19 Eigenvalues, Eigenvectors, Ordinary Differential Equations, and Control

This section introduces eigenvalues and eigenvectors of a matrix, and discusses the role of the eigenvalues in determining the behavior of solutions of systems of ordinary differential equations. An application to linear control theory is described.

19.1 Linear control theory: feedback

Consider the initial value problem

\[ \frac{d}{dt} x(t) = ax(t), \quad t \geq 0, \quad x(0) = x_0, \quad (1) \]

where \( x(t) \) is a real-valued function, \( a \) is a real constant, and the initial value \( x_0 \) is positive. We will refer to the variable \( t \) as “time.” The solution

\[ x(t) = \exp(at) x_0 \]

remains bounded as \( t \) grows only when the constant \( a \) is zero or negative. When \( a \) is positive, the solution will grow without bound. In applications to control problems, the solution \( x(t) \) is often referred to as the system state.

Models of control problems often include an additional term in equation (1), known as the feedback. The following equations illustrate a typical linear control problem with the feedback term \( f x(t) \):

\[ \frac{d}{dt} x(t) = ax(t) - bx(t), \quad t \geq 0, \quad x(0) = x_0, \quad b \neq 0, \quad (2) \]

\[ y(t) = cx(t), \quad c \neq 0. \]

The constants \( a, b, \) and \( c \) are determined by the model. The function \( y(t) \) is called the system output; in this example it is a multiple of the system state \( x(t) \). In real-world applications, the system output represents an observable or measurable property of the system. The parameter \( f \) is called the feedback gain. The purpose of the feedback term is to prevent unbounded growth of the solution \( x(t) \) as time increases.

The solution of equation (2) is

\[ x(t) = \exp ((a - bf)t) x_0. \]
Whether the system state $x(t)$ is bounded depends on the quantity $a - bf$. The scalar-valued linear control problem amounts to choosing a value of $f$, such that $a - bf \leq 0$. In most applications one requires

$$a - bf < 0. \quad (4)$$

In many control problems of interest, $x(t)$ and $f$ are column vectors, $a$ is a matrix, and $b$ and $c$ are row vectors, e.g., $x(t), f, b^T, c^T \in \mathbb{R}^n$ and $a \in \mathbb{R}^{n \times n}$. Then $a - bf$ is an $n \times n$ matrix, and the condition (4) translates into the requirement that the matrix $a - bf$ only have negative eigenvalues. The following section reviews results on eigenvalue and eigenvector. Thereafter, we will return to control problems.

### 19.2 Matrices, eigenvalues, and eigenvectors

Let $A$ be a square $n \times n$ matrix. A scalar $\lambda$ and a nonzero vector $v$ that satisfy the equation

$$Av = \lambda v \quad (5)$$

are called an eigenvalue and eigenvector of $A$, respectively. The eigenvalue may be a real or complex number, and the eigenvector may have real or complex entries. The eigenvectors are not unique; see Exercises 19.5 and 19.7 below.

Equation (5) may be rewritten as

$$(\lambda I - A)v = 0, \quad (6)$$

which shows that the nonzero eigenvector $v$ lies in the null space of the matrix $\lambda I - A$. Matrices with nontrivial null spaces are, by definition, singular, and therefore, $\det(\lambda I - A) = 0$. The function $p_A(z) := \det(zI - A)$ is a polynomial of degree $n$. This can be verified by expanding the determinant along rows or columns. The polynomial $p_A(z)$ is called the characteristic polynomial of the matrix $A$.

Example 19.1. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Its characteristic polynomial is given by

$$p_A(z) = (z - 1)(z - 1)(z - 2).$$
The eigenvalue $\lambda = 1$ is said to be of \textit{algebraic multiplicity} 2, because it is a zero of of $p_A(z)$ of multiplicity 2. The eigenvalue $\lambda = 2$ is of algebraic multiplicity 1. \hfill \Box

Example 19.2. Expanding the characteristic polynomial for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

along the last row yields

$$p_A(z) = \det \begin{bmatrix} z - 1 & -2 \\ -1 & z - 3 \end{bmatrix} (z - 1) = (z^2 - 4z + 1)(z - 1).$$

This shows that the eigenvalues of the matrix are

$$\lambda_1, \lambda_2 = 2 \pm \sqrt{3}, \quad \lambda_3 = 1.$$

If we replace the entry 2 in position $(1,2)$ of the matrix (7) by any real number strictly smaller than $-1$, then the matrix has one pair of complex conjugate eigenvalues. For instance, setting the $(1,2)$-entry to $-2$ yields the eigenvalues

$$\lambda_1 = 2 + i, \quad \lambda_2 = 2 - i, \quad \lambda_3 = 1,$$

where $i = \sqrt{-1}$ is the imaginary unit, i.e., $i^2 = -1$. In particular, $\lambda_1$ and $\lambda_2$ are complex-valued eigenvalues. Both $\lambda_1$ and $\lambda_2$ are said to have real part 2; $\lambda_1$ has imaginary part 1 and $\lambda_2$ has imaginary part $-1$. Thus, the imaginary part is the coefficient of $i$. Since $\lambda_1$ and $\lambda_2$ have the same real parts and have imaginary parts of opposite sign, these eigenvalues are said to be complex conjugate; see below for further comments on complex numbers. \hfill \Box

**Proposition 1** \textit{The roots of $p_A(z)$ are the eigenvalues of $A$.}

\textit{Proof.} We already have seen that $p_A(z)$ vanishes at the eigenvalues of $A$. Conversely, assume that $p_A(\lambda) = 0$ for some scalar $\lambda$. Then the matrix $\lambda I - A$ is singular. Let $v$ be a nontrivial solution of the homogeneous linear system of equations (6). Then $v \neq 0$ satisfies (5). Thus, $v$ is an eigenvector and $\lambda$ an eigenvalue of $A$. \hfill \Box

By the Fundamental Theorem of Algebra, a polynomial of degree $n$ has precisely $n$ zeros, counting multiplicities. Therefore, every $n \times n$ matrix has $n$ eigenvalues. Some of the zeros of $p_A(z)$ may be complex numbers. It follows that eigenvalues may be complex. However, the important class of symmetric matrices only have real eigenvalues; see Section 20.
Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the eigenvalues of the \( n \times n \) matrix \( A \), and let \( v_1, v_2, \ldots, v_n \) denote the corresponding eigenvectors, i.e.,

\[
Av_j = \lambda_j v_j, \quad j = 1, 2, \ldots, n.
\] (8)

A matrix is said to be diagonalizable if the \( n \) eigenvectors \( v_1, v_2, \ldots, v_n \) can be chosen to be linearly independent. Assume this is the case and introduce the eigenvector matrix

\[
V = [v_1, v_2, \ldots, v_n].
\]

This matrix is nonsingular since its columns are linearly independent. Define the diagonal matrix determined by the eigenvalues of \( A \),

\[
D = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n].
\] (9)

Then the equations (8) can be expressed compactly as

\[
AV = VD.
\]

Since the matrix \( V \) is nonsingular, the above equation yields

\[
A = VDV^{-1}.
\] (10)

This formula shows that the matrix \( A \) is diagonal when expressed in terms of the eigenvector basis \( \{v_1, v_2, \ldots, v_n\} \). This is the reason for the importance of eigenvalues and eigenvectors. The set of eigenvalues of a matrix is sometimes referred to as the spectrum of the matrix, and the factorization (10) as the spectral factorization. Most, but not all, square matrices are diagonalizable.

Example 19.3. The matrix

\[
A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.
\]

has the eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = 1 \), but only one linearly independent eigenvector. This follows from equation (6), which can be expressed as

\[
\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} v = 0.
\]

The above equation shows that all solutions are of the form \( v = [\alpha, 0]^T \), where \( \alpha \) is a nonvanishing scalar. Thus, all eigenvectors of \( A \) are a multiple of the axis vector \( e_1 = [1, 0]^T \).

Perturbing any one of the diagonal entries of \( A \) slightly gives a matrix with distinct eigenvalues. Matrices with pairwise distinct eigenvalues have linearly independent eigenvectors. We conclude that there are matrices arbitrarily close to \( A \) with linearly independent eigenvectors; see Exercise 19.8 for an illustration. \( \square \)
Exercise 19.1
Let
\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}. \]
Determine the characteristic polynomial \( p_A(z) \) of \( A \). □

Exercise 19.2
Let
\[ A = \begin{bmatrix} 0.5 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}. \]
Determine the characteristic polynomial \( p_A(z) \) of \( A \). □

Exercise 19.3
Give a simple expression for \( A^2 \) in terms of the matrices \( V \) and \( D \) by using the spectral factorization (10). What is the corresponding expression for \( A^k \) when \( k \) is a positive integer? Assume that \( A \) is nonsingular. What is the corresponding expression for \( A^k \) when \( k \) is a negative integer? □

Exercise 19.4
Let \( A \) be an \( n \times n \) matrix with nonnegative eigenvalues. Give an expression for \( A^{1/2} \) by using the spectral factorization (10). □

Exercise 19.5
Eigenvectors are not unique. Let \( v \) be an eigenvector of \( A \). Show that any nonzero multiple of \( v \) is also an eigenvector of \( A \). □

Exercise 19.6
Consider the matrix
\[ A = \text{diag}[1, 2, 3]. \]
What are the eigenvalues? Describe all eigenvectors, e.g., by investigating the solution set of (6) for \( \lambda = 1, 2, 3 \). □
Exercise 19.7
Consider the matrix
\[ A = \text{diag}[1, 1, 3]. \]

Describe all eigenvectors, e.g., by investigating the solution set solutions of (6) for \( \lambda = 1, 3. \)

Exercise 19.8
Compute eigenvectors associated with the distinct eigenvalues of the matrix
\[ A = \begin{bmatrix} 1.01 & 2 \\ 0 & 1 \end{bmatrix}. \]

Are they linearly independent? Are they almost parallel? Cf. the discussion of Example 19.3.

Exercise 19.9
Use the MATLAB/Octave command `magic` to determine a 4 \( \times \) 4 matrix, whose entries form a magic square. Use the MATLAB/Octave command `eig` to compute the spectral factorization of the matrix. How are the eigenvectors normalized? One of the eigenvectors has all entries equal. Can this be expected? Hint: What is the corresponding eigenvalue?

19.3 Systems of linear ordinary differential equations
Consider the system of linear ordinary differential equations (ODEs) in time \( t \geq 0: \)
\[
\begin{align*}
\frac{d}{dt} x_1(t) & = a_{11} x_1(t) + a_{12} x_2(t) + a_{13} x_3(t), \\
\frac{d}{dt} x_2(t) & = a_{21} x_1(t) + a_{22} x_2(t) + a_{23} x_3(t), \\
\frac{d}{dt} x_3(t) & = a_{31} x_1(t) + a_{32} x_2(t) + a_{33} x_3(t).
\end{align*}
\]

(11)

This system has a unique solution when the initial values
\[
 x_1(0) = x_1, \quad x_2(0) = x_2, \quad x_3(0) = x_3
\]

are prescribed. We can write the system of ODEs (11) as
\[
\frac{d}{dt} x(t) = Ax(t),
\]

(13)
where

\[ A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}, \quad x(t) = \begin{bmatrix}
  x_1(t) \\
  x_2(t) \\
  x_3(t)
\end{bmatrix}. \]

The solution of (13) for \( t \geq 0 \) is given by

\[ x(t) = \exp(At)x(0), \]

where \( \exp(At) \) is the \textit{matrix exponential function}.

Let \( A \) be an \( n \times n \) matrix with spectral factorization (10). We define

\[ \exp(A) := V \exp(D)V^{-1} \] (14)

and

\[ \exp(D) := \text{diag}[\exp(\lambda_1), \exp(\lambda_2), \ldots, \exp(\lambda_n)], \]

where \( D \) is given by (9). The solution of the system of differential equations (13), where now \( A \) is this \( n \times n \) matrix, can be expressed as

\[ x(t) = \exp(At)x(0) = V \exp(Dt)V^{-1}x(0) = V \text{diag}[\exp(\lambda_1t), \exp(\lambda_2t), \ldots, \exp(\lambda_nt)]V^{-1}x(0), \]

where \( x(0) \) is the \( n \)-vector of initial values.

We often are interested in whether solution \( x(t) \) decreases to zero or grows in magnitude as \( t \) increases. Therefore the norm

\[ \|x(t)\| = \|V \exp(Dt)V^{-1}x(0)\| \leq \kappa(V)\|x(0)\| \max_{1 \leq j \leq n}\{|\exp(\lambda_jt)|\} \] (15)

is of interest. Here \( \kappa(V) := \|V\|\|V^{-1}\| \) is the condition number of the eigenvector matrix \( V \).

Note that neither \( \kappa(V) \) nor \( x(0) \) depend on \( t \). Therefore, the growth or decay of the norm of the solution depends entirely on the factor \( \max_{1 \leq j \leq n}\{|\exp(\lambda_jt)|\} \); see Exercises 19.12-19.14.

**Exercise 19.10**

Let \( A \) be an \( n \times n \) matrix. MATLAB and Octave allow the commands \( \exp(A) \) and \( \text{expm}(A) \). Which command yields the matrix exponential? What does the other command compute? Hint: Compare with (14). □
Exercise 19.11
Show the bound (15). □

Exercise 19.12
Assume that all the eigenvalues \( \lambda_j \) are strictly negative. Does the norm of the solution \( x(t) \) of (13) increase or decrease as \( t \) becomes large? What does this imply for each component \( x_j(t), j = 1, 2, 3, \) of \( x(t) \)? □

Exercise 19.13
Assume that all the eigenvalues \( \lambda_j \) are strictly positive. Does the norm of the solution \( x(t) \) of (13) increase or decrease as \( t \) becomes large? □

Exercise 19.14
Assume that the \( 3 \times 3 \) matrix \( A \) has the eigenvalues \( \lambda_1 = \lambda_2 = -1 \) and \( \lambda_3 = 0 \). How can we expect the solution \( x(t) \) of (13) to behave as \( t \) increases? □

Exercise 19.15
Let \( A \) be the matrix from Exercise 19.9 and let \( x(0) = [1, 1, 1, 1]^T \). Plot the solution \( x(t) \) of (13) for \( 0 \leq t \leq 0.2 \). How does the solution behave? □

Exercise 19.16
Let the matrix \( A \) be obtained by subtracting \( 34.5I \) from the matrix used in Exercise 19.15, where \( I \) denotes the identity matrix. Let \( x(0) = [1, 1, 1, 1]^T \) and plot the solution \( x(t) \) of (13) for \( 0 \leq t \leq 5 \). How does the solution behave? Explain! □

19.4 Linear control theory

The subject of linear control theory is concerned with choosing a feedback gain vector \( f \in \mathbb{R}^n \), so that the solution \( x(t) \) of the control system

\[
\frac{d}{dt}x(t) = Ax(t) - bf^Tx(t), \quad A \in \mathbb{R}^{n \times n}, \quad x(t), b \in \mathbb{R}^n, \quad t \geq 0, \tag{16}
\]

\[
y(t) = c^Tx(t), \quad c \in \mathbb{R}^n.
\]
remains bounded as time $t$ increases. However, only the output $y(t)$ can be observed. The initial state $x(0)$ and the vectors $b$ and $c$ are defined by the model, as is the matrix $A$, which we assume to be diagonalizable. The solution to the initial value problem (16) is given by

$$x(t) = \exp ((A - bf^T)t)x(0);$$

compare this formula with the analogous scalar expression in equation (3).

The problem (16) is said to be controllable if a feedback gain vector $f$ can be found, such that $\|x(t)\|$ remains bounded as time $t$ increases. This entails choosing $f$ so that the eigenvalues of the matrix $A - bf^T$ are non-positive (or, more generally, have non-positive real part). There are several algorithms available for determining such a vector $f$. Some of these are called eigenvalue assignment methods, because they seek to determine a feedback gain vector $f$, such that the matrix $A - bf^T$ has specified eigenvalues. For many matrix-vector pairs $\{A, b\}$, a gain vector, such that $A - bf^T$ has a prescribed spectrum can be found. However, the computation of this vector is for some pairs $\{A, b\}$ too ill-conditioned to yield useful results in finite precision arithmetic.

The gain vector $f$ indicates to engineers how a structure, such as the space station, should be reinforced to avoid harmful undamped oscillations. Another example is provided below.

19.5 A control example

The electro-magnetic suspension of a ferrous ball is a classic example in the control theory literature. The physical goal of the example is to maintain the position of the suspended ball. Our mathematical goal is to express the physical model as a linear control problem of the form (16), and to solve for a feedback gain vector to stabilize the system and keep the ball suspended in a fixed position. We present a slightly modified version of the standard example in order to simplify the computations.

The example is illustrated in Figure 1. A voltage $v$ is applied to the coil at the top of the illustration. The current $i(t)$ flowing through the coil at time $t$ generates a magnetic force $F$ that pulls the ball up. At the same time, the force of gravity $G$ is pulling the ball back down towards the ground. We denote the distance between the end of the coil and the top of the ball at time $t$ as $h(t)$. The ball will remain suspended in midair whenever the forces $F$ and $G$ balance out. The dotted line at $h(0)$ indicates the set position at which we desire to suspend the ball (as illustrated, the ball is well below the set point). The deviation from this position is denoted by $\tilde{h}(t) := h(t) - h(0)$. Similarly, the deviation from the current $i(0)$ is written $\tilde{i}(t) := i(t) - i(0)$.

A standard model of the motion of the ball is given by

$$\frac{d^2}{dt^2}h(t) = \frac{1}{m}(G - F), \quad (17)$$

9
where $m$ is the mass of the ball, $r$ is the resistance of the wire, and $\ell$ is the impedance of the coil. The gravitational force $G$ is constant. The magnetic force $F = ki(t)^2/h(t)$ is a nonlinear function of the distance $h(t)$ and current $i(t)$ ($k$ is a constant).

The nonlinearity of $F$ prevents us from directly setting up a linear control problem of the form (16). We can, however, assume that the model is approximately linear near the set point $h(0)$, and use the Taylor series approximation:

$$
F \approx F(h(0), i(0)) + \frac{\partial F}{\partial h}_{i(0)}(h(t) - h(0)) + \frac{\partial F}{\partial i}_{h(0)}(i(t) - i(0))
$$

Noting that $\frac{d^2}{dt^2}h(t) = \frac{d^2}{dt^2}\tilde{h}(t)$, we can substitute the linear approximation of $F$ into equation (17),

$$
\frac{d^2}{dt^2}\tilde{h}(t) = \frac{G}{m} - k_1 + k_2\tilde{h}(t) + k_3\tilde{i}(t),
$$

where we have simplified the notation by introducing the constants

$$
k_1 := \frac{F(h(0), i(0))}{m}, \quad k_2 := -\left.\frac{\partial F}{m\partial h}\right|_{i(0)}, \quad \text{and}, \quad k_3 := -\left.\frac{\partial F}{m\partial i}\right|_{h(0)}.
$$
Define the state vector
\[
    x(t) := \begin{bmatrix} \tilde{h}(t), \frac{d}{dt}\tilde{h}(t), \tilde{i}(t) \end{bmatrix}^T
\]
and let the system output be the deviation from the set point,
\[
    y(t) := \tilde{h}(t).
\]
Let the constants have the values
\[
k_2 = 1, \quad k_3 = -1, \quad r/\ell = 10, \quad G/m - k_1 = 1, \quad v/\ell = 1.
\]
The definition of \(x(t), y(t)\), and the constants, together with (19) and (18), yield the control system
\[
    \frac{d}{dt} x(t) = \begin{bmatrix} 0 & 1 & 0 \\ k_2 & 0 & k_3 \\ 0 & 0 & -r/\ell \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ G/m - k_1 \\ v/\ell \end{bmatrix}
\]
\[
    = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -10 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\]
\[
    = Ax(t) + b, \quad y(t) = c^T x(t)
\]
of the form (16), where \(c := [1, 0, 0]^T\).

The system output \(y(t)\) measures deviation from the desired set position; this value should be driven to zero in order to suspend the ball at the set position \(h(0)\).

**Exercise 19.17**

What are the eigenvalues of the matrix \(A\) in (20)? Solve the system of differential equations (20). Graph the system output \(y(t)\) is a function of time \(t\). \(\square\)

You will find from the above exercise that the system as described is not stable; the system output diverges over time and the ball will not remain suspended at the set point. We can control the system output by adding a feedback term \(-f^T x(t)\) to equation (20), transforming it into a linear control problem of the form (16). Specifically, we would like to determine a feedback gain vector \(f\), such that the solution \(x(t)\) converges to 0 as \(t\) increases.

The matrix \(A - bf^T\) may have two complex-valued eigenvalues, say, \(\lambda_1\) and \(\lambda_2\), for certain vectors \(f\). We therefore have to discuss how \(\exp(\lambda_j t)\) behaves as \(t\) increases for complex-valued \(\lambda_j\).
Let $\alpha \in \mathbb{R}$ and $i = \sqrt{-1}$. We define
\[
\exp(i\alpha) = \cos(\alpha) + i\sin(\alpha).
\] (21)

We refer to $\cos(\alpha)$ as the real part and $\sin(\alpha)$ as the imaginary part of the complex number $\exp(i\alpha)$. Complex numbers can be thought of as vectors in $\mathbb{R}^2$. They differ from ordinary vectors in $\mathbb{R}^2$ only in that complex numbers can be multiplied, using a special rule, while vectors in $\mathbb{R}^2$ cannot. Thus, we may identify the complex number $\exp(i\alpha)$ with the point $(\cos(\alpha), \sin(\alpha))$ in $\mathbb{R}^2$. This point lives on the unit circle in $\mathbb{R}^2$. We define the magnitude of the complex number $\exp(i\alpha)$ to be the length of the corresponding vector $(\cos(\alpha), \sin(\alpha))$ in $\mathbb{R}^2$. We have
\[
|\exp(i\alpha)| = \sqrt{\cos^2(\alpha) + \sin^2(\alpha)} = 1.
\]

We therefore say that $\cos(\alpha) + i\sin(\alpha)$ lives on the unit circle in the complex plane.

Returning to the eigenvalues of $A - bf^T$, express the complex eigenvalue $\lambda_1$ in the form
\[
\lambda_1 = \lambda_{1,1} + i\lambda_{1,2}, \quad \lambda_{1,1}, \lambda_{1,2} \in \mathbb{R}, \quad i = \sqrt{-1},
\]
where $\lambda_{1,1}$ is the real part and $\lambda_{1,2}$ the imaginary part of $\lambda_1$. Using that the exponential of a sum is the product of the exponential of each term yields
\[
\exp(\lambda_1) = \exp(\lambda_{1,1} + i\lambda_{1,2}) = \exp(\lambda_{1,1}) \exp(i\lambda_{1,2}).
\]

From (21) we now obtain
\[
\exp(\lambda_1) = \exp(\lambda_{1,1})(\cos(\lambda_{1,2}) + i\sin(\lambda_{1,2})).
\]

It follows that
\[
|\exp(\lambda_1)| = \exp(\lambda_{1,1})|\cos(\lambda_{1,2}) + i\sin(\lambda_{1,2})| = \exp(\lambda_{1,1}),
\]
where we have used that $\exp(\lambda_{1,1}) > 0$ and $|\cos(\lambda_{1,2}) + i\sin(\lambda_{1,2})| = 1$.

When we study the stability of solutions of systems of ODEs, we are interested in whether expressions of the form $|\exp(\lambda_j t)|$ increase or decrease as $t$ increases; cf. (15). Our discussion above shows that
\[
|\exp(\lambda_j t)| = \exp(\lambda_{j,1} t),
\]
where $\lambda_{j,1}$ denotes the real part of the eigenvalue $\lambda_j$. We therefore only are concerned with the sign of the real part of the eigenvalues when determining whether the solution $x(t)$ converges to zero as $t$ increases.
Let the matrix $A$ and vector $b$ be defined by (20), and let $V$ be the eigenvector matrix and $D$ the eigenvalue matrix of the $A$; cf. (10). Introduce the vectors

$$\tilde{b} = [\tilde{b}_1, \tilde{b}_2, \tilde{b}_3]^T := V^{-1}b, \quad \tilde{f} := [\alpha, 0, 0]^T, \quad f := V^{-T}\tilde{f},$$

(22)

where $\alpha$ is a scalar to be determined. Then

$$A - bf^T = V \left( D - \tilde{b}\tilde{f}^T \right) V^{-1}.$$  

(23)

**Exercise 19.18**

Let the vectors $\tilde{b}$ and $\tilde{f}$, as well as the matrix $D$ be the same as in (22). What are the eigenvalues of the matrix $D - \tilde{b}\tilde{f}^T$? □

**Exercise 19.19**

Equation (23) and Exercise 19.18 provide us with an algorithm for solving the linear control problem associated with this example:

1. Compute $[V,D]=\text{eig}(A)$

2. Let $b=[0,1,1]'$ and compute $\texttt{btilde}=V\backslash b$

3. Let $\texttt{ftilde}=[1,0,0]'$

4. Find a value of $\alpha$ so that the eigenvalues of $D - \alpha\texttt{btilde}^*\texttt{ftilde}'$ are all have negative real part.

5. Compute $f = V'\backslash \texttt{ftilde}$

Code the above algorithm in MATLAB or Octave, solve the example control problem, and make a new plot of the system output $y(t)$ over time for your solution. □