently did, AB resorts to tracing back to the x-axis and y-axis for a *point-wise analysis*:

AB: Um... for the most part, I'd say, it looks like, um, this it the, uh, the 10 second mark that I drew on the x-axis,...10 meters, maybe. At the 10second mark that I drew, it looks like B would have been at 10 meters I drew in and A would have been at 20, so it looks like A would have been winning for the most part. because A slacked off..

Episode 3.

AB had no difficulty with *pointwise analysis* in the following task. He noted the significant value on the one axis, then traced back to the corresponding point on the other axis. He did so confidently and quickly.

INT: [TASK] This graph shows how long it takes students to leave the building during a fire drill. How many students were in the building before the fire drill (Leinhardt, Zaslavsky, & Stein)? (See Episode 3 in Appendix.)

Episode 4.

AB did an *across-time analysis* with ease while answering the task in Episode 4. Note the independent variable is time and that the task involves only two variables (time and plant height) as opposed to the three mentioned in Episode #2 (speed, distance, and time). He did not resort to tracing back to the axes at all, but nicely and efficiently matched the graph with its corresponding statement. The fact that the choices were provided may have enabled him to do so without *point-wise analysis.*

Episode 5.

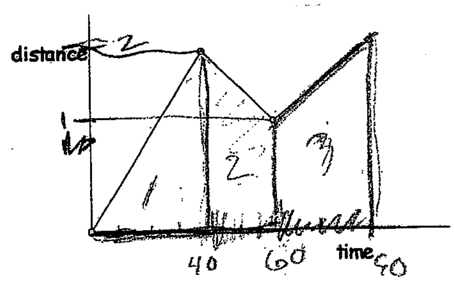
Even when corresponding statements were not given AB performed an accurate *across time analysis*. Here the situation involved three variables -- time, rate, and distance -- and he apparently did not conflate the three, as did LD.

INT: [TASK] Which of the graphs below could represent a journey, a trip? Describe what happens in each case (Leinhardt, Zaslavsky, & Stein). (See Episode 5 in Appendix.)
AB: A and B don't make any sense, because like right here time is going on and you're distance is staying the same, you're just sitting there, but all of a sudden, time,...In a very short amount of time you go from no distance, to a great distance,..

AB: No, that can't happen. And then B doesn't make any sense, because time is going on, starting here and it looks like it goes backwards, which can't happen, and then,..so I'd have to go with C. Time is going forward and the distance is getting greater, here you go backwards a little bit, as time goes on you go forwards a little bit more as time keeps going, so...

When interpreting changes in slope, however, he reverted to tracing values back to the axis, rather than comparing the overall slant of the line:

INT: Can you tell how fast you're traveling at different times?
AB: It's like between the second and third points would be it looks like a shorter, a greater distance over a shorter time.
INT: How can you tell that?
AB: I drew the lines down from the second and third point and it's taking up less amount, or less time than the other two sections that I drew. And it looks like, maybe equal between the first section and maybe the third section.



(c)

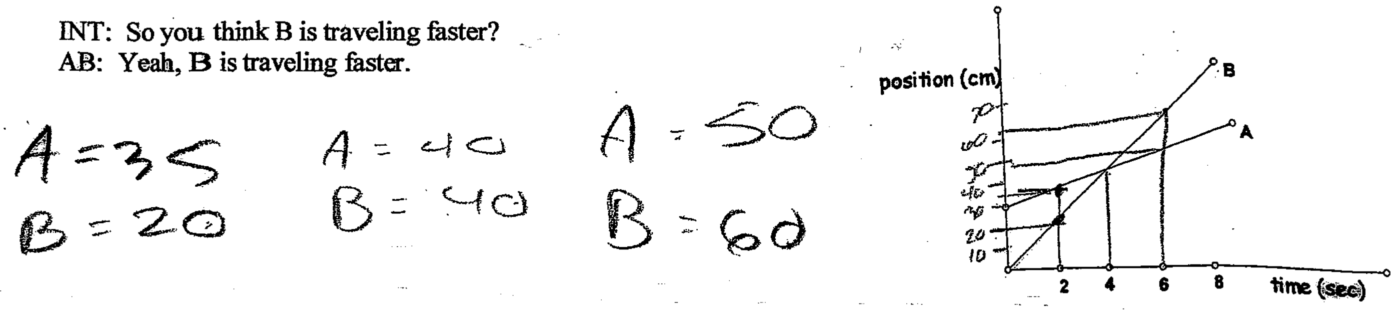
AB used the same strategy while comparing rates in the task below.

Episode 7.

INT [TASK]: This is a position vs. time graph. At the instant when t = 2 seconds, is the speed of object A greater than, less than, or equal to the speed of object B? (Clement, 1989). (See Episode 7 in Appendix.)

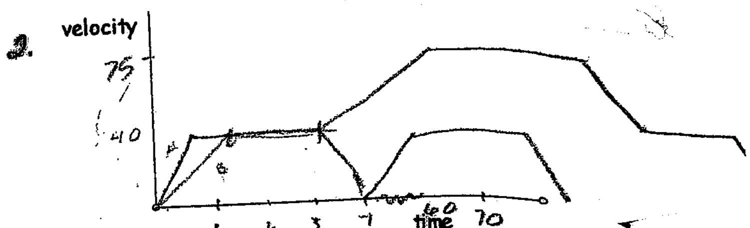
AB: Time = 2 seconds, looking at object A, here's it's position, and then, here's object B, um...it's position is greater than. Because, if I were to draw in lines again in the position, here's 10, 20, 30, 40 ,50 units. At 2 seconds, object B would be at 20 and object A would be at 35, so A is 25, B is 20. So A's distance would be greater than B's at 2.
INT: A's distance would be greater than B's at 2, okay, but at that particular time, 2 seconds, is the *speed* of the object A greater than, less than or equal to the *speed* of object B?
AB: oh the speed o f the object,... so this one, this is the distance, let's go to 4, (*writing points*), ...I'm just trying to plot points and then see how far in time, how far they've traveled at a certain amount. So... A = 50 seconds and B would be 60. It looks like B is traveling faster,
INT: You think B is traveling faster?
AB...because it's from the graph its slope is greater, and because of the numbers I've figured out it looks like B goes 20, 40, and 60 and A only goes 35, 40, and 50.

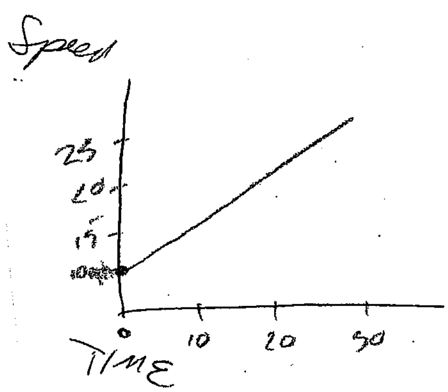
INT: So you think B is traveling faster?
AB: Yeah, B is traveling faster.



AB has an overall sense of the *global* characteristics of a graph, but uses *point-wise* analysis to confirm his hunches. In Bell & Janvier's (1981) terminology, he *measured* intervals or slopes in order to *compare* them.

AB's tendency toward linearity is noticeable in the following three sketches. In the first, I asked him to draw a graph of speed vs. time for a bike rider coasting over the edge of the hill; in the second he was to sketch a graph of two different cars travelling at different speeds vs. the time that they traveled. In the third, I asked him to draw a graph of the number of passengers on a commuter train as a function of the time of day.





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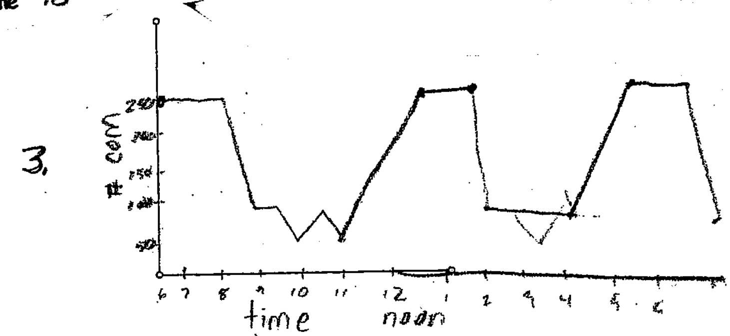


In graph #1 above, AB did not confuse the shape of the graph with that of the hill being climbed i.e. he avoided the common "graph as picture" error noted so frequently in the literature (Clement 1989, Bell & Janvier 1981, Monk 1992, and Sierpinski 1992). He graphed an increasing speed. The fact that he appears to have a clear understanding of the concepts of distance, rate, and time may have contributed to ability to draw an appropriate graph. That he draws a linear graph may be due to his tendency to graph linearly, rather than his wrestling with more the complex concept of acceleration:

AB: Since he's going down the hill, the speed would increase, because he's going faster, and then time would keep going on, so I have marked it like, 30 seconds on down the hill, he'd be going 25, it might not be accurate, but...
INT: You've drawn it that the speed is increasing at a constant rate, do you have any idea if that's true or not?

AB: Yeah, unless he, um… fell off, or hit, a you know a slope coming back up
INT: ..a mogul..?
AB: yeah . It would just gradually increase at a constant rate.

Perhaps I could have probed more deeply here.

In the second graph AB avoids making the *local correspondence* error that LD did:

INT: What does it mean here, *(pointing to where the two lines join)*
AB: Where they intersect? It shows where they hit that speed at that particular time it's just that one driver B hit 40 mph, like he finally hit 40, driver A had already been there for a minute and was coasting.
INT: Can you make any conclusions about where they are in their trip? Like, are they close to each other, far away from each other or can't you make any conclusions?
AB: At first A is farther away, if they start at the same time. Because that person already hit 40 mph and that person is already long gone and driver B is still behind, trying to get 40 mph and then they hit it.
INT: They hit what?
AB: They hit the 40-mph that this person already gone.
INT: Does this mean they're traveling together?
AB: Yeah, well…my story's got…
They'd be ah… they're both at 40 mph, where the two lines intersect, they're just at 40 mph.
INT: Okay
AB: They went off their separate ways.

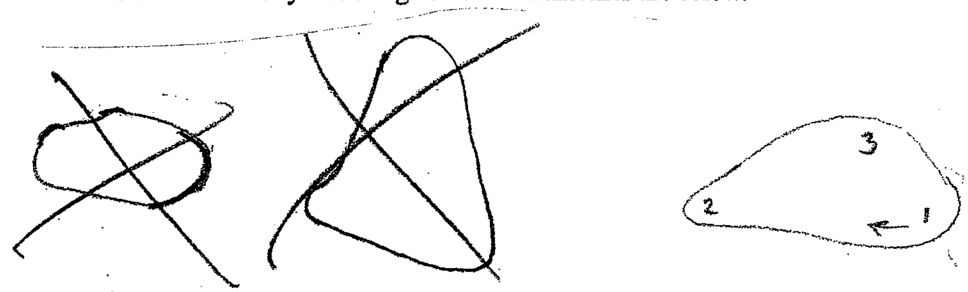
AB's apparent understanding of the related concepts of time, distance, and rate permit him to avoid making the common "graph as picture" error in the following scenario as well.

Episode 9.

INT [TASK]: The following graph represents the speed of a racing car as a function of its distance along the track. This time, this is the distance along the track. This is [on] a racetrack of some type, this is the speed at which it's going up along the y-axis here. The speed is a function of its distance along the track. I want you to study it for a bit. Can you tell me, from the graph, how many bends there are along the track on which the car was driving (Bell & Janvier, 1981)? (See Episode 9 in Appendix.)

AB: The car's going 100 mph to start out, then 0 distance, then it decreases its speed to go around a curve, it looks like, then it goes right back up. At that first dip, and then it does it again, so I'm saying there's probably 3 bends.
INT: Three bends in the road?
AB: Yeah.
INT: Tell me again, why you think there's three bends?
AB: Because it shows here that it decreases… you'd want to decrease to go around the curve, then increase when you're coming out of it. Then that's what it showed, three times... this second one, is bigger, because it decreases more while he's going around it and then he has to increase.

When asked to make a sketch of the actual track, he first drew a typical oval track, then settled on an appropriate three-sided track, accurate even in the degree of sharpness in the turns. He concluded that turn #2 was the sharpest because the decrease in velocity was the greatest. His sketches are below.

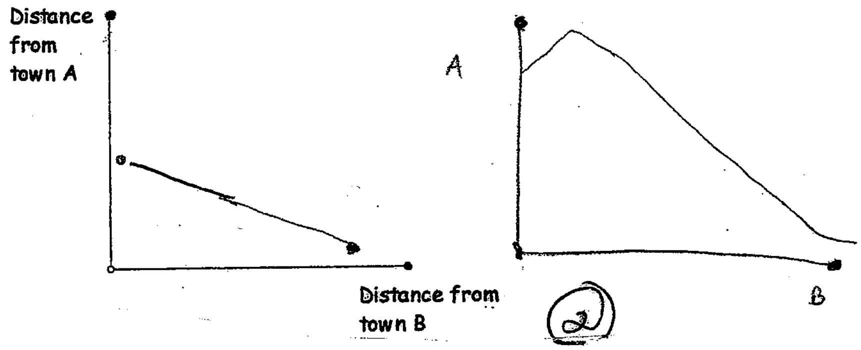


The following task reveals AB's difficulty in constructing a graph when *time* is NOT the independent variable.

Episode 10.

INT [TASK]: This is a car, at C, this little hexagon is town A, this hexagon is town B, and the car is traveling along the road in the direction of the arrow. It's following along the road in a straight path at a constant speed. What I'd like you to do is draw a graph here. Along one axis is distance from town A and the other axis is distance from town B (Leinhardt, Zaslavsky, & Stein, 1990). (See Episode 10 in Appendix.)

INT: And this is *(pointing to endpoint of y axis)*
AB: I'm going to say, town A.
INT: ..and this is
AB: town B.
INT: And what's measured along this...axis?



AB: The distance to town A? And this is the distance to town B. I'm just not sure...You're actually farther away from town A. He's far away, then getting closer to town A, closer, then farther away *(DRAWING PATH, I.E. GRAPH IS PATH OF CAR)*. So, he's going right in front of town A, then further away. So this IS town A *(pointing to point A)* and this IS town B *(pointing to town B)*. *(Speaks decidedly)* He starts off, getting, he's farther away from town A getting closer, town A, while still getting closer to town B. He's getting closer, then all of a sudden he's right parallel, then he starts getting farther away.
INT: Farther away from... A?
AB: Yeah, and then getting closer to B. I'm going with this one (*decidedly pointing to graph #2)*.
INT: Graph #2?
AB: Yeah.

AB finally succumbs to the "pictorial distractor", to the "graph as picture" error. The endpoints of the axes were the towns and his graph traced the path of the car. When *time* is not the independent variable, AB had the same difficulty LD did earlier.

Case Study 3: JS

Of the three subjects, JS displayed the most sophisticated interpretations, yet illustrated many of the common conceptions that both LD and AB did. One of these is the (apparently) widely held model that *time* is always the independent variable.

Episode 1. (See Appendix)

JS: Time is on the bottom, in minutes, temperature, in celsius.
INT: What made you put the time on the bottom? and celsius on the y axis?
JS: I always thought time goes on the bottom.
INT: Any particular reason?
JS: I just thought that that's the way it always went.

JS, like AB, appears to be following a set procedure, or rule, rather than thinking about the independent vs. dependent characteristics of the variables. While thinking about the data in table #2, he also assumed that *time* was independent, even though neither the situation nor the arrangement in the table suggested it:

Unlike AB, however, JS appeared to understand what causes a graph to be a straight line. Note his comments while trying to match the data in table #2 to its graph:

INT: Are there some you could eliminate right off the bat and why?
JS: The ones with the curves.
INT: Why could you eliminate the curves right off the bat?
JS: Cuz it's a constant growth.

He apparently was able to recognize the *global features* of a graph, without resorting to a *pointwise analysis*:

INT: So you could eliminate the curves.
JS: Right, anything with the curve. So it would just be a, b, or e, but since it's growing it can't be e. and....I don't think it could be A either, because it doesn't do linear at zero. It doesn't start there.....

He also appeared to understand the relationship between a constant rate of change and the shape of the graph:

INT: You think it might be I? *(Referring to table #3).*
JS: Yeah, because the slope increases. It starts off at 5 then goes to 7, then goes to 8, then it goes back down to 6, and then it stays constant, so it goes off into a line.

When asked about table #4, he again noticed the constant rate of change and related it correctly to the graph:

INT: After three pints of beer, number 4..
JS: I think that's E ...because there's a constant decrease and then it ends in zero, so it would be on the line, the x axis.
INT: So the constant decrease means....what are you looking for?
JS: Slope that's going down to the right.
INT: Could it be a curve, or does it have to be...
JS: No, it can't be a curve cuz it's constant.

Similarly he identified the correct graph for table #5:

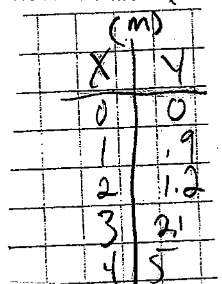
INT: Number of Bird Species on a Volcanic Island
JS: I think that would be K, because the slope slowly increases and then it goes to a constant. So the constant is level, and that's the only one that levels off.

Like both LD and AB, JS reverted to *pointwise analysis*, tracing back to each axis, when asked to develop a table of values for the graphs given in Episode 1B: (See Appendix.)

JS: I'm just labeling the x and y axis in the ones....*(long pause)*
INT: What did you come up with. Would you describe what you've come up with?
JS: I just labeled the x and y axes with numbers, then I just went up the different points and across to the other one to where it's about... and then wrote across what it is.
INT: Did your points change at all as you were working on it?
JS: Yeah. Here's a steady increase, *(pointing to first graph)* and then there's a little bit more between the two y

points.. . .in between there, the y points.

Interestingly, he did not choose coordinates that matched his thinking. He used his values marked on the axes, which were not necessarily accurate. His points in his table (see below) did NOT reflect a constant rate of change.



He made similar errors for the other tables. Constructing an accurate table of values is difficult for JS, even though he correctly recognized the *global features* of the graph. A *construction* task, is more difficult for JS than an *interpretation* task, as it was for AB earlier.

Episode 2.

INT: [TASK] The rough sketch in the figure below describes what happens when three athletes, A,B, and C enter a 400-meter hurdles race. Imagine that you are the race commentator. Describe what is happening as carefully as you can. You do not need to measure anything accurately (Bush & Greer, 1999). (See Episode 2 in Appendix.)

In this task JS, like AB, had little difficulty integrating the concepts of distance, rate, and time as he accurately interpreted and compared *global* features of the graph without resorting to a *pointwise* analysis:

JS: T the beginning of the race, C starts out in the lead, with A in second and B in third,...
INT: How are you knowing that?
JS: Because it's ...it has the higher slope, it's kind of like, slanted...Um...it has a greater slope
NT: And then what happened?
JS: Um...C slowly slows down....
INT: What could have caused that?
JS: He must have tripped over a hurdle.
INT: Why do you think that?
JS: Because his distance didn't go anywhere over some seconds. Whereas A and B are still going and I guess C gets back up and he's just running. But, the race is between A and B, where A is leading most of the way and B overtakes him at the end. B gets faster toward the end, while A starts slowing down a little bit.
INT: So you think B is the winner?
JS: Yes. B came in first, A is in second, no,...yeah, ... B is in first, A is in second, C is in third.

Though JS incorrectly uses the term *slope* for *any* rate of change, even nonlinear, he correctly interprets the rates of change for non-linear sections of the graph. He has a clear conception of the concept of rate of change and how it affects the shape of the graph. He did NOT use a pointwise analysis, instead using Bell & Janovier's (1981) highest level of recognition: that of gradient reading:

INT: Let's look at A and B a little bit. What can you say about how fast they're running and are they running faster, slowing down, going the same rate, what do you think?

JS: Um…They're both going pretty much the same the whole race, except for A started out running fast, then he started slowing down.
INT: How do you know he slowed down?
JS: Because it's a curved slope, where the top starts curving more like ah, less distance for the time whereas, B runs about the same until one point where his slope starts curving upwards near the end.
INT: Do you know what you call that in a race? You call that a "kick" at the end. It looks like, what was he doing? For the whole race, he was running…
JS: ..constant the whole race. and then at the end he got a burst of energy.

Like both LD and AB, JS easily performed tasks requiring point reading (Episode 3) and also those requiring graph interpretation when the independent variable is *time* (Episode 4). Further evidence of his sound understanding of the meaning of slope of a line and its graphical interpretation are his comments in Episode 5 (See Appendix).

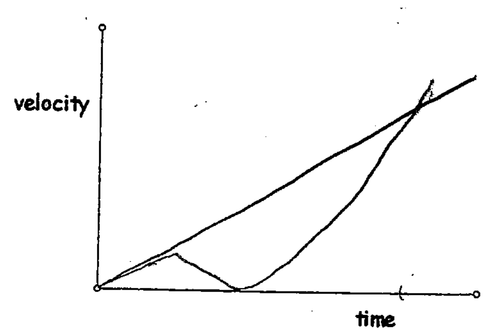
INT: Does the graph tell you anything about how fast you're traveling?
JS: Um… not specifically, but it shows how fast like….it doesn't show any numbers, you could just say that you're going faster because it's more, um….the slope is bigger…

Interestingly, he rarely resorts to *pointwise analysis*, requiring neither the vertical lines of LD in Episode 6 nor those of AB in Episode 7. To use Bell & Janvier's (1981) terminology, he recognizes the *global features* of graphs readily using *gradient reading* as appropriate:

Episode 6.
INT: At what age do girls put on weight the most rapidly?
JS: Between 12-15.
INT: How do you know that?
JS: The angle of the slope is greater. (SLOPE IS RATE OF CHANGE).
INT: The slope of what? This is a curve….
JS: The curve is going more up, it's straighter, so it's like, more weight over time.

Episode 7.
INT: This is a position vs. time graph. At the instant when t = 2 seconds, is the speed of the object A greater than, less than, or equal to that of object B?
JS: A? It would be less than.
INT: Why do you think that?
JS: Because A is going a shorter position in the same amount of time.
INT: A is going a shorter position?
JS: The position isn't as great as…it doesn't cover as much distance in the same amount of time.
INT: But the position is here, isn't this farther along? *(pointing to line A at t = 2)*
JS: Right, but the speed between 0 and 2, though,…
INT: How can you tell the speed?
JS: Cuz it has the rise over run.

I then asked JS to make a sketch of speed vs. time for two different cars. His sketch is as follows:



Initially, he made a *local correspondence error,* i.e. he interprets his graph literally as the path of his cars:

JS: The car starts out pretty slowly and then you could say that, he stops at a stop sign, so it has to go back down to 0, and then, he gets on the freeway, where he increases very quickly where he matches the speed of traffic. Whereas the other car goes another way, doesn't have to hit the stop sign, and he can go just right on the freeway and just go straight.
INT: What does it mean, when you have the two lines here, intersect?
JS: Um...there at the same spot.
INT: There at the same spot? SO they're passing each other?
JS: Yeah.... they would be passing each other.

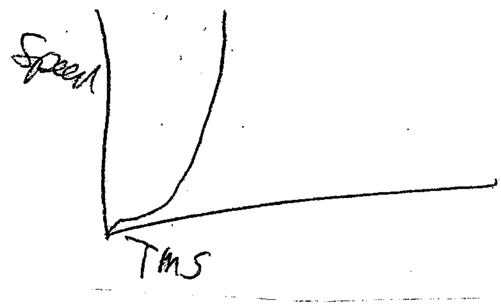
In a later interview he made the same mistake when given a graph of the same scenario: Episode 7B.

JS: They're at the same spot.
INT: They're at the same spot? How do you know that?
JS: Cuz....distance is rate over time.
INT: Distance is rate over time
JS: ....which is velocity, so...the distances are the same.

Through an intensive and guided *pointwise analysis* discussed in a situation familiar to JS, he eventually was able to note the error of his ways:

INT: Suppose we do know the actual speeds. *(Marks speeds and times on axes.)*
JS: 50 miles an hour.
INT: Now, let's do this again. This is the velocity. It's not a picture. It's the velocity. This car *(pointing to A)* is going really, really slow. So she, say it's your mom, left your house, going really, really slow. Then all of a sudden she's speeding up. Is she ever traveling as fast as you are?...if this is you *(points to other car's velocity)* except for right here? *(pointing to point of intersection).*
JS: Is he ever traveling as fast as me?
INT: Yes. At any given time, how do your speeds compare?
JS: The other one is going faster..
INT: Always?
JS: Right, always. (PERTURBATION)
INT: Does that make sense?
JS: Yeah, I see it.
INT: SO all of a sudden here, according to the graph...what's this graph telling you about car A? What do I know about car A?
JS: The velocity is the same?
INT: And what do I know about car B?
JS: The velocity is the same.
INT: After one hour, what's happening?
JS: Just the velocity is the same? Not the distance....?
INT: Now you've come up with two different things. Now which makes the most sense.
JS: The velocity is the same.
INT: DO you think the position would be the same?
JS: No.

It took some directed, pointwise analysis for JS to make the correct interpretation in this scenario. I was curious to see if he made Clement's (1989) *global correspondence* error, as LD did, confusing a graph as a whole with a different problem situation. I gave him task 8: Draw the shape of a the graph of speed vs. time for a bike rider coasting over the edge of a hill. His picture follows:

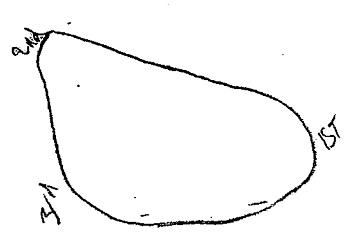


As did AB, his apparent understanding of distance, rate, and time permitted him to avoid the "graph as picture" error:

JS: Um…okay…when it's coasting over the edge of the hill, it's like going the same, then it's slowly increasing…the speed over time.
INT: Why is the speed increasing?
JS: Because of the pull of gravity.

In a similar fashion, when given the race track task (Episode 9), his familiarity with the race track scenario (when asked if he was "into" race cars, he responded "Yeah, that's the only video game we play") enabled him to an immediate correct interpretation:

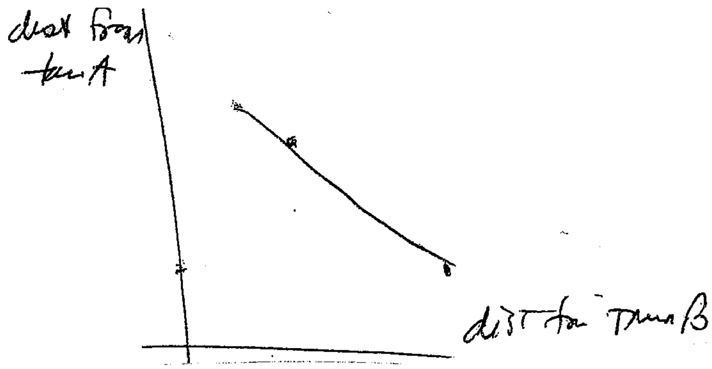
INT: Study the graph a bit. Can you tell me, from looking at the graph, how many bends there are along the track on which the car was driving?
JS: Three *(responds almost immediately).*
INT: How can you tell?
JS: Cuz they had to slow down for every one.
INT: They slowed down for every one?
JS: Yeah.
INT: Which bend would you say is the worst?
JS: The second one.
INT: And why do you say that that's the worst?
JS: Because here's *(points to second dip in graph)* where the drop in speed is the most.
INT: Which would be the easiest?
JS: The last one *(responds with no hesitation)*
INT: Why would you say the last one?
JS: Uh, it doesn't drop quite as much,….the speed.
INT: Can you sketch a graph of the track?
JS: Okay….



Episode 10.

As did both LD and AB, JS initially had difficulty sketching a graph when *time* is NOT the independent variable.

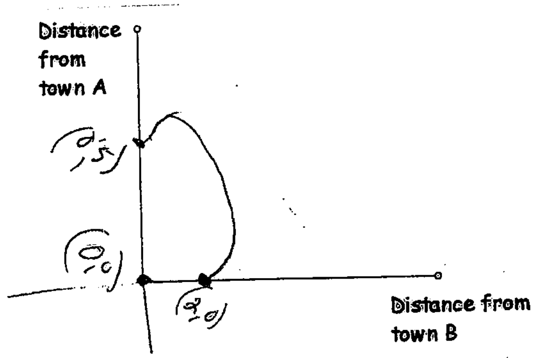
He, like the others, thought of his graph as a literal picture of the scenario and initially sketched the following:



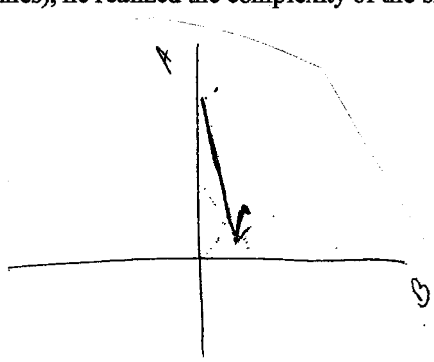
JS: Okay. ... So you're about ? (six) feet away from like, A. As you're going, you're getting a little closer to A, but then you're slowly going away from A, towards B. ...distance from A. Yeah

He apparently is thinking that the endpoint of the axes are the locations of the towns. He does not seem convinced, however, as he attempts to make a second sketch:

INT: Okay. Are you pretty sure?
JS: I don't know. (PERTURBATION)
INT: Just asking...
JS: Because you get closer here to B....*(pause)*. This would be, like, a little loop type thing.
INT: A little loop type thing?
JS: *(pauses, then graphs #2 below.)*



Unlike the LD and AB, however, he developed a much deeper insight into the problem, and eventually was able to sketch the following graph, which at least indicates a somewhat accurate representation of the *distances* from the two towns. Significantly, it was a careful *pointwise analysis,* carefully directed by the interviewer, that led JS to understand, and correctly sketch the graph. When questioned about the *rate of change* of the distances, i.e. about the *shape of his graph* (he drew straight lines), he realized the complexity of the situation.



Summary and Closing Discussion

The research literature points to common errors students make in the interpretation of graphs and the reasons behind these difficulties. One common difficulty experienced by all three subjects in this study was that of interpreting a graph as a literal *picture of the situation* as opposed to a sketch of the *relationship between two variables.* The degree of difficulty varied with each student, but all three exhibited this difficulty somewhat. Bell & Janvier (1981) noted a progression of student understanding from point reading to interval reading to gradient (slope) reading. Monk (1992) found that *across time*, or *qualitative* analysis is more difficult for students than *point-wise analysis.* It is perhaps significant to note that all three students reverted to *pointwise analysis* when confronting a problem that was difficult for them. Even JS, the most mathematically sophisticated of the three students, depended upon such an analysis in Episode 10, the most difficult of the tasks presented. Not only did they USE this technique, but they USED IT WELL, i.e. IT HELPED THEM UNDERSTAND THE GLOBAL FEATURES OF THE GRAPH. It would appear that a sound point-wise understanding of graphs is prerequisite to a higher level of understanding. This observation has implications for teaching, particularly in this age of calculator and computer technology. Including experiences in the classroom that require *point by point analysis or interpretation* is sound pedagogy.

Graphs in which *time* was the independent variable were the easiest for students to sketch and interpret. All three subjects exhibited greater competence and confidence with *interpretation tasks* rather than *construction tasks.* Only one of the subjects revealed a sound understanding of the notion of independent vs. dependent variables. The others appeared to depend upon a rote procedure for deciding which was which. Students exhibited varying degrees of understanding of the notion of *constant rate of change* as it relates to tables and to graphs. Some students tended to favor a linear interpretation even if the situation clearly did not warrant it.

It appeared that students had an easier time interpreting a graph when the situation was familiar to them and they understood related concepts well. Recall LD's continued difficulties with the distance, rate, and time scenarios, but her ease in interpretation of boys and girls rates of growth over time. Recall also JS initial difficulty with Episode 10, and then his later understanding through pointwise analysis in a familiar situation. This observation has implications for teaching in the Deweyian sense of locating education within a student's experience.

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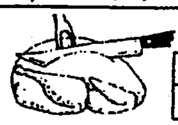
Appendix

Appendix

Episode 1

1. Cooling Coffee

Time (minutes)	0	5	10	15	20	25	30
Temperature (C°)	90	79	70	62	55	49	44

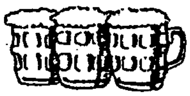


2. Cooking Times for Turkey

Weight (lb)	6	8	10	12	14	16	18	20
Time (hours)	2½	3	3½	4	4½	5	5½	6

3. How a Baby Grew Before Birth

Age (months)	2	3	4	5	6	7	8	9
Length (cm)	4	9	16	24	30	34	38	42



4. After Three Pints of Beer . . .

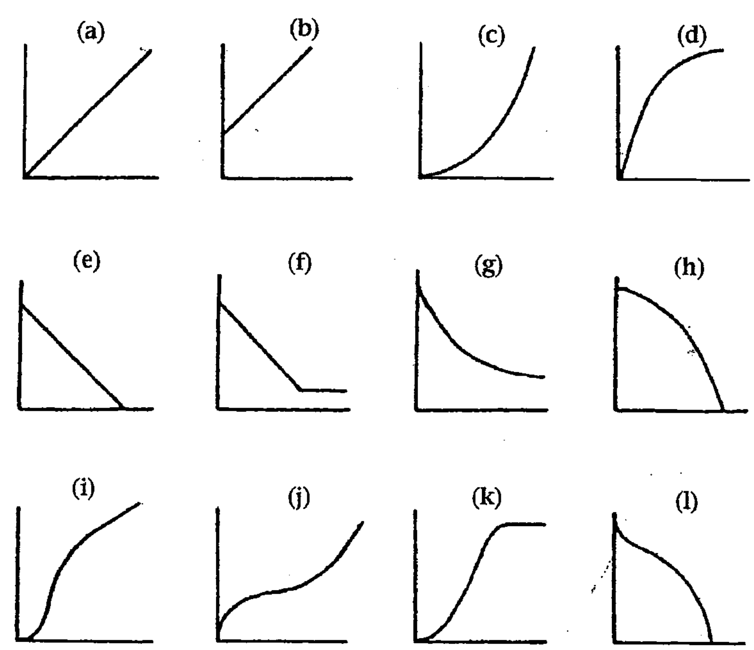
Time (hours)	1	2	3	4	5	6	7
Alcohol in the blood (mg/100ml)	90	75	60	45	30	15	0

5. Number of Bird Species on a Volcanic Island

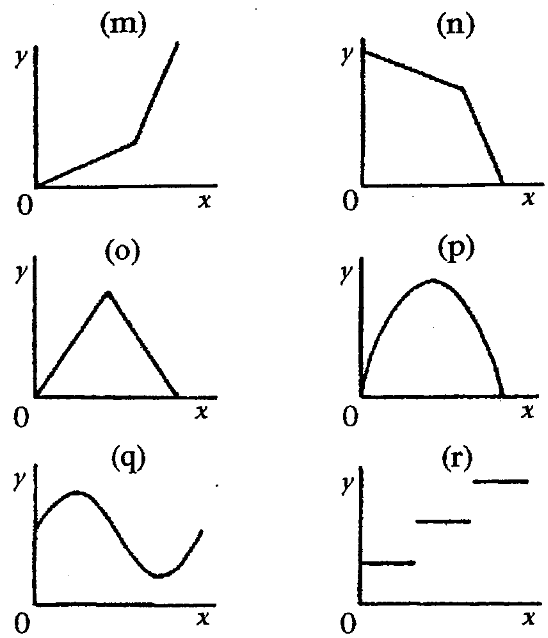
Year	1880	1890	1900	1910	1920	1930	1940
Number of Species	0	1	5	17	30	30	30

6. Life Expectancy

Age (years)	Number of Survivors	Age (years)	Number of Survivors
0	1000	50	913
5	979	60	808
10	978	70	579
20	972	80	248
30	963	90	32
40	950	100	1

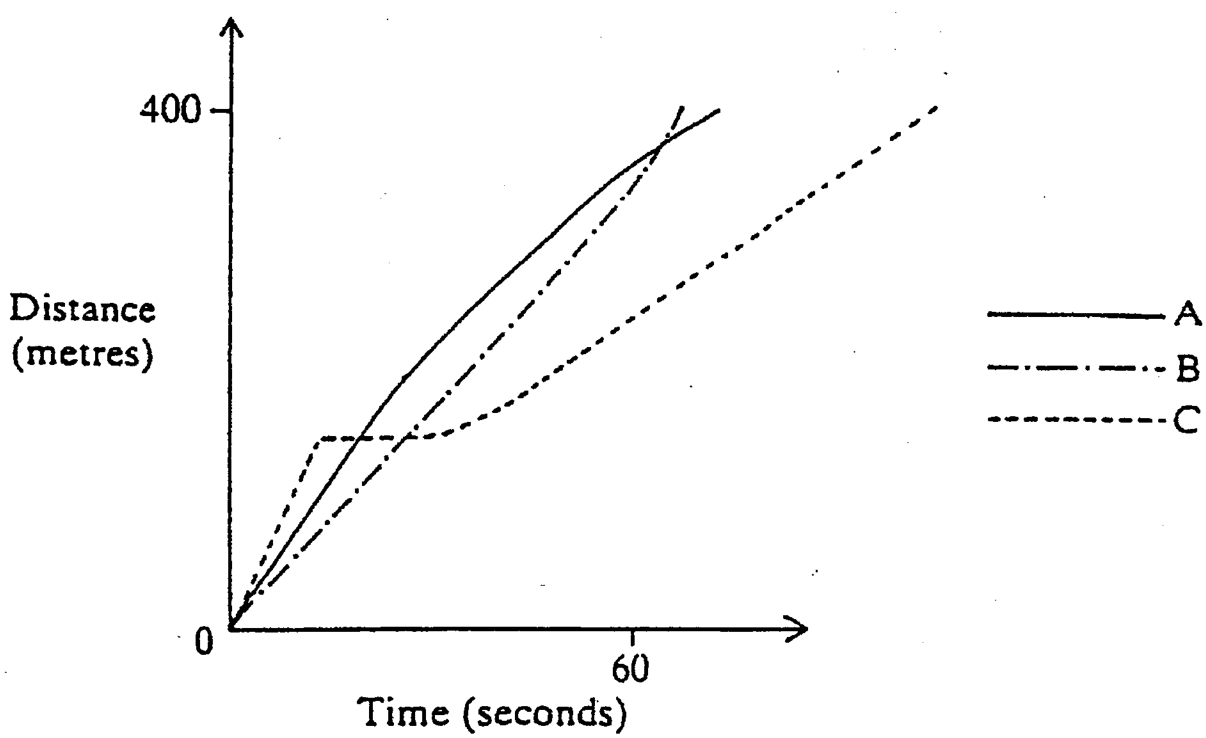


Episode 1B



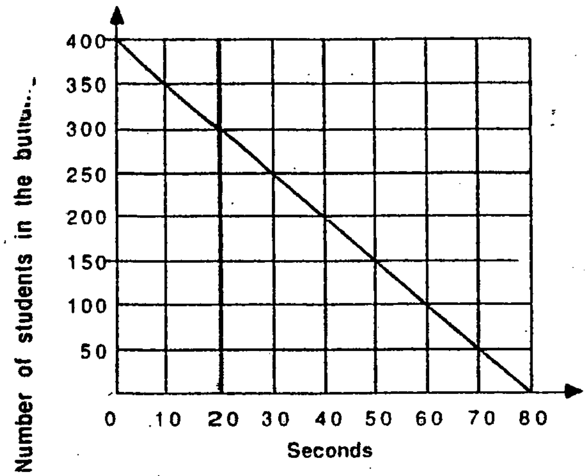
THE HURDLES RACE GRAPH

Episode 2



Episode 3

Functions, Graphs, and Graphing

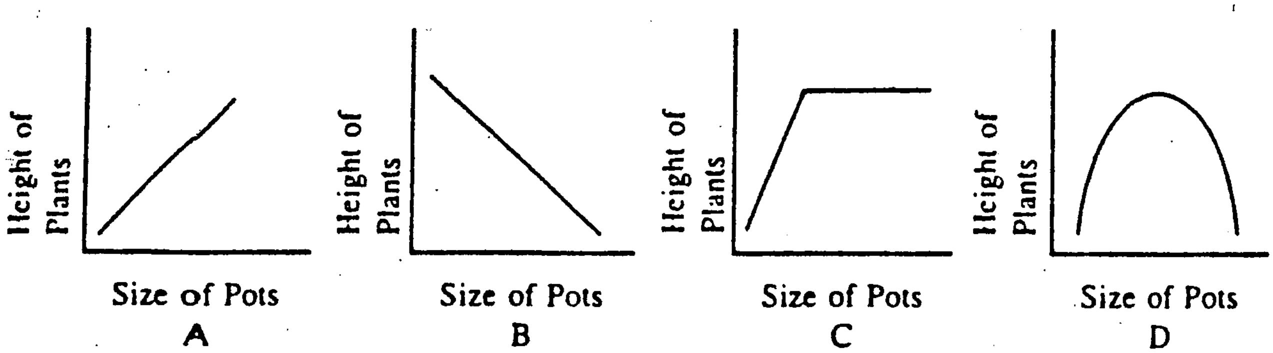


This graph shows how long it takes students to leave the building during a fire drill.

1. How many students were in the building before the fire drill?
2. How many students were in the building after
   a. 10 seconds?
   b. 30 seconds?
   c. 40 seconds?
3. How many seconds had passed when there were only 50 students left in the building?
4. How many seconds did it take for all of the students to leave the building?

Episode 4

Dick plans to study the effect of growing sunflowers in different size pots. The graphs below show four possible outcomes of his experiment.

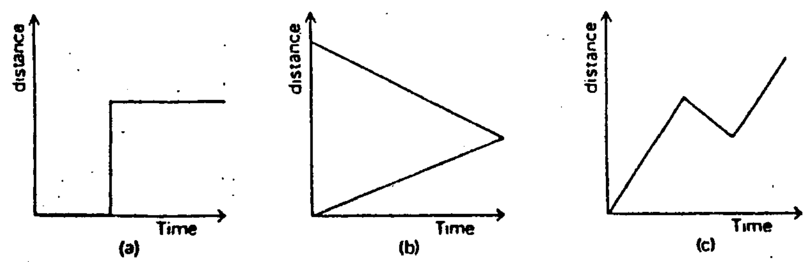


Which graph is *best* described by each of the following statements.

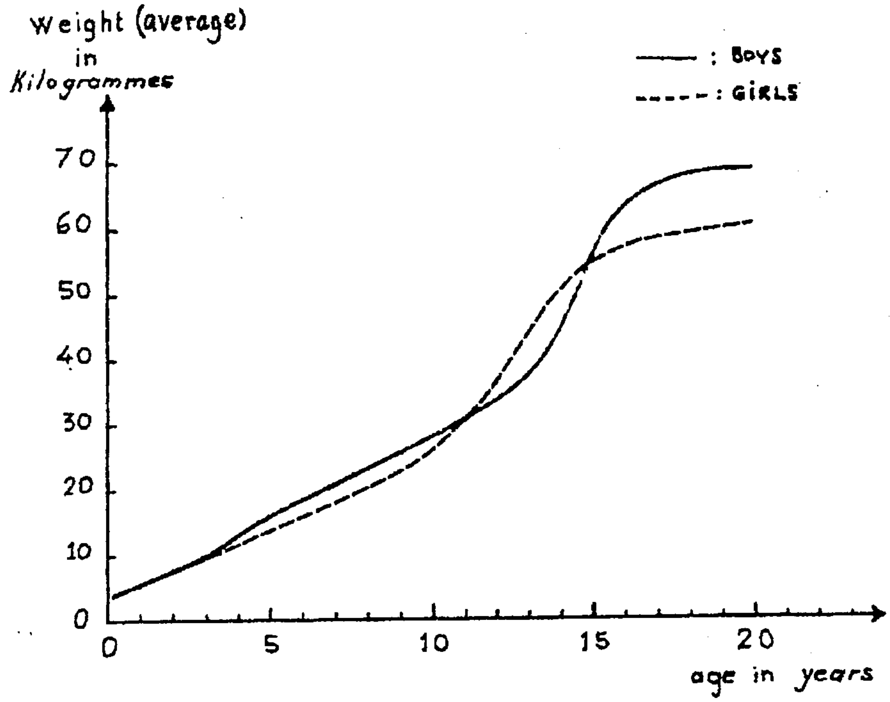
16. As the pot size increases, the plant height decreases.
17. As the pot size increases the plant height increases up to a certain pot size. With larger pots, plant height remains the same.

Episode 5

Which of the graphs below represent journeys? Describe what happens in each case.



Episode 6



1. The average weight of boys at age 9 is ....

2. The average weight of girls at age 17 is ....

3. From what age do the <u>boys</u> on average weigh more than 55 kilogrammes? ....

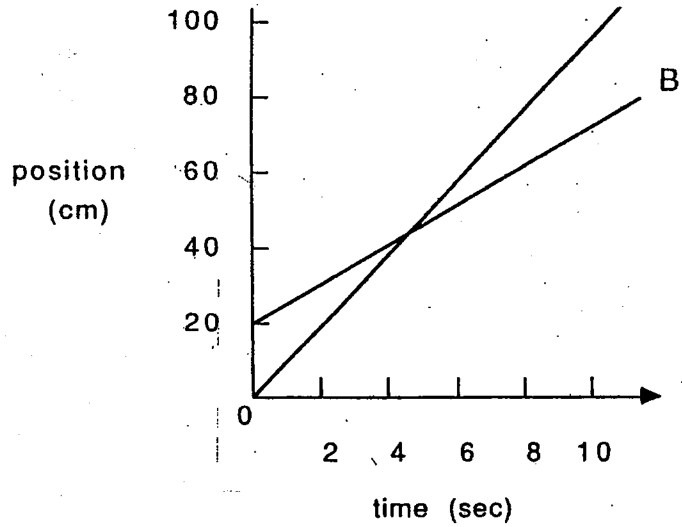
4. From what age do the <u>girls</u> on average weigh more than 20 kilogrammes? ....

5. When (at what ages) do the girls weigh more than the boys? ....

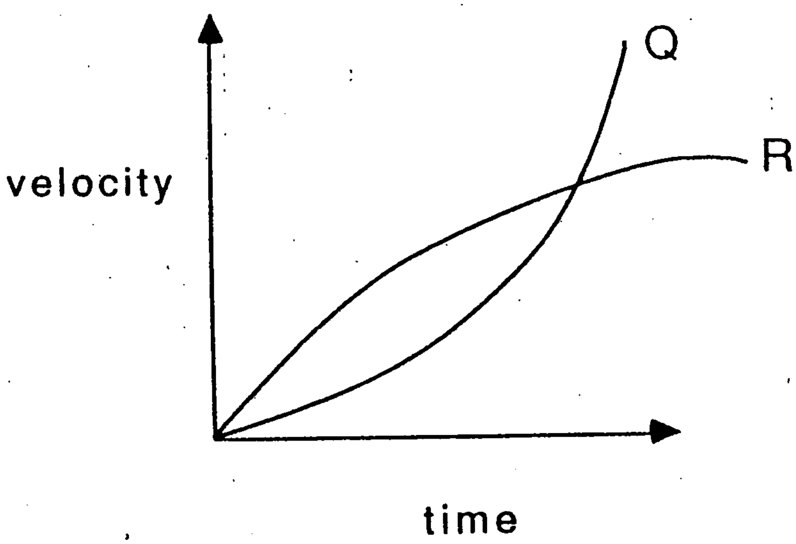
6. By how many kilogrammes does the average weight of girls increase between age 3 and age 8? ....

7. At what age do the girls put on weight most rapidly? ....

Episode 7

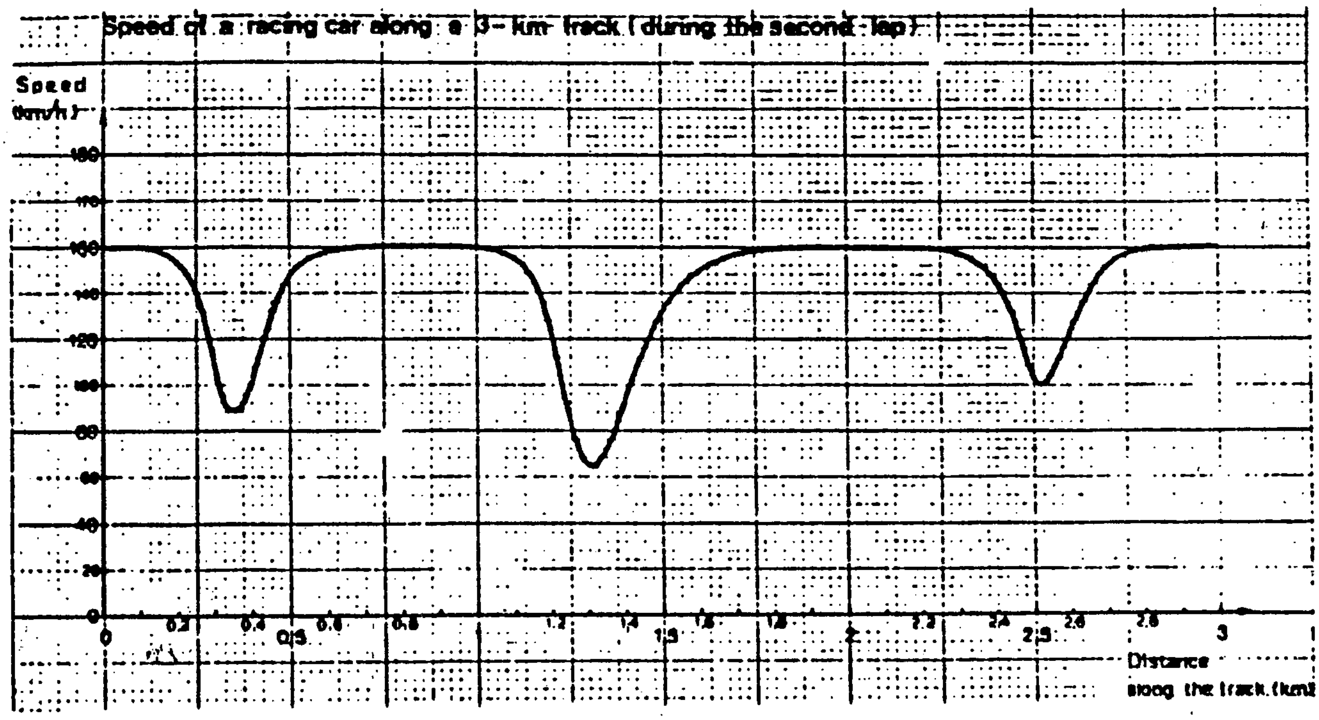


Episode 7B



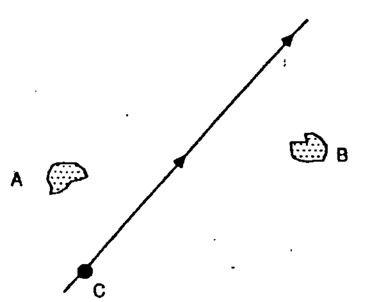
Episode 8: Draw the shape of a graph of speed vs. time for a bike rider coasting over the edge of a hill.

Episode 9



Can you tell me, from the graph, how many bends there are along the track on which the car was driven?

Episode 10:



The car C travels at a constant speed along a road in the direction of the arrows. Complete the sketch of the graph.