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Perron communicability and sensitivity of multilayer networks

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Dedicated to Claude Brezinski on the occasion of his 80th birthday.

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Abstract XXX

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1 Introduction

Many complex systems can be modeled as networks. Informally, a network is a collection of objects, referred to as *nodes* or *vertices*, that are connected to each other in some fashion; the connections are referred to as *edges*. The edges may be directed or undirected, and may be equipped with positive weights that correspond to their importance. The nature of the nodes, edges, and weights depends on the application. Some modeling situations require more than one kind of nodes or more than one type of edges.

Multilayer networks are networks that consist of different kinds of edges and possibly different types of nodes. This kind of networks arise when one seeks to model a complex system that contains connections and objects with different properties. For instance, when modeling train and bus connections

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in a country, the train routes and bus routes define edges with distinctive properties, and the train and bus stations may make up nodes with diverse properties. The connections between a train station and an adjacent bus station give rise to yet another kind of edges, along which travelers walk. Edge weights may be chosen proportional to the number of travelers along an edge, proportional to the distance between the nodes that the edge connects, or proportional to the cost of traveling along an edge. Whether it is meaningful to distinguish between the different kinds of edges and nodes, and using edge weights, depends on the nature and purpose of the network model.

It is often of interest to determine the ease of communication between nodes in a network, as well as how important a node is in some well-defined sense. Also, it is desirable to be able to assess the sensitivity of the measure of communication between the nodes to perturbations in the edge weights. For instance, if the nodes represent cities, and the edges represent roads between the cities, with edge weights proportional to the amount of traffic on each road, then one may be interested in which road should be widened or made narrower to increase or reduce, respectively, communication in the network the most. The available data may be contaminated by measurement errors. We are then interested in how sensitive to errors in the data our choice of road to widen or make narrower is.

The investigation of the importance of nodes and edges, as well as the sensitivity of the communicability within a network to changes in the edge weights of the network with only one kind of nodes and edges has received considerable attention in the literature; see, e.g., [4–8, 11, 13] and references therein. The present paper extends the communicability and sensitivity analysis in [4, 13] to multilayer networks. Since multilayer networks typically have a large number of nodes and edges, we focus on techniques that are well suited for large-scale networks.

We consider multilayer networks that are represented by graphs which share the same set of vertices $V_N = \{1, 2, \dots, N\}$ and have edges both within a layer and between layers. We will simply refer to this kind of networks as *multilayer networks*. Nice recent discussions on multilayer networks are provided by Bergermann and Stoll [2] and Tudisco et al. [15]. De Domenico et al. [3] describe how multilayer networks with L layers can be modeled by a fourth order tensor and introduce a *supra-adjacency matrix* $B \in \mathbb{R}^{NL\times NL}$ for the representation of such networks. In detail, let $A^{(\ell)} = [w_{ij}^{(\ell)}]_{i,j=1}^{N} \in \mathbb{R}^{N\times N}$ be the non-negative adjacency matrix for the graph in layer ℓ for $\ell = 1, 2, \dots, L$. Thus, the entry $w_{i,j}^{(\ell)} \geq 0$ is the "weight" of the edge between node i and node j in layer ℓ. If the graph is "unweighted", then all nonzero entries of $A^{(\ell)}$ are set to one. The matrix $B \in \mathbb{R}^{NL\times NL}$ is a block matrix with $N \times N$ blocks. The ℓth diagonal block is the adjacency matrix $A^{(\ell)} \in \mathbb{R}^{N\times N}$ for layer ℓ, for $\ell = 1, 2, \dots, L$; the off-diagonal $N \times N$ block in position (ℓ_1, ℓ_2), with $1 \leq \ell_1, \ell_2 \leq L$ and $\ell_1 \neq \ell_2$ represents the inter-layer connections between the layers ℓ_1 and ℓ_2; see Section 4 for details.

We may consider B an adjacency matrix for a complex network with NL nodes. Assuming this network is strongly connected, the Perron-Frobenius theory applies, from which it follows that B has a unique eigenvalue $\rho > 0$ of largest magnitude (the *Perron root*) and that the associated right and left eigenvectors, $\mathbf{x}$ and $\mathbf{y}$, respectively, can be normalized to be of unit Euclidean norm with all components positive. These normalized eigenvectors are commonly referred to as the right and left *Perron vectors*, respectively. Thus,

$$B\mathbf{x} = \rho\mathbf{x}, \qquad \mathbf{y}^T B = \rho\mathbf{y}^T. \tag{1}$$

We will assume throughout this paper that the Perron vectors $\mathbf{x}$ and $\mathbf{y}$ have been normalized in the stated manner.

Following [4], we introduce the *Perron network communicability*,

$$C^{\mathrm{PN}}(B) = \exp_0(\rho)\mathbf{1}_{NL}^T\mathbf{y}\mathbf{x}^T\mathbf{1}_{NL} = \exp_0(\rho)\left(\sum_{j=1}^{NL} y_j\right)\left(\sum_{j=1}^{NL} x_j\right), \tag{2}$$

where

$$\exp_0(t) = \exp(t) - 1, \quad \mathbf{x} = [x_1, x_2, \ldots, x_{NL}]^T, \quad \mathbf{y} = [y_1, y_2, \ldots, y_{NL}]^T,$$

and $\mathbf{1}_{NL} \in \mathbb{R}^{NL}$ denotes the vector of all entries one. The communicability measure (2) is analogous to the total network communicability for a general adjacency matrix $M \in \mathbb{R}^{NL\times NL}$ with NL nodes,

$$C^{\mathrm{TN}}(M) = \mathbf{1}_{NL}^T \exp(M)\mathbf{1}_{NL},$$

introduced by Benzi and Klymko [1], but is cheaper to compute for networks with many nodes or layers, i.e., when NL is large.

Due to the normalization of the Perron vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$1 \leq \sum_{j=1}^{NL} x_j \leq \sqrt{NL}, \qquad 1 \leq \sum_{j=1}^{NL} y_j \leq \sqrt{NL}.$$

Therefore,

$$\exp_0(\rho) \leq C^{\mathrm{PN}}(B) \leq NL\exp_0(\rho). \tag{3}$$

Typically, $\exp_0(\rho) \gg NL$. It then follows that the quantity $\exp_0(\rho)$ is a fairly accurate indicator of the Perron communicability of the graph represented by B in the sense that it suffices to consider $\exp_0(\rho)$ to determine whether the total Perron communicability of a network is large or small. The right-hand side bound in (3) will be sharpened slightly in Proposition 2 below.

Following the approach in [3], we form the leading eigentensors $Y \in \mathbb{R}^{N\times L}$ and $X \in \mathbb{R}^{N\times L}$ for the multilayer network associated with B by reshaping the Perron vectors $\mathbf{y}$ and $\mathbf{x}$, respectively. Thus, the first column of the matrix Y is made up of the first N components of the vector $\mathbf{y}$, the second column of Y are the next N components of the vector $\mathbf{y}$, etc. The *joint eigenvector centrality* of node i in layer ℓ is given by the entry in position (i, ℓ) of Y. The rows of Y represent the *eigenvector versatility* of the nodes. Moreover, the (scalar) *versatility* of node i is given by

$$\nu_i = (Y\mathbf{1}_L)_i, \quad i = 1, 2, \ldots, N. \tag{4}$$

The vector $\mathbf{1}_L$ may be replaced by some other vector in $\mathbb{R}^L$ with nonnegative entries if another weighting of the columns of Y is desired.

Remark 1 *If we replace the matrix B in* (1) *by BB^T, then we obtain analogously to* (2) *the* Perron hub communicability

$$C^{PN}(BB^T) = \exp_0(\rho_{\mathrm{BB^T}})\mathbf{1}_{NL}^T\mathbf{x}\mathbf{x}^T\mathbf{1}_{NL},$$

where $\rho_{\mathrm{BB^T}}$ is the Perron root for BB^T and $\mathbf{x}$ is the Perron vector for BB^T. Similarly, if we replace the matrix B in (1) *by B^TB, then we obtain the* Perron authority communicability

$$C^{PN}(B^TB) = \exp_0(\rho_{\mathrm{B^TB}})\mathbf{1}_{NL}^T\mathbf{x}\mathbf{x}^T\mathbf{1}_{NL},$$

where $\rho_{\mathrm{B^TB}} = \rho_{\mathrm{BB^T}}$ is the Perron root for BB^T and $\mathbf{x}$ is the Perron vector for BB^T.

We turn to special multilayer networks with L layers that do not have edges between nodes in different layers, and we will refer to these kinds of networks as *multiplex networks*. They can be represented by a third-order tensor. The graph for layer ℓ is associated with the non-negative adjacency matrix $A^{(\ell)} \in \mathbb{R}^{N\times N}$, $\ell = 1, 2, \ldots, L$, and a mode-1 unfolding of the third-order tensor that represents the network yields an L-vector of these adjacency matrices:

$$\mathcal{A} = [A^{(1)}, A^{(2)}, \ldots, A^{(L)}] \in \mathbb{R}^{N\times NL}. \tag{5}$$

The supra-adjacency matrix $B \in \mathbb{R}^{NL\times NL}$ for the multiplex network associated with (5) has the diagonal blocks $A^{(\ell)}$, $\ell = 1, 2, \ldots, L$, and every $N \times N$ off-diagonal block is the identity matrix $I_N \in \mathbb{R}^{N\times N}$; see, e.g., [3]. Hence, the coupling is diagonal and uniform. One may introduce a parameter $\gamma \geq 0$ that determines how strongly the layers influence each other. This yields the matrix

$$B := B(\gamma) = \mathrm{diag}[A^{(1)}, A^{(2)}, \ldots, A^{(L)}] + \gamma(\mathbf{1}_L\mathbf{1}_L^T \otimes I_N - I_{NL}), \tag{6}$$

where $\otimes$ denotes the Kronecker product; see [2].

Due to the potentially large size of the matrices B in (1) and (6) one typically computes their right and left Perron vectors by an iterative method, which only require the evaluation of matrix-vector products with the matrices B and B^T. Clearly, does not have to store B, but only the matrix (5) to evaluate matrix-vector products with the matrix (6) and its transpose.

Remark 2 *If one is interested in the Perron hub or authority communicability of the network, then the matrices $A^{(\ell)}$ in* (5) *should be replaced by $A^{(\ell)}(A^{(\ell)})^T$ or $(A^{(\ell)})^TA^{(\ell)}$, respectively, for $\ell = 1, 2, \ldots, L$.*

Following [14, Definition 3.5], we define for future reference the L-dimensional vectors of the *marginal layer Y-centralities* and the *marginal layer X-centralities*

$$\mathbf{c}_Y = Y^T \mathbf{1}_N \quad \text{and} \quad \mathbf{c}_X = X^T \mathbf{1}_N, \tag{7}$$

respectively.

It is the purpose of the present paper to investigate the Perron network communicability of multilayer networks that can be represented by a supra-adjacency matrix $B \in \mathbb{R}^{NL\times NL}$, as well as the special case of multiplex networks that are represented by the matrix (5). We also are interested in the sensitivity of the communicability to errors or changes in the entries of the supra-adjacency matrix B and in the entries of (5) in the case of a multiplex network. The particular structure of B in (6) for multiplex networks will be exploited.

The organization of this paper is as follows. The Wilkinson perturbation for a supra-adjacency matrix is defined in Section 2. This perturbation forms the basis for our sensitivity analysis of multilayer networks. Section 3 discusses some properties of the Perron and total network communicabilities. A sensitivity analysis for multilayer networks based on the Wilkinson perturbation is presented in Section 4. Both Sections 3 and 4 first discuss multilayer networks that can be defined by general supra-adjacency matrices, and subsequently describe simplifications that ensue for multiplex networks that can be defined by (5). Section 5 presents a few computed examples, and Section 6 contains concluding remarks.

2 Wilkinson perturbation for supra-adjacency matrices

Let $B \in \mathbb{R}^{NL\times NL}$ be the supra-adjacency matrix in (1). We assume that B is irreducible. This is equivalent to that the graph associated with B is strongly connected; see, e.g., [9]. Let $\rho > 0$ be the Perron root of B, and let $\mathbf{x}$ and $\mathbf{y}$ be the associated right and left normalized Perron vectors. Thus, all entries of $\mathbf{x}$ and $\mathbf{y}$ are positive, and $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$. Throughout this paper $\|\cdot\|_2$ denotes the Euclidean vector norm or the spectral matrix norm, and $\|\cdot\|_F$ stands for the Frobenius norm. The vectors $\mathbf{x}$ and $\mathbf{y}$ are uniquely determined.

Let $E \in \mathbb{R}^{NL\times NL}$ be a nonnegative matrix such that $\|E\|_2 = 1$, and let $\varepsilon > 0$ be a small constant. Denote the Perron root of $B + \varepsilon E$ by $\rho + \delta\rho$. Then

$$\delta\rho = \varepsilon \frac{\mathbf{y}^T E \mathbf{x}}{\mathbf{y}^T \mathbf{x}} + O(\varepsilon^2); \tag{8}$$

see [10]. Moreover,

$$\frac{\mathbf{y}^T E \mathbf{x}}{\mathbf{y}^T \mathbf{x}} = \frac{|\mathbf{y}^T E \mathbf{x}|}{\mathbf{y}^T \mathbf{x}} \leq \frac{\|\mathbf{y}\|_2 \|E\|_2 \|\mathbf{x}\|_2}{\mathbf{y}^T \mathbf{x}} = \frac{1}{\cos\theta}, \tag{9}$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. The quantity $1/\cos\theta$ is referred to as the *condition number* of ρ and denoted by $\kappa(\rho)$; see Wilkinson [16, Section 2].

Note that when B is symmetric, we have $\mathbf{x} = \mathbf{y}$ and, hence, $\theta = 0$. In this situation ρ is well-conditioned. Equality in (9) is achieved for the *Wilkinson perturbation*

$$E = \mathbf{y}\mathbf{x}^T \in \mathbb{R}^{NL\times NL}, \tag{10}$$

which we will refer to as W. For $E = W$, the perturbation (8) of the Perron root is $\delta\rho = \varepsilon\kappa(\rho) + O(\varepsilon^2)$.

We conclude this section by observing that all the above statements hold true if everywhere the spectral norm is replaced by the Frobenius norm.

3 Some properties of the Perron and total network communicabilities

This section discusses a few properties of the Perron communicability and how it relates to the total network communicability.

Proposition 1

$$C^{\mathrm{PN}}(B) = \exp_0(\rho)\mathbf{c}_Y^T\mathbf{c}_X, \tag{11}$$

where $\mathbf{c}_X$ *is the vector of the marginal layer* X*-centralities and* $\mathbf{c}_Y$ *is the vector of the marginal layer* Y*-centralities in* (7)*.*

Proof The proof follows by (2), observing that

$$\mathbf{1}_{NL}^T\mathbf{y}\mathbf{x}^T\mathbf{1}_{NL} = \mathbf{1}_N^T Y X^T\mathbf{1}_N = \mathbf{c}_Y^T\mathbf{c}_X.$$

Remark 3 *When the network is undirected, according the definition in* (7)*, one has that* $\mathbf{c}_X = \mathbf{c}_Y$*, because* $\mathbf{x} = \mathbf{y}$*. This gives, by* (11)*, the symmetric Perron communicability*

$$C^{\mathrm{PN\,sym}}(B) = \exp_0(\rho)\|\mathbf{c}_Y\|_2^2.$$

Proposition 2

$$C^{\mathrm{PN}}(B) \le NL\exp_0(\rho)\cos\phi,$$

where ϕ *is the angle between the vector* $\mathbf{c}_Y$ *of the marginal layer* Y*-centralities and the vector* $\mathbf{c}_X$ *of the marginal layer* X*-centralities in* (7)*.*

Proof One has

$$\mathbf{c}_Y^T\mathbf{c}_X = \|\mathbf{c}_X\|_2\|\mathbf{c}_Y\|_2\cos\phi,$$

where ϕ is the angle between $\mathbf{c}_Y$ and $\mathbf{c}_X$. Let $\|\cdot\|_1$ denote the vector 1-norm. It is evident that

$$\|\mathbf{c}_X\|_1 = \sum_{j=1}^{NL} x_j = \|\mathbf{x}\|_1, \quad \|\mathbf{c}_Y\|_1 = \sum_{j=1}^{NL} y_j = \|\mathbf{y}\|_1.$$

Since

$$\|\mathbf{c}_X\|_2 \le \|\mathbf{c}_X\|_1 = \|\mathbf{x}\|_1 \le \sqrt{NL}\|\mathbf{x}\|_2, \quad \|\mathbf{c}_Y\|_2 \le \|\mathbf{c}_Y\|_1 = \|\mathbf{y}\|_1 \le \sqrt{NL}\|\mathbf{y}\|_2,$$

we have the bound

$$\|\mathbf{c}_X\|_2\|\mathbf{c}_Y\|_2 \leq NL\|\mathbf{x}\|_2\|\mathbf{y}\|_2 = NL,$$

which gives the proof, by using (11).

Remark 4 *When the network is undirected, by Remark 3, Proposition 2 reads*

$$C^{\mathrm{PN\,sym}}(B) \leq NL\exp_0(\rho),$$

which is the same bound as (3).

Matrix function-based communicability measures have been generalized in [2] to the case of layer-coupled multiplex networks that can be represented by supra-adjacency matrices B of the form (6), i.e., by $\mathcal{A}$ defined by (5). Following the argument in [4], assume that the Perron root ρ of a supra-adjacency matrix B of the form (6) is significantly larger than the magnitude of its the other eigenvalues. Then

$$C_0^{\mathrm{TN}}(\mathcal{A}) \approx \kappa(\rho)C^{\mathrm{PN}}(\mathcal{A}),$$

where $C_0^{\mathrm{TN}}(\mathcal{A}) = \mathbf{1}_{NL}^T\exp_0(B)\mathbf{1}_{NL}$ and $C^{\mathrm{PN}}(\mathcal{A})$ refers to the Perron network communicability (2) when B is of the form (6). Thus, the multiplex total network communicability depends on the conditioning of the Perron root.

Remark 5 *It is straightforward to see that if the network represented by the matrix B of the form* (6) *is undirected, and the Perron root ρ is significantly larger than the magnintude of the other eigenvalues of B, then one has*

$$C_0^{\mathrm{TN\,sym}}(\mathcal{A}) \approx C^{\mathrm{PN\,sym}}(\mathcal{A}).$$

Indeed, the Perron vectors $\mathbf{x}$ and $\mathbf{y}$ coincide so that $\kappa(\rho) = 1$.

4 Multilayer network Perron root sensitivity

Let the supra-adjacency matrix $B \in \mathbb{R}^{NL\times NL}$ be associated with an L-layer network as described above. Then an edge from node i in layer ℓ_i to node j in layer ℓ_j, with $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$, and $\ell_i, \ell_j \in \{1, 2, \ldots, L\}$, is associated with the (i, j)th entry $w_{ij}^{(\ell_i,\ell_j)} > 0$ of the (ℓ_i, ℓ_j) block of order $N \times N$ of the matrix B.

Consider increasing [decreasing] the weight $w_{ij}^{(\ell_i,\ell_j)}$ of an existing edge by $\varepsilon > 0$ [$\varepsilon < 0$], or introducing a new edge from node i in layer ℓ_i to node j in layer ℓ_j with weight $\varepsilon > 0$. This corresponds to perturbing the supra-adjacency matrix B by the matrix εE, where the matrix $E \in \mathbb{R}^{NL\times NL}$ has entries zero everywhere, except for the entry one in position (i, j) in the block (ℓ_i, ℓ_j). It follows from (8) that the impact on the Perron root of this perturbation is

$$\delta\rho = \varepsilon\kappa(\rho)\, y_{N(\ell_i-1)+i}\, x_{N(\ell_j-1)+j} + O(\varepsilon^2).$$

The notion of *multilayer network Perron root sensitivity* with respect to the direction $(i, \ell_i) \longrightarrow (j, \ell_j)$, defined by

$$S^{\mathrm{PR}}_{i,\,j,\,\ell_i,\,\ell_j}(B) := \kappa(\rho)\, y_{N(\ell_i-1)+i}\, x_{N(\ell_j-1)+j}, \tag{12}$$

is helpful for determining which edge(s) to insert in, or remove from, a multilayer network.

Remark 6 *Notice that the largest entries of* $\mathbf{x}$ *and* $\mathbf{y}$ *are strictly smaller than* 1*, hence the multilayer network Perron root sensitivity* (12) *with respect of any direction is less than* $\kappa(\rho)$*. Indeed,* $\mathbf{x}$ *and* $\mathbf{y}$ *are unit vectors with positive entries so that, if, e.g.,* $x_{N(\ell_j-1)+j} = 1$*, this would imply that* $x_k = 0$ *for all* $k \neq N(\ell_j - 1) + j$*, which is not possible.*

We also introduce the *multilayer network Perron root sensitivity matrix* associated with B, denoted by $S^{\mathrm{PR}}(B)$, whose entries are given by the quantities $S^{\mathrm{PR}}_{i,\,j,\,\ell_i,\,\ell_j}(B)$. One easily derives the following result.

Proposition 3 *The multilayer Perron root sensitivity matrix is given by*

$$S^{\mathrm{PR}}(B) = \kappa(\rho) W \in \mathbb{R}^{NL\times NL}, \tag{13}$$

where W *is the Wilkinson perturbation defined by* (10)*.*

Remark 7 *Notice that both the spectral norm and the Frobenius norm of the multilayer network Perron root sensitivity matrix are equal to the condition number of the Perron root. Moreover, the Perron communicability* (2) *reads*

$$C^{\mathrm{PN}}(B) = \frac{\exp_0(\rho)}{\rho} \mathbf{1}_{NL}^T S^{\mathrm{PR}}(B) \mathbf{1}_{NL}.$$

Remark 8 *Following [13, Eqs (2.1)-(2.2)], the spectral impact of each existing edge in* B *can be analyzed by means of the matrix*

$$-\frac{1}{\rho} B \circ S^{\mathrm{PR}}(B) \in \mathbb{R}^{NL\times NL},$$

where $\circ$ *denotes the Hadamard product.*

The exponential of the spectral radius of the graph associated with B often is a fairly accurate relative measure of the Perron network communicability of the graph; cf. (3). If we would like to modify the graph by adding an edge that increases the Perron network communicability as much as possible, then we should choose the indices i, j, ℓ_i, and ℓ_j for the new edge so that

$$x_{N(\ell_j-1)+j} = \max_{1\le k\le NL} x_k, \qquad y_{N(\ell_i-1)+i} = \max_{1\le k\le NL} y_k.$$

Conversely, assume that we wish to simplify a graph by removing an edge, and we would like this simplification not to affect the Perron network communicability much. We then should choose the indices i, j, ℓ_i, and ℓ_j for the edge to be discarded so that

$$x_{N(\ell_j-1)+j} = \min_{1\le k\le NL} x_k, \qquad y_{N(\ell_i-1)+i} = \min_{1\le k\le NL} y_k,$$

where we only minimize over XXX: We remark that the perturbation bound (8) only is valid for ε of small enough magnitude.

Example 4.1. Consider a monoplex network made up of a directed circular graph, whose associated adjacency matrix $B = [w_{ij}]_{i,j=1}^N$ is the circulant matrix defined by

$$w_{i,i+1} = 1, \quad i = 1, 2, \dots, N-1, \qquad w_{N,1} = 1,$$

with all other entries zero. The eigenvalues of B are equidistant on the unit circle in the complex plane with Perron root $\rho = 1$. The perturbation $E = -\mathbf{e}_N \mathbf{e}_1^T$ gives the downshift matrix; all its eigenvalues vanish. Thus, $\rho(B+E) = 0$. This perturbation is too large for the expansion (8) to be meaningful. □

Example 4.2. Removing an edge from a graph may make the graph disconnected. Then the bound (8) might not apply even if the removed edge only has a tiny positive weight. This can be seen considering the graph in Example 4.1 with the weight $w_{N,1}$ reduced to 10^{-4}. □

Despite the limitations of the analysis illustrated by Examples 4.1 and 4.2, the perturbation result (8) is useful for choosing which edge(s) to remove to simplify the multilayer graph. This is illustrated in Section 5. It may be desirable that the graph obtained after removing an edge is connected. The connectedness has to be verified separately.

Remark 9 *Notice that when the network is undirected, it may be meaningful to require the perturbation of the network also be symmetric. Thus, instead of considering the network sensitivity* (12) *with regard to the direction* $(i, \ell_i) \longrightarrow (j, \ell_j)$*, we investigate the sensitivity of the network with regard to perturbations in the directions* $(i, \ell_i) \longrightarrow (j, \ell_j)$ *and* $(j, \ell_j) \longrightarrow (i, \ell_i)$*. This results in the expression*

$$\begin{aligned} S_{i,j,\ell_i,\ell_j}^{\mathrm{PR\,sym}}(B) &:= \kappa(\rho)\,(y_{N(\ell_i-1)+i}\, x_{N(\ell_j-1)+j} + y_{N(\ell_j-1)+j}\, x_{N(\ell_i-1)+i}) \\ &= 2x_{N(\ell_i-1)+i}\, x_{N(\ell_j-1)+j}, \end{aligned}$$

where we have used that $\mathbf{x} = \mathbf{y}$*. This expression is analogous to* (12)*.*

We conclude this section with some comments on multiplex networks. In such a network, an edge from node i to node j in layer ℓ, with $i, j \in \{1, 2, \dots, N\}$, $i \neq j$, and $\ell \in \{1, 2, \dots, L\}$ is associated with the entry $w_{ij}^{(\ell)} \geq 0$ of the adjacency matrix $A^{(\ell)}$. Increasing the weight $w_{ij}^{(\ell)} > 0$ of an existing edge by $\varepsilon > 0$, or introducing a new edge by setting a zero weight w_{ij} to $\varepsilon > 0$, means perturbing $\mathcal{A}$ in (5) by $\varepsilon\mathcal{P}$, where

$$\mathcal{P} = [O_N, \dots, O_N, P_{ij}^{(\ell)}, O_N, \dots, O_N] \in \mathbb{R}^{N \times NL} \quad \text{with} \quad P_{ij}^{(\ell)} = \mathbf{e}_i \mathbf{e}_j^T \in \mathbb{R}^{N \times N}. \tag{14}$$

Here $O_N \in \mathbb{R}^{N \times N}$ denotes the zero matrix. The perturbation $\varepsilon\mathcal{P}$ corresponds to perturbing the supra-adjacency matrix B by an $NL \times NL$ block matrix with all null $N \times N$ blocks except for the ℓth diagonal block $A^{(\ell)}$ in which the (i, j)-entry is set equal to ε.

Introduce the *multiplex Perron root sensitivity* $S^{\mathrm{PR}}_{i,j,\ell}(\mathcal{A})$ with respect to the direction (i,j) in layer ℓ,

$$S^{\mathrm{PR}}_{i,j,\ell}(\mathcal{A}) := \kappa(\rho)\, y_{N(\ell-1)+i}\, x_{N(\ell-1)+j}.$$

which is analogous to the quantity (12) for more general multilayer networks. Thus, if $\mathcal{P}$ is defined by (14) and $\mathcal{A}$ by (5), one has, from (8), $\delta\rho \approx \varepsilon S^{\mathrm{PR}}_{i,j,\ell}(\mathcal{A})$. Analogously, consider reducing the (i,j)th entry $w^{(\ell)}_{ij} > 0$ of the adjacency matrix $A^{(\ell)}$ by ε and assume that ε, $0 < \varepsilon \ll 1$, is small enough so that the multiplex associated with $\mathcal{A} - \varepsilon\mathcal{P}$ is nonnegative and connected. Then $\delta\rho \approx -\varepsilon S^{\mathrm{PR}}_{i,j,\ell}(\mathcal{A})$.

Moreover, as shown in Remark 9, considering an undirected multiplex, results in the expression

$$S^{\mathrm{PR\,sym}}_{i,j,\ell}(\mathcal{A}) := 2\, x_{N(\ell-1)+i}\, x_{N(\ell-1)+j}.$$

Recall that the Perron root sensitivity matrix (13) for general multilayer networks depends on the maximal perturbation $W \in \mathbb{R}^{NL\times NL}$ of the supra-adjacency B as well as on the condition number $\kappa(\rho)$. By assuming that B is of the type in (6), the results in the following section will lead to analogous properties of the *multiplex Perron root sensitivity matrix* $S^{\mathrm{PR}}(\mathcal{A})$, whose non-null entries entries are given by the quantities $S^{\mathrm{PR}}_{i,j,\ell}(\mathcal{A})$.

4.1 Exploiting the multiplex stucture

Consider the cone $\mathcal{D}$ of all nonnegative block-diagonal matrices in $\mathbb{R}^{NL\times NL}$ with L blocks in $\mathbb{R}^{N\times N}$ and let $M|_{\mathcal{D}}$ denote the matrix in $\mathcal{D}$ that is closest to a given matrix $M \in \mathbb{R}^{NL\times NL}$ with respect to the Frobenius norm. It is straightforward to verify that $M|_{\mathcal{D}}$ is obtained by replacing all the entries outside the block-diagonal structure by zero.

Let $E \in \mathcal{D}$ be such that $\|E\|_F = 1$, and let $\varepsilon > 0$ be a small constant. Then

$$\frac{\mathbf{y}^T E\mathbf{x}}{\mathbf{y}^T\mathbf{x}} = \frac{|\mathbf{y}^T E\mathbf{x}|}{\mathbf{y}^T\mathbf{x}} \leq \frac{\|\mathbf{y}\|_2 \|\mathbf{y}\mathbf{x}^T|_{\mathcal{D}}\|_F \|\mathbf{x}\|_2}{\mathbf{y}^T\mathbf{x}} = \frac{\|\mathbf{y}\mathbf{x}^T|_{\mathcal{D}}\|_F}{\mathbf{y}^T\mathbf{x}}, \tag{15}$$

with equality for the *$\mathcal{D}$-structured analogue of the Wilkinson perturbation*

$$E = \frac{\mathbf{y}\mathbf{x}^T|_{\mathcal{D}}}{\|\mathbf{y}\mathbf{x}^T|_{\mathcal{D}}\|_F}; \tag{16}$$

see [12]. The quantity $\|\mathbf{y}\mathbf{x}^T|_{\mathcal{D}}\|_F/\mathbf{y}^T\mathbf{x} = \kappa(\rho)\|\mathbf{y}\mathbf{x}^T|_{\mathcal{D}}\|_F$ will be referred to as the *$\mathcal{D}$-structured condition number* of ρ and denoted by $\kappa_{\mathcal{D}}(\rho)$. For E in (16), the perturbation (8) of the Perron root is $\delta\rho = \varepsilon\kappa_{\mathcal{D}}(\rho) + O(\varepsilon^2)$.

Thus, the $\mathcal{D}$-structured analogue of the Wilkinson perturbation is the maximal perturbation for the Perron root ρ of a supra-adjacency matrix of the type in (6). It is intuitive that the multiplex Perron root sensitivity matrix should consist in the projection into $\mathcal{D}$ of (13) and, in fact, it is straightforward deriving the following results.

Proposition 4 *The multiplex Perron root sensitivity matrix is given by*

$$S^{\mathrm{PR}}(\mathcal{A}) = \kappa(\rho) W|_{\mathcal{D}}\,,$$

where W is the Wilkinson perturbation defined by (10) *and $\mathcal{D}$ is the cone of all nonnegative block-diagonal matrices in $\mathbb{R}^{NL\times NL}$ with L blocks in $\mathbb{R}^{N\times N}$.*

Analogously to (13), the multiplex Perron root sensitivity matrix is the product of the maximal admissible perturbation and the relevant condition number of the Perron value. That is to say that $S^{\mathrm{PR}}(\mathcal{A})$ is given by the product of the $\mathcal{D}$-structured condition number of ρ, $\kappa_{\mathcal{D}}(\rho)$, and the $\mathcal{D}$-structured analogue of the Wilkinson perturbation W:

$$S^{\mathrm{PR}}(\mathcal{A}) = \kappa(\rho)\|W|_{\mathcal{D}}\|_F \frac{W|_{\mathcal{D}}}{\|W|_{\mathcal{D}}\|_F}.$$

Hence, the Frobenius norm of the multiplex Perron root sensitivity matrix is equal to the structured condition number $\kappa_{\mathcal{D}}(\rho)$ of the Perron root; see Remark 7 for the general case of a multilayer network.

The above argument quantitatively shows that the Perron communicability in multiplexes is less sensitive, both component-wise and norm-wise, than the Perron communicability in more general multilayer networks.

Remark 10 *Following the argument in Remark 7, we may claim that the effective Perron communicability in a multiplex is*

$$C^{\mathrm{PN}}(\mathcal{A}) = \frac{\exp_0(\rho)}{\rho}\mathbf{1}_{NL}^T S^{\mathrm{PR}}(\mathcal{A})\mathbf{1}_{NL}.$$

Moreover, observing that

$$\mathbf{1}_{NL}^T S^{\mathrm{PR}}(\mathcal{A})\mathbf{1}_{NL} \leq NL\|S^{\mathrm{PR}}(\mathcal{A})\|_F = NL\,\kappa_{\mathcal{D}}(\rho),$$

gives the following upper bound, which is sharper than (3)*,*

$$C^{\mathrm{PN}}(\mathcal{A}) \leq NL\exp_0(\rho)\|W|_{\mathcal{D}}\|_F.$$

We conclude by observing that the multiplex Perron root sensitivity matrix may have the following alternative representation:

$$S^{\mathrm{PR}}(\mathcal{A}) = \kappa(\rho)\mathcal{W}\,,$$

where

$$\mathcal{W} := [W^{(1)}, W^{(2)}, \ldots, W^{(L)}] \in \mathbb{R}^{N\times NL}. \tag{17}$$

Here $W^{(\ell)} \in \mathbb{R}^{N\times N}$ is constructed by multiplying the ℓth column of Y by the ℓth row of X^T, for $\ell = 1, 2, \ldots, L$, where the matrices $X, Y \in \mathbb{R}^{NL}$ are determined by reshaping the right and left Perron vectors $\mathbf{x}$ and $\mathbf{y}$ of B; see Section 1 for the definition of X and Y.

Remark 11 *Analogously to Remark 8, one notices that the analysis of the spectral impact of each existing edge in* $\mathcal{A}$ *can be made, according to [13, Eqs (2.1)-(2.2)], by means of*

$$-\frac{1}{\rho}\mathcal{A} \circ S^{\mathrm{PR}}(\mathcal{A}).$$

4.2 Exploiting the sparsity structure of multiplexes

When considering perturbation of *existing* edges, in fact we take into account the projection of W into the cone $\mathcal{S}$ of all matrices in $\mathcal{D}$ having the same sparsity structure as the given $\mathrm{diag}(A^{(1)}, \dots, A^{(L)})$. The argument that lead to the *structured* results in (15) and (16) holds true for any (further) sparsity structure of the matrix $\mathrm{diag}(A^{(1)}, \dots, A^{(L)})$. Moreover, $\kappa_{\mathcal{S}}(\rho) \le \kappa_{\mathcal{D}}(\rho) \le \kappa(\rho)$. Thus, we easily derive the following result for the *multiplex Perron root structured sensitivity matrix* $S^{\mathrm{PR\,struct}}(\mathcal{A})$, whose non-null entries entries are given by the quantities $S^{\mathrm{PR}}_{i,\,j,\,\ell}(\mathcal{A})$ relevant to the positive entries in B.

Proposition 5 *The multiplex Perron root structured sensitivity matrix is given by*

$$S^{\mathrm{PR\,struct}}(\mathcal{A}) = \kappa(\rho) W|_{\mathcal{S}}\,,$$

where W *is the Wilkinson perturbation defined by* (10) *and* $\mathcal{S}$ *is the cone of all nonnegative block-diagonal matrices in* $\mathbb{R}^{NL\times NL}$ *with* L *blocks in* $\mathbb{R}^{N\times N}$ *having the same sparsity structure as the diagonal block matrices of the given* B (6)*.*

One has the following component-wise and norm-wise inequalities:

$$S^{\mathrm{PR\,struct}}(\mathcal{A}) \le S^{\mathrm{PR}}(\mathcal{A}),$$

$$\|S^{\mathrm{PR\,struct}}(\mathcal{A})\|_F \le \|S^{\mathrm{PR}}(\mathcal{A})\|_F.$$

Remark 12 *Following the argument in Remark 10, we are in a position to introduce the notion of* structured *Perron communicability in a multiplex and claim that*

$$C^{\mathrm{PN\,struct}}(\mathcal{A}) = \frac{\exp_0(\rho)}{\rho}\mathbf{1}_{NL}^T S^{\mathrm{PN\,struct}}(\mathcal{A})\mathbf{1}_{NL},$$

obtaining, by

$$\mathbf{1}_{NL}^T S^{\mathrm{PR\,struct}}(\mathcal{A})\mathbf{1}_{NL} \le NL\|S^{\mathrm{PR\,struct}}(\mathcal{A})\|_F = NL\,\kappa_{\mathcal{S}}(\rho),$$

the following sharper upper bound

$$C^{\mathrm{PN\,struct}}(\mathcal{A}) \le NL\exp_0(\rho)\|W|_{\mathcal{S}}\|_F \le NL\exp_0(\rho)\|W|_{\mathcal{D}}\|_F.$$

Finally, one may alternatively represent $S^{\mathrm{PR\,struct}}(\mathcal{A})$ as

$$S^{\mathrm{PR\,struct}}(\mathcal{A}) = \kappa(\rho)\mathcal{W}||_{\mathcal{S}}\,,$$

where $\mathcal{W}||_{\mathcal{S}}$ is obtained from $\mathcal{W}$ in (17), by projecting each matrix $W^{(\ell)}$ into the cone $\mathcal{S}^{(\ell)}$ of all nonnegative matrices in $\mathbb{R}^{N\times N}$ having the same sparsity structure as the given $A^{(\ell)}$, for $\ell = 1, 2, \ldots L$.

4.3 About the possible symmetry pattern of multiplexes

Assume the network be represented by a symmetric supra-adjacency matrix B of the type in (6). Applying the arguments in the preceding subsections to the cone of all the symmetric matrices in $\mathcal{D}$ [all the symmetric matrices in $\mathcal{S}$] would lead to the same structured analogue of the Wilkinson perturbation as $W|_{\mathcal{D}}$ [as $W|_{\mathcal{S}}$]. Indeed, as the network is undirected the Perron vectors coincide, so that the Wilkinson perturbation $W = \mathbf{y}\mathbf{x}^T = \mathbf{y}\mathbf{y}^T$ is a symmetric matrix itself.

5 Computed examples

5.1 Example 1 : Small synthetic multilayer network

We construct a small unweighted general multilayer network with $n = 4$ nodes and $L = 3$ layers. The block diagonal matrices are obtained as the adjacency matrices of the the graphs representing each layer. The off diagonal blocks represent the edges connecting nodes from different layers. Some of the edges are directed, which result in a non-symmetric supra adjacency matrix. The perron root of the supra adjacency matrix is $\rho(B) = 2.3471$ and its condition number is $\kappa(\rho(B)) = 1.0248$ Let $\epsilon = 0.4$. The new Perron root $\rho(B + \epsilon W)$, where W is the matrix in (10), is 2.7516. Thus the spectral radius increases by 0.4045 as expected since $\epsilon\kappa(\rho) = 0.4099$ If we replace the matrix W by the matrix of all ones, normalized to be of unit Frobenius norm, then the spectral radius increases by 0.3452. Clearly, this is not an accurate estimate of the actual worst-case sensitivity of $\rho(B)$ to perturbations.
Now, we compute the largest entry of the Perron root sensitivity matrix $S^{PR}_{2,3,4,2}(B) = 0.2134$ Increasing the flow of the edge connecting node 2 from layer 3 and node 4 in layer 2 will result in an important change in the Perron root.

5.2 Example 2 : ScotlandYard data set

In this example, we consider the Scotland Yard transportation network created by the authors of [2]. It consists of $N = 199$ nodes representing public transport stops in the city of London and $L = 4$ layers representing different mode of transportation (boat, underground, bus and taxi). The edges are weighted and undirected. We compute the perron root of the supra adjacency matrix $\rho(B) = 17.6055$ and its condition number is $\kappa(\rho(B)) = 1$ Let $\epsilon = 0.5$. The new Perron root $\rho(B + \epsilon W)$, where W is the matrix in (10), is 18.1055. Thus the spectral radius increases by 0.5 as expected since $\epsilon\kappa(\rho) = 0.5$ If we replace the matrix W by the matrix of all ones, normalized to be of unit Frobenius norm, then the spectral radius increases by 0.009. Clearly, this is not an accurate estimate of the actual worst-case sensitivity of $\rho(B)$ to perturbations. Now, we compute the largest entry of the Perron root sensitivity matrix $S^{PR}_{89,2,67,2}(B) = 0.2407$ Increasing the weight of the edge connecting nodes 89 and 67 from layer 2 will result in an important change in the Perron root comparing to increasing the weight of a random edge. $\rho(B)$ increase by 0.2445 when Increasing the weight of the edge connecting nodes 89 and 67 from layer 2 by 0.5 On the other hand, the Perron root $\rho(B)$ does not change when setting the entry $(162, 560)$ of B to zero. This corresponds to removing the edge connecting the node 162 from layer 1 to its corresponding node 162 from layer 3. This edge corresponds to the smallest entry of the Perron root sensitivity matrix $S^{PR}_{162,1,162,3}(B) = 3.2279.10^{-5}$.

5.3 Example 3 : European Airlines data set

European Airlines data set consists of 450 nodes representing European airports and 37 layers representing different airlines operating in Europe, with 3588 total number of edges. This network can be represented by a supra adjacency matrix B as in (6) where the block diagonal matrices contain ones if an airline offers a flight connection between two airports and zeros otherwise. Each off diagonal block is the identity matrix, it reflects the effort added by changing airlines on connecting flights. As mentioned in [], we only include $N = 417$ nodes from the the largest connected component of the network that is associated with the sum of the layers adjacency matrices to guarantee the existence of the Perron value. We compute the largest eigenvalue $\rho(B) = 38.3714$. Let $\epsilon = 0.5$, the new Perron root $\rho(B + \epsilon W)$, where W is the matrix in (10), is 38.8714. Thus the spectral radius increases by 0.5 as expected since $\epsilon\kappa(\rho) = 0.5$.

If we replace the matrix W by the matrix of all ones, normalized to be of unit Frobenius norm, then the spectral radius increases by 0.1616. Clearly, this is not an accurate estimate of the actual worst-case sensitivity of $\rho(B)$ to perturbations.

Now, we compute the smallest entry of the Perron root sensitivity matrix $S^{PR}_{202,31,202,28}(B) = 5.1845.10^{-13}$. This suggest that the cost of changing from Czech airline to Niki airline at Valan Airport could be avoided without influencing the communicability of the network.

On the other hand, we compute two largest entries of the Perron root sensitivity matrix $S^{PR}_{38,1,2,1}(B) = 0.004$ and $S^{PR}_{157,1,2,1}(B) = 0.0036$. This suggests that the Perron root may be increased the most by increasing the flow of flights operated by Lufthansa airline between Munich airport and Frankfurt Am Main airport and between Dusseldorf airport and Frankfurt Am Main airport.

5.4 Example 4 :General multilayer

We consider an example of a general multilayer with 180 nodes and 6 layers. We compute the Perron root of the supra adjacency matrix $\rho(B) = 8.1324$ with condition number $\kappa(\rho) = 1.3277$. Let $\epsilon = 0.5$, the Perron root of $B+\epsilon W$, where W is the Wilnkson matrix increased by 0.644 as expected since $\epsilon\kappa(\rho) = 0.6639$. We compute the largest entry of the Perron root sensitivity matrix $S^{PR}_{6,1,24,1}(B) = 0.1922$ Increasing the weight of the edge connecting node 6 and node 24 from layer 1 by 0.5 increased in the Perron root by 0.1643.

6 Conclusion

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