Internality of generalized averaged Gaussian quadrature rules and truncated variants for measures induced by Chebyshev polynomials

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Abstract

Generalized averaged Gaussian quadrature rules and truncated variants associated with a nonnegative measure with support on a real open interval $\{t : a < t < b\}$ may have nodes outside this interval, in other words the rules may fail to be internal. Such rules cannot be applied when the integrand is defined on $\{t : a < t < b\}$ only. This paper investigates whether generalized averaged Gaussian quadrature rules and truncated variants are internal for measures induced by Chebyshev polynomials. Our results complement those of Notaris [13] for Gauss-Kronrod quadrature formulas for the same kind of measures.

Keywords: Gauss quadrature, averaged Gauss quadrature, truncated generalized averaged Gauss quadrature, internality of quadrature, measures induced by Chebyshev polynomials

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1. Introduction

We consider the approximation of integrals of the form

$$I[f] := \int_a^b f(t)\, d\lambda(t),$$

with a nonnegative measure $d\lambda$ supported on the real interval $\{t : a < t < b\}$ by an interpolatory s-node quadrature formula

$$Q_s[f] = \sum_{j=1}^{s} \omega_j f(x_j).$$

For clarity, we will generally denote the nodes x_j and weights ω_j by $x_j(Q_s)$ and $\omega_j(Q_s)$, respectively. The nodes are assumed to be real and ordered, i.e.,

$$x_1(Q_s) < x_2(Q_s) < \cdots < x_s(Q_s).$$

We say that $x_1(Q_s)$ and $x_s(Q_s)$ are the first and last nodes of the quadrature rule Q_s, respectively.

Let p_k denote the monic polynomial of degree k that is orthogonal to $\mathbb{P}_{k-1}$ (the set of all polynomials of degree less than or equal to $k-1$) relative to the measure $d\lambda$, i.e.,

$$\int_a^b t^j p_k(t)\, d\lambda(t) = 0, \quad j = 0, 1, \ldots, k-1.$$

Recall that the polynomials p_k satisfy a three-term recurrence relation of the form

$$p_{k+1}(t) = (t - \alpha_k)p_k(t) - \beta_k p_{k-1}(t), \quad k = 0, 1, \ldots, \tag{1.1}$$

where $p_{-1}(t) \equiv 0$, $p_0(t) \equiv 1$, and the coefficients β_k are positive. It is well known that the unique interpolatory quadrature rule with n nodes and the highest possible algebraic degree of precision, $2n-1$, is the Gaussian rule relative to the measure $d\lambda$,

$$G_n[f] = \sum_{j=1}^{n} \omega_j(G_n) f(x_j(G_n)).$$

The nodes and weights of G_n can easily be computed by the Golub-Welsch algorithm [11] in only $\mathcal{O}(n^2)$ arithmetic floating point operations (flops). This algorithm is based on the observation that the nodes are the

eigenvalues and the weights are proportional to the square of the first component of the eigenvectors of the symmetric tridiagonal matrix

$$J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ 0 & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{1.2}$$

determined by the recursion coefficients (1.1). This matrix is known as the Jacobi matrix associated with G_n.

An important problem in computational mathematics is to find another quadrature rule, Q_s, that is of higher algebraic degree of precision than $G_n[f]$, for estimating the error $|I[f] - G_n[f]|$ by calculating

$$|Q_s[f] - G_n[f]|\,, \tag{1.3}$$

i. e., $Q_s[f]$ plays the role of the true value of the integral. For instance, Q_s can be chosen as the $(2n+1)$-node Gauss-Kronrod quadrature rule, H_{2n+1}, associated with G_n, when it exists. In this case, the degree of precision of H_{2n+1} is at least $3n+1$. However, Gauss-Kronrod rules are known not to exist for several of the classical weight functions, including the Hermite and Laguerre weight functions; see [7, 18, 19]. A nice recent survey of Gauss-Kronrod rules is provided by Notaris [14].

The non-existence of certain Gauss-Kronrod rules inspired Laurie [12] to develop anti-Gaussian quadrature rules. These rules always exist and have real nodes, at most two of which are outside the convex hull of the support of the measure $d\lambda$. Moreover, all quadrature weights are positive and the anti-Gauss rules easily can be constructed by computing the eigenvalues and squares of the first component of suitably normalized eigenvectors of a symmetric tridiagonal matrix closely related to the matrix (1.2).

Spalević [22] (by following Peherstorfer [16]; see also [17]) proposed a simple numerical method for constructing generalized averaged Gaussian formulas $\widehat{G}_{2n+1}$, which are well suited for estimating the error in G_n; cf. (1.3). The nodes of the quadrature rules $\widehat{G}_{2n+1}$ are the zeros of the polynomial

$$t_{2n+1} := p_n \cdot F_{n+1},$$

where

$$F_{n+1} = p_{n+1} - \bar{\beta}_{n+1} \cdot p_{n-1} \tag{1.4}$$

and the p_k are the orthogonal polynomials (1.1). It is shown in [22] that $\widehat{G}_{2n+1}$ has algebraic degree of precision at least $2n+2$ when $\bar{\beta}_{n+1} = \beta_{n+1}$.

In this case, we denote this formula by $\widehat{G}^S_{2n+1}$. Details on these rules and discussions on applications can be found in the recent papers [3, 6, 20, 21, 22, 23].

When we instead let $\bar{\beta}_{n+1} = \beta_n$ in (1.4), the generalized averaged Gaussian formula $\widehat{G}_{2n+1}$ becomes the quadrature rule $\widehat{G}^L_{2n+1}$ introduced by Laurie [12]. It has algebraic degree of precision at least $2n+1$. The rules $\widehat{G}^L_{2n+1}$ are closely connected with the so called anti-Gaussian quadrature rules; see [1, 2, 12, 14, 15] for details on these rules. Truncated versions of the quadrature formula $\widehat{G}^S_{2n+1}$, with algebraic degree of precision at least $2n+2$ and with only $2n-r+1$ nodes, have been considered in [21]. We denote the truncated rules by $Q^{(n-r)}_{2n-r+1}$ $(n \geq 2)$, $r = 1, 2, \ldots, n-1$. The simplest truncated generalized averaged Gaussian quadrature formula is

$$Q^{(1)}_{n+2}[f] = \sum_{j=1}^{n+2} \omega_j(Q^{(1)}_{n+2}) f(\tau_j(Q^{(1)}_{n+2}));$$

see [3, Eq. (4.1)] for discussions. Its nodes are the zeros of the polynomial

$$t_{n+2}(t) = (t - \alpha_{n-1})p_{n+1}(t) - \beta_{n+1}p_n(t). \tag{1.5}$$

Let $\widehat{T}_{2n+1} \in \mathbb{R}^{(2n+1)\times(2n+1)}$ denote the Jacobi matrix associated with $\widehat{G}^S_{2n+1}$. Then its leading principal submatrix of order $2n-r+1$ is the Jacobi matrix associated with $Q^{(n-r)}_{2n-r+1}$. The nodes and weights of $Q^{(n-r)}_{2n-r+1}$ can be computed by applying the Golub-Welsch algorithm to this submatrix.

The internality of the quadrature rules $\widehat{G}^L_{2n+1}$, $\widehat{G}^S_{2n+1}$, and $Q^{(1)}_{n+2}$ $(n \geq 2)$ for the classical weight functions is investigated in [12], [22], and [3], respectively; for Bernstein-Szegő weight functions the internality is discussed in [4]. The latter weight functions were introduced by Gautschi and Notaris [10], who studied the internality of Gauss-Kronrod rules for Bernstein-Szegő weight functions.

It is the purpose of the present paper to investigate the internality of the quadrature rules $\widehat{G}^L_{2n+1}$, $\widehat{G}^S_{2n+1}$ $(n \geq 1)$, and $Q^{(1)}_{n+2}$ $(n \geq 2)$ for measures induced by Chebyshev polynomials. These measures were introduced by Gautschi and Li [9]. The beginning of Section 2 recalls, from [9, Section 3], some results for monic orthogonal polynomials for the mentioned measures. An application to constrained least-squares polynomial approximation described by Gautschi [8] is outlined at the end of the section. Section 3 studies the internality of generalized averaged Gaussian rules and truncated variants for these measures. Numerical examples illustrate the theory and the usefulness of these quadrature rules for estimating the error in Gaussian rules. Concluding remarks can be found in Section 4.

2. Monic orthogonal polynomials for measures induced by Chebyshev polynomials

This section reviews results from [9, Theorems 3.1–3.7] and [13, Section 2] for the sake of completeness. For fixed $n \geq 1$, we define the measures

$$d\hat{\lambda}_n^{[i]}(t) = \left[\pi_n^{[i]}(t)\right]^2 d\lambda^{[i]}(t), \quad -1 < t < 1, \quad i = 1, 2, 3, 4, \tag{2.1}$$

where

$$\begin{aligned} d\lambda^{[1]}(t) &= \frac{dt}{\sqrt{1-t^2}}\, dt, \qquad d\lambda^{[2]}(t) = \sqrt{1-t^2}\, dt, \\ d\lambda^{[3]}(t) &= \sqrt{\frac{1+t}{1-t}}\, dt, \qquad d\lambda^{[4]}(t) = \sqrt{\frac{1-t}{1+t}}\, dt \end{aligned} \tag{2.2}$$

are the four Chebyshev measures and

$$\begin{aligned} \pi_n^{[1]}(t) &= \overset{\circ}{T}_n\,(t) = 2^{1-n} T_n(t), \qquad \pi_n^{[2]}(t) = \overset{\circ}{U}_n\,(t) = 2^{-n} U_n(t), \\ \pi_n^{[3]}(t) &= \overset{\circ}{V}_n\,(t) = 2^{-n} V_n(t), \qquad \pi_n^{[4]}(t) = \overset{\circ}{W}_n\,(t) = 2^{-n} W_n(t) \end{aligned} \tag{2.3}$$

are the corresponding nth-degree monic Chebyshev polynomials. The latter can be expressed with the well-known formulas

$$\begin{aligned} T_n(\cos\theta) &= \cos n\theta, & U_n(\cos\theta) &= \frac{\sin(n+1)\theta}{\sin\theta}, \\ V_n(\cos\theta) &= \frac{\cos(n+1/2)\theta}{\cos(\theta/2)}, & W_n(\cos\theta) &= \frac{\sin(n+1/2)\theta}{\sin(\theta/2)}. \end{aligned}$$

Let

$$\hat{\pi}_{m,n}^{[i]}(\cdot) = \pi_m\left(\cdot; d\hat{\lambda}_n^{[i]}\right), \quad m = 0, 1, 2, \ldots, \quad i = 1, 2, 3, 4,$$

denote the monic orthogonal polynomials relative to the measures (2.1).

Theorem 2.1. *For any $n \geq 1$, there holds*

$$\hat{\pi}_{m,n}^{[1]}(t) = \overset{\circ}{T}_m\,(t), \quad 0 \leq m \leq n. \tag{2.4}$$

Theorem 2.2. *If $n = 1$, then*

$$\hat{\pi}_{m,1}^{[1]}(t) = \begin{cases} \displaystyle\sum_{i=0}^{m/2} (-1)^i 4^{-i}\ \overset{\circ}{T}_{m-2i}\,(t), & m \text{ (even)} \geq 2, \\ \displaystyle\sum_{i=0}^{(m-1)/2} (-1)^i 4^{-i} \frac{m-2i}{m}\ \overset{\circ}{T}_{m-2i}\,(t), & m \text{ (odd)} \geq 1. \end{cases} \tag{2.5}$$

Moreover, the recurrence relation for the polynomials $\hat{\pi}^{[1]}_{m,1}$, $m = 0, 1, 2, \ldots$, *is given by*

$$\begin{aligned} \hat{\pi}^{[1]}_{-1,1}(t) &= 0, \quad \hat{\pi}^{[1]}_{0,1}(t) = 1, \\ \hat{\pi}^{[1]}_{k+1,1}(t) &= t\,\hat{\pi}^{[1]}_{k,1}(t) - \hat{\beta}^{[1]}_{k,1}\hat{\pi}^{[1]}_{k-1,1}(t), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{2.6}$$

where

$$\hat{\beta}^{[1]}_{0,1} = \frac{\pi}{2}, \quad \hat{\beta}^{[1]}_{k,1} = \begin{cases} \dfrac{1}{4}\left(1 - \dfrac{2}{k+1}\right), & k \text{ (even)} \geq 2, \\ \dfrac{1}{4}\left(1 + \dfrac{2}{k}\right), & k \text{ (odd)} \geq 1. \end{cases} \tag{2.7}$$

Theorem 2.3. *For* $n \geq 2$, *the polynomials* $\hat{\pi}^{[1]}_{m,n}$, $m = 0, 1, 2, \ldots$, *satisfy*

$$\begin{aligned} \hat{\pi}^{[1]}_{-1,n}(t) &= 0, \quad \hat{\pi}^{[1]}_{0,n}(t) = 1, \\ \hat{\pi}^{[1]}_{k+1,n}(t) &= t\,\hat{\pi}^{[1]}_{k,n}(t) - \hat{\beta}^{[1]}_{k,n}\hat{\pi}^{[1]}_{k-1,n}(t), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{2.8}$$

where

$$\hat{\beta}^{[1]}_{k,n} = \begin{cases} \dfrac{\pi}{2^{2n-1}}, & \text{if } k = 0, \\ \dfrac{1}{4}\left(1 - \dfrac{(-1)^{k/n}}{1 + k/n}\right), & \text{if } k = 0 \pmod{n},\ k \neq 0, \\ \dfrac{1}{4}\left(1 + \dfrac{(-1)^{(k-1)/n}}{1 + (k-1)/n}\right), & \text{if } k = 1 \pmod{n}, \\ \dfrac{1}{4}, & \text{otherwise.} \end{cases} \tag{2.9}$$

Theorem 2.4. *For any* $n \geq 1$, *there holds*

$$\hat{\pi}^{[2]}_{m,n}(t) = \overset{\circ}{T}_m(t), \quad 0 \leq m \leq n+1. \tag{2.10}$$

Theorem 2.5. *For* $n \geq 1$, *the polynomials* $\hat{\pi}^{[2]}_{m,n}$, $m = 0, 1, 2, \ldots$, *satisfy*

$$\begin{aligned} \hat{\pi}^{[2]}_{-1,n}(t) &= 0, \quad \hat{\pi}^{[2]}_{0,n}(t) = 1, \\ \hat{\pi}^{[2]}_{k+1,n}(t) &= t\,\hat{\pi}^{[2]}_{k,n}(t) - \hat{\beta}^{[2]}_{k,n}\hat{\pi}^{[2]}_{k-1,n}(t), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{2.11}$$

where

$$\hat{\beta}^{[2]}_{k,n} = \begin{cases} \dfrac{\pi}{2^{2n+1}}, & \text{if } k=0, \\ \dfrac{1}{4}\left(1-\dfrac{1}{1+k/(n+1)}\right), & \text{if } k=0 \pmod{n+1},\ k\neq 0, \\ \dfrac{1}{4}\left(1+\dfrac{1}{1+(k-1)/(n+1)}\right), & \text{if } k=1 \pmod{n+1}, \\ \dfrac{1}{4}, & \text{otherwise.} \end{cases} \tag{2.12}$$

Theorem 2.6. *For any $n \geq 1$, there holds*

$$\hat{\pi}^{[3]}_{m,n}(t) = \overset{\circ}{T}_m(t), \quad 0 \leq m \leq n. \tag{2.13}$$

Theorem 2.7. *For $n \geq 1$, the polynomials $\hat{\pi}^{[3]}_{m,n}$, $m = 0, 1, 2, \ldots,$ satisfy*

$$\begin{aligned} \hat{\pi}^{[3]}_{-1,n}(t) &= 0, \quad \hat{\pi}^{[3]}_{0,n}(t) = 1, \\ \hat{\pi}^{[3]}_{k+1,n}(t) &= \left(t - \hat{\alpha}^{[3]}_{k,n}\right)\hat{\pi}^{[3]}_{k,n}(t) - \hat{\beta}^{[3]}_{k,n}\hat{\pi}^{[3]}_{k-1,n}(t), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{2.14}$$

where

$$\hat{\alpha}^{[3]}_{k,n} = \begin{cases} \dfrac{1}{4}\,\dfrac{1}{1+(k-n)/(2n+1)}, & \text{if } k=n \pmod{2n+1}, \\ -\dfrac{1}{4}\,\dfrac{1}{1+(k-n-1)/(2n+1)}, & \text{if } k=n+1 \pmod{2n+1}, \\ 0, & \text{otherwise} \end{cases} \tag{2.15}$$

and

$$\hat{\beta}^{[3]}_{k,n} = \begin{cases} \dfrac{\pi}{2^{2n}}, & \text{if } k=0, \\ \dfrac{1}{4}\left(1-\dfrac{1}{1+2k/(2n+1)}\right), & \text{if } k=0 \pmod{2n+1},\ k\neq 0, \\ \dfrac{1}{4}\left(1+\dfrac{1}{1+2(k-1)/(2n+1)}\right), & \text{if } k=1 \pmod{2n+1}, \\ \dfrac{1}{4}\left(1-\dfrac{1}{4[1+(k-n-1)/(2n+1)]^2}\right), & \text{if } k=n+1 \pmod{2n+1}, \\ \dfrac{1}{4}, & \text{otherwise.} \end{cases} \tag{2.16}$$

For any $n \geq 1$, we have

$$\hat{\pi}_{m,n}^{[4]}(t) = (-1)^m \, \hat{\pi}_{m,n}^{[3]}(-t), \quad m = 0, 1, 2, \ldots;$$

cf. [9, Eq. (3.15)] and [13, Eq. (2.21)]. Therefore, Theorems 2.6 and 2.7 can be used to obtain formulas for the recursion coefficients for $\hat{\pi}_{m,n}^{[4]}$ that are analogous to those for $\hat{\pi}_{m,n}^{[3]}$.

We conclude this section with a discussion on an application of orthogonal polynomials relative to inner products that are defined by measures of the form (2.1). This application is described by Gautschi [8, Section 8]. Consider the task of approximating a given continuously differentiable function f by a polynomial $p \in \mathbb{P}_N$ in the open interval $-1 < t < 1$ in the least-squares sense

$$\min_{p \in \mathbb{P}_N} \int_{-1}^{1} (f(t) - p(t))^2 d\lambda^{[i]}(t), \tag{2.17}$$

where $d\lambda^{[i]}$ for some $i \in \{1, 2, 3, 4\}$ is one of the measures (2.2). Assume that we would like the polynomial p to satisfy the interpolation conditions

$$p(t_j) = f(t_j), \qquad 1 \leq j \leq n, \tag{2.18}$$

where $n < N$ and the t_j are the zeros of the orthogonal polynomial $\pi_n^{[i]}$ defined by (2.3). This is a constraint least-squares problem. This kind of problem is difficult to solve numerically. Gautschi [8, Section 8] observed that the desired polynomial can be expressed as

$$p(t) = p_{n-1}(t; f) + \pi_n^{[i]}(t) q(t),$$

where $p_{n-1}(\cdot; f) \in \mathbb{P}_{n-1}$ solves the interpolation problem (2.18), $\pi_n^{[i]}$ is given by (2.3), and $q \in \mathbb{P}_{N-n}$. Then

$$\begin{aligned} \int_{-1}^{1} [f(t) - p(t)]^2 \, d\lambda^{[i]}(t) &= \int_{-1}^{1} \left[f(t) - p_{n-1}(t; f) - \pi_n^{[i]}(t) q(t) \right]^2 d\lambda^{[i]}(t) \\ &= \int_{-1}^{1} \left[\frac{f(t) - p_{n-1}(t; f)}{\pi_n^{[i]}(t)} - q(t) \right]^2 \left[\pi_n^{[i]}(t) \right]^2 d\lambda^{[i]}(t). \end{aligned}$$

Letting

$$g(t) = \frac{f(t) - p_{n-1}(t; f)}{\pi_n^{[i]}(t)} \tag{2.19}$$

gives the unconstrained least-squares problem

$$\min_{q\in\mathbb{P}_{N-n}} \int_{-1}^{1} [g(t)-q(t)]^2 \left[\pi_n^{[i]}(t)\right]^2 d\lambda^{[i]}(t)$$

with a measure of the form (2.1). The solution of this unconstrained least-squares problem is much easier than the solution of the constrained least-squares problem (2.17)-(2.18). The former can be solved by expressing q as a linear combination of the orthogonal polynomials $\pi_{0,n}^{[i]}, \pi_{1,n}^{[i]}, \ldots, \pi_{N-n,n}^{[i]}$. The determination of the coefficients of this linear combination requires the calculation of integrals of the form

$$\int_{-1}^{1} g(t)\pi_{j,n}^{[i]}(t) \left[\pi_n^{[i]}(t)\right]^2 d\lambda^{[i]}(t), \quad j=0,1,\ldots,N-n. \tag{2.20}$$

The form of the function g (2.19) typically makes it necessary to evaluate the integrals (2.20) by a quadrature rule, such as a Gaussian rule relative to the measure $\left[\pi_n^{[i]}(t)\right]^2 d\lambda^{[i]}$. The following section discusses how the quadrature error when using a Gaussian rule relative to such a measure can be estimated accurately and inexpensively.

3. Internality of generalized Gaussian rules and their truncated variants

Notaris [13, Section 3] presented explicit formulas for the monic Stieltjes polynomials

$$\hat{\pi}_{n+1,n}^{*[i]}(\cdot) = \pi_{n+1}^{*}\left(\cdot; d\hat{\lambda}_n^{[i]}\right), \quad i=1,2,3,4,$$

relative to the measures (2.1) in terms of monic Chebyshev polynomials of the first and second kinds. This allowed him to analyze the internality of the corresponding Gauss-Kronrod quadrature formulas. We will denote the $(2n+1)$-node Gauss-Kronrod rule that is associated to an n-node Gauss rule by H_{2n+1}. In this section we are concerned with generalized averaged Gaussian quadrature rules $\widehat{G}_{2n+1}^{S}$ and truncated variants $Q_{2n-r+1}^{(n-r)}$ $(n\geq 2)$, $r=1,2,\ldots,n-1$, as well as with the rules $\widehat{G}_{2n+1}^{L}$. We will not discuss the evaluation of these quadrature rules since this already has been done in [4]. Here we only note that the nodes and weights of $\widehat{G}_{2n+1}^{S}$ and $\widehat{G}_{2n+1}^{L}$ can be computed in only $\mathcal{O}(n^2)$ flops by the Golub-Welsch algorithm applied to a symmetric tridiagonal matrix analogous to (1.2) of order $2n+1$. The trailing nontrivial entries of this matrix are the leading nontrivial entries of the matrix (1.2); see [4] for details.

3.1. The measures $d\hat{\lambda}_n^{[1]}$ $(n \geq 2)$

It follows from (2.1), (2.2), and (2.3) that

$$d\hat{\lambda}_n^{[1]}(t) = \left[\overset{\circ}{T}_n\,(t)\right]^2 (1-t^2)^{-1/2}dt, \quad -1 < t < 1, \quad n \geq 2.$$

Using (2.4) and the relations $\hat{\beta}_{n,n}^{[1]} = 3/8$, $\hat{\beta}_{n+1,n}^{[1]} = 1/8$, which follow from (2.9), we obtain from (2.8) that

$$\begin{aligned} \hat{\pi}_{n+1,n}^{[1]}(t) &= t\hat{\pi}_{n,n}^{[1]}(t) - \hat{\beta}_{n,n}^{[1]}\hat{\pi}_{n-1,n}^{[1]}(t) \\ &= t\,\overset{\circ}{T}_n\,(t) - \frac{3}{8}\,\overset{\circ}{T}_{n-1}\,(t). \end{aligned}$$

The polynomial F_{n+1} defined by (1.4) can be written as

$$\begin{aligned} F_{n+1,n}^{[1]S}(t) &= \hat{\pi}_{n+1,n}^{[1]}(t) - \hat{\beta}_{n+1,n}^{[1]}\hat{\pi}_{n-1,n}^{[1]}(t) \\ &= t\,\overset{\circ}{T}_n\,(t) - \frac{1}{2}\,\overset{\circ}{T}_{n-1}\,(t). \end{aligned}$$

In the sequel, we will need the well-known property

$$\overset{\circ}{T}_k\,(\mp 1) = \frac{(\mp 1)^k}{2^{k-1}}, \quad k \in \mathbb{N}. \tag{3.1}$$

The conditions of internality of the first and last zeros of the polynomial (1.4), and therefore of the corresponding generalized averaged Gaussian quadrature formula $\widehat{G}_{2n+1}$, can be expressed as (cf. [3, 4, 5])

$$(-1)^{n+1}F_{n+1}(-1) \geq 0, \quad F_{n+1}(1) \geq 0. \tag{3.2}$$

For the polynomial $F_{n+1,n}^{[1]S}$ of the present section, we have

$$F_{n+1,n}^{[1]S}(1) = \frac{1}{2^{n-1}} - \frac{1}{2}\frac{1}{2^{n-2}} = 0 \tag{3.3}$$

and

$$(-1)^{n+1}F_{n+1,n}^{[1]S}(-1) = (-1)^{n+1}\left[\frac{(-1)^{n+1}}{2^{n-1}} - \frac{1}{2}\frac{(-1)^{n-1}}{2^{n-2}}\right] = 0. \tag{3.4}$$

Since the quantities in equations (3.3) and (3.4) vanish, the first and last zeros of the polynomial $F_{n+1,n}^{[1]S}$ are ∓ 1, and it follows that the formula $\widehat{G}_{2n+1}^{S}$ is of Lobatto-type.

Proposition 3.1. *The generalized averaged Gaussian quadrature formulas* $\widehat{G}^S_{2n+1}$ *relative to the measure* $d\hat{\lambda}^{[1]}_n(t)$ *and truncated variants* $Q^{(n-r)}_{2n-r+1}$ *(*$n \geq 2$*),* $r = 1, 2, \ldots, n-1$*, are internal.*

Proof. The inequalities (3.3) and (3.4) show (3.2) from which the internality of $\widehat{G}^S_{2n+1}$ follows. As mentioned above, the zeros of $\widehat{G}^S_{2n+1}$ are eigenvalues of a symmetric tridiagonal matrix $\widehat{T}_{2n+1} \in \mathbb{R}^{(2n+1)\times(2n+1)}$ and the zeros of $Q^{(n-r)}_{2n-r+1}$ are the eigenvalues of the leading principal submatrix of $\widehat{T}_{2n+1}$ of order $2n-r+1$. The convex hull of the eigenvalues of this submatrix is a subset of the convex hull of the eigenvalues of $\widehat{T}_{2n+1}$. This shows that the rules $Q^{(n-r)}_{2n-r+1}$ are internal. $\square$

The first $4n-1$ entries of the Jacobi matrix associated with $\widehat{G}^S_{2n+1}$ ($n \geq 2$) coincide with the first $4n-1$ entries of the Jacobi matrix associated with G_{2n+1}. This implies that $\widehat{G}^S_{2n+1}$ is exact for polynomials in $\mathbb{P}_{4n-1}$. As $4n-1 \geq 3n+1$ for $n \geq 2$, we conclude that $\widehat{G}^S_{2n+1}$ agrees with the Gauss-Kronrod quadrature formula H_{2n+1}; see [24, Theorem 3.1].

We note that

$$F^{[1]S}_{n+1,n}(t) = t\,\overset{\circ}{T}_n\,(t) - \frac{1}{2}\,\overset{\circ}{T}_{n-1}\,(t) = \overset{\circ}{T}_{n+1}\,(t) - \frac{1}{4}\,\overset{\circ}{T}_{n-1}\,(t) = (t^2-1)\,\overset{\circ}{U}_{n-1}\,(t).$$

Notaris [13, Eq. (3.1)] derived this form of the Stieltjes polynomial for the Gauss-Kronrod rule H_{2n+1} in another manner. The last equality can easily be derived by using the trigonometric representations of $\overset{\circ}{T}_m$ and $\overset{\circ}{U}_m$; see (2.3) and the formulas that follow (2.3).

We turn to the generalized averaged Gaussian formula $\widehat{G}^L_{2n+1}$. It follows from

$$\begin{aligned} F^{[1]L}_{n+1,n}(t) &= \hat{\pi}^{[1]}_{n+1,n}(t) - \hat{\beta}^{[1]}_{n,n}\,\hat{\pi}^{[1]}_{n-1,n}(t) \\ &= t\,\overset{\circ}{T}_n\,(t) - \frac{3}{4}\,\overset{\circ}{T}_{n-1}\,(t) \end{aligned}$$

that

$$F^{[1]L}_{n+1,n}(1) = \frac{1}{2^{n-1}} - \frac{3}{4}\frac{1}{2^{n-2}} = -\frac{1}{2^n} < 0$$

and

$$(-1)^{n+1}F^{[1]L}_{n+1,n}(-1) = (-1)^{n+1}\left[\frac{(-1)^{n+1}}{2^{n-1}} - \frac{3}{4}\frac{(-1)^{n-1}}{2^{n-2}}\right] = -\frac{1}{2^n} < 0.$$

This shows the following proposition.

Proposition 3.2. *The generalized averaged Gaussian quadrature formulas* $\widehat{G}^L_{2n+1}$ *relative to the measure* $d\hat{\lambda}^{[1]}_n$ *are not internal. Both their first and last nodes are outside of the interval* $[-1, 1]$.

One can show that $\widehat{G}^L_{2n+1}$ is exact for all polynomials in $\mathbb{P}_{2n+1}$.

$x_i(\widehat{G}^S_{2n+1})$	$x_i(\widehat{G}^S_{2n+1})$
$\mp 1.(+0)$	$\mp 9.749279122(-1)$
$\mp 9.009688679(-1)$	$\mp 7.818314825(-1)$
$\mp 6.234898019(-1)$	$\mp 4.338837391(-1)$
$\mp 2.225209340(-1)$	0

Table 3.1: The nodes $x_i(\widehat{G}^S_{2n+1})$ relative to the measure $d\hat{\lambda}^{[1]}_n$ for $n = 7$.

$x_i(Q^{(n-r)}_{2n-r+1})$	$x_i(Q^{(n-r)}_{2n-r+1})$
$\mp 9.917125037(-1)$	$\mp 8.863626717(-1)$
$\mp 6.311748192(-1)$	$\mp 2.872035373(-1)$
0	

Table 3.2: The nodes $x_i(Q^{(n-r)}_{2n-r+1})$ relative to the measure $d\hat{\lambda}^{[1]}_n$ for $n = 7$ and $r = 6$.

$x_i(Q^{(n-r)}_{2n-r+1})$	$x_i(Q^{(n-r)}_{2n-r+1})$
$\mp 9.972037972(-1)$	$\mp 9.308737486(-1)$
$\mp 8.660254038(-1)$	$\mp 6.801727378(-1)$
$\mp 5.633200581(-1)$	$\mp 2.947551744(-1)$
$\mp 1.490422662(-1)$	

Table 3.3: The nodes $x_i(Q^{(n-r)}_{2n-r+1})$ relative to $d\hat{\lambda}^{[1]}_n$ for $n = 7$ and $r = 1$.

Example 3.1. We illustrate Propositions 3.1 and 3.2 with some computations. All computations reported in this paper have been carried out in MATLAB with quadruple precision arithmetic. We consider quadrature rules relative to the measure $d\hat{\lambda}^{[1]}_n$ for $n = 7$. Table 3.1 is concerned with the

rule $\widehat{G}^S_{2n+1}$ for $n = 7$ and displays the nodes $x_i(\widehat{G}^S_{2n+1})$. The quadrature rule is seen to be internal. One of the nodes is at the origin due to symmetry of the measure. The notation $\mp 1.(+0)$ in the table signifies that the smallest and largest nodes are very close to ∓ 1.

Table 3.2 shows the nodes of the quadrature rule $Q^{(n-r)}_{2n-r+1}$ for $n = 7$ and $r = 6$. This rule is seen to be internal as well. Due to symmetry, one of the nodes is at the origin. The nodes of the rule $Q^{(n-r)}_{2n-r+1}$ for $n = 7$ and $r = 1$ are displayed in Table 3.3, which shows the rule to be internal.

$x_i(\widehat{G}^L_{2n+1})$	$x_i(\widehat{G}^L_{2n+1})$
$\mp 1.062612326(+0)$	$\mp 9.749279122(-1)$
$\mp 9.423918316(-1)$	$\mp 7.818314825(-1)$
$\mp 6.543343398(-1)$	$\mp 4.338837391(-1)$
$\mp 2.336378837(-1)$	0

Table 3.4: The nodes $x_i(\widehat{G}^L_{2n+1})$ relative to $d\hat{\lambda}^{[1]}_n$ for $n = 7$.

n	$x_1(\widehat{G}^L_{2n+1}),\ x_{2n+1}(\widehat{G}^L_{2n+1})$	n	$x_1(\widehat{G}^L_{2n+1}),\ x_{2n+1}(\widehat{G}^L_{2n+1})$
2	$\mp 1.118033989(+0)$	3	$\mp 1.087663874(+0)$
4	$\mp 1.074480571(+0)$	5	$\mp 1.067889603(+0)$
6	$\mp 1.064435166(+0)$	8	$\mp 1.061659220(+0)$
9	$\mp 1.061167193(+0)$	10	$\mp 1.060915990(+0)$
15	$\mp 1.060668259(+0)$	20	$\mp 1.060660425(+0)$
25	$\mp 1.060660180(+0)$	30	$\mp 1.060660172(+0)$

Table 3.5: The nodes $x_1(\widehat{G}^L_{2n+1})$ and $x_{2n+1}(\widehat{G}^L_{2n+1})$ relative to $d\hat{\lambda}^{[1]}_n$ for some $2 \le n \le 30$.

We turn to the quadrature rules $\widehat{G}^L_{2n+1}$. Table 3.4 displays the nodes $x_i(\widehat{G}^L_{2n+1})$ relative to $d\hat{\lambda}^{[1]}_n$ for $n = 7$. This rule is seen not to be internal. Due to symmetry, one of the nodes is at the origin. Table 3.5 shows the smallest and largest nodes of quadrature rules $\widehat{G}^L_{2n+1}$ for several values of n. None of these rules are internal.

Consider the integral of a function f relative to the measure $d\hat{\lambda}^{[1]}_n$ $(n \ge 2)$,

$$I_n[f] = \int_{-1}^{1} f(t)\, d\hat{\lambda}^{[1]}_n(t).$$

n	$e_n^S[f_1]$	$e_n^{(1)}[f_1]$	Error	$I_n[f_1]$
2	1.454(−1)	1.485(−1)	1.478(−1)	1.742...(+0)
3	1.221(−2)	1.221(−2)	1.221(−2)	4.194...(−1)
4	2.232(−4)	2.232(−4)	2.232(−4)	1.058...(−1)
5	2.740(−7)	2.744(−7)	2.740(−7)	2.646...(−2)
6	1.052(−8)	1.052(−8)	1.052(−8)	6.617...(−3)
9	4.699(−16)	4.699(−16)	4.699(−16)	1.033...(−4)
10	6.355(−19)	6.355(−19)	6.355(−19)	2.584...(−5)

Table 3.6: Estimates $e_n^S[f_1]$ and $e_n^{(1)}[f_1]$ of the magnitude of the exact error $|I_n[f_1] - G_n[f_1]|$, denoted by "Error", and the exact values of the integral $I_n[f_1]$ for some n.

We would like to estimate the magnitude of the error of the Gaussian quadrature rule, $|I_n[f] - G_n[f]|$, which we denote by "Error" in Table 3.6, by means of the quantities

$$e_n^S[f] = \left|\widehat{G}_{2n+1}^S[f] - G_n[f]\right| \quad \text{and} \quad e_n^{(1)}[f] = \left|Q_{n+2}^{(1)}[f] - G_n[f]\right|.$$

Table 3.6 shows error estimates for the integrand

$$f_1(t) = e^{3t}\sin(2t)$$

for several values of n. The exact values of the integral $I_n[f_1]$ also are reported. The table shows both the quadrature rules $\widehat{G}_{2n+1}^S[f_1]$ and $Q_{n+2}^{(1)}[f_1]$ to give accurate estimates of the magnitude of the quadrature error obtained with the Gauss formula $G_n[f_1]$. In fact, Table 3.6 indicates that it may be advantageous to use $\widehat{G}_{2n+1}^S[f_1]$ or $Q_{n+2}^{(1)}[f_1]$, instead of $G_n[f_1]$, as approximations of $I[f_1]$ with error estimates $e_n^S[f_1]$ or $e_n^{(1)}[f_1]$, respectively.

3.2. The measure $d\hat{\lambda}_1^{[1]}$

We consider the measure

$$d\hat{\lambda}_1^{[1]}(t) = \left[\overset{\circ}{T}_1(t)\right]^2 (1-t^2)^{-1/2}\,dt = t^2\,(1-t^2)^{-1/2}\,dt, \quad -1 < t < 1,$$

and first regard the polynomials $F^{[1]S}_{m+1,1}$. It follows from Theorem 2.2 and (2.5), (2.6), and (2.7) that for m (even) ≥ 2,

$$\begin{aligned} F^{[1]S}_{m+1,1}(t) &= \hat{\pi}^{[1]}_{m+1,1}(t) - \hat{\beta}^{[1]}_{m+1,1}\,\hat{\pi}^{[1]}_{m-1,1}(t) \\ &= \sum_{i=0}^{m/2}(-1)^i 4^{-i}\frac{m+1-2i}{m+1}\,\overset{\circ}{T}_{m+1-2i}\,(t) \\ &- \frac{1}{4}\left(1+\frac{2}{m+1}\right)\sum_{i=0}^{m/2-1}(-1)^i 4^{-i}\frac{m-1-2i}{m-1}\,\overset{\circ}{T}_{m-1-2i}\,(t). \end{aligned}$$

Since for any integer $n \geq 0$,

$$\sum_{i=0}^{n}(-1)^i = \frac{1+(-1)^n}{2} \quad \text{and} \quad \sum_{i=0}^{n}(-1)^i\, i = (-1)^n\left[\frac{n+1}{2}\right],$$

where $[x]$ denotes the integer part of x, we have for m (even) ≥ 2 that

$$\sum_{i=0}^{m/2}(-1)^i(m+1-2i) = (m+1)\frac{1+(-1)^{m/2}}{2} - 2\cdot(-1)^{m/2}\left[\frac{m+2}{4}\right] = \frac{m+2}{2}.$$

Similarly,

$$\sum_{i=0}^{m/2-1}(-1)^i(m-1-2i) = \frac{m}{2}.$$

Now, we can easily compute

$$F^{[1]S}_{m+1,1}(1) = (-1)^{m+1}F^{[1]S}_{m+1,1}(-1) = -\frac{1}{2^m(m-1)}, \quad m \text{ (even)} \geq 2,$$

which shows that the generalized averaged formula $\widehat{G}^S_{2m+1}$ is not internal for m (even) ≥ 2.

We turn to the situation when m is odd. It follows from Theorem 2.2 and (2.5), (2.6), and (2.7) that for m (odd) ≥ 3, we have

$$\begin{aligned} F^{[1]S}_{m+1,1}(t) &= \hat{\pi}^{[1]}_{m+1,1}(t) - \hat{\beta}^{[1]}_{m+1,1}\,\hat{\pi}^{[1]}_{m-1,1}(t) \\ &= (-1)^{\frac{m+1}{2}}4^{-\frac{m+1}{2}} + \sum_{i=0}^{(m+1)/2-1}(-1)^i 4^{-i}\,\overset{\circ}{T}_{m+1-2i}\,(t) \\ &- \frac{1}{4}\left(1-\frac{2}{m+2}\right)\left[(-1)^{\frac{m-1}{2}}4^{-\frac{m-1}{2}} + \sum_{i=0}^{(m-1)/2-1}(-1)^i 4^{-i}\,\overset{\circ}{T}_{m-1-2i}\,(t)\right]. \end{aligned}$$

Now, we can easily compute

$$F^{[1]S}_{m+1,1}(1) = (-1)^{m+1} F^{[1]S}_{m+1,1}(-1) = \frac{1}{2^m(m+2)}, \quad m \text{ (odd)} \geq 3,$$

which shows that the generalized averaged formula $\widehat{G}^S_{2m+1}$ is internal for m (odd) ≥ 3.

Since $\hat{\pi}^{[1]}_{0,1}(t) = 1$, we have for $m = 1$ that

$$F^{[1]S}_{2,1}(t) = \overset{\circ}{T}_2\,(t) - \frac{1}{3}.$$

Therefore,

$$F^{[1]S}_{2,1}(\mp 1) = \frac{1}{6} \geq 0.$$

We turn to the polynomials $F^{[1]L}_{m+1,1}$. For m (even) ≥ 2, we have

$$F^{[1]L}_{m+1,1}(t) = \hat{\pi}^{[1]}_{m+1,1}(t) - \hat{\beta}^{[1]}_{m,1}\,\hat{\pi}^{[1]}_{m-1,1}(t), \tag{3.5}$$

where $\hat{\beta}^{[1]}_{m,1} = \frac{1}{4}(1 - 2/(m+1))$. We conclude in a similar manner as above that

$$F^{[1]L}_{m+1,1}(1) = (-1)^{m+1} F^{[1]L}_{m+1,1}(-1) = \frac{1}{2^m(m+1)}, \quad m \text{ (even)} \geq 2,$$

which shows that the generalized averaged formula $\widehat{G}^L_{2m+1}$ is internal for m (even) ≥ 2.

For m (odd) ≥ 3, eq. (3.5) holds with $\hat{\beta}^{[1]}_{m,1} = \frac{1}{4}(1 + 2/m)$. We conclude similarly as above that

$$F^{[1]L}_{m+1,1}(1) = (-1)^{m+1} F^{[1]L}_{m+1,1}(-1) = -\frac{1}{2^m \cdot m}, \quad m \text{ (odd)} \geq 3,$$

which immediately shows that the generalized averaged formula $\widehat{G}^L_{2m+1}$ is not internal for m (odd) ≥ 3.

For $m = 1$, we have

$$F^{[1]L}_{2,1}(t) = \overset{\circ}{T}_2\,(t) - 1$$

and, therefore,

$$F^{[1]L}_{2,1}(\mp 1) = -\frac{1}{2} < 0.$$

We have shown the following result.

Theorem 3.3. *Let the measure be $d\hat{\lambda}_1^{[1]}$. The generalized averaged Gaussian quadrature formulas $\widehat{G}_{2m+1}^S$ are not internal for $m \geq 2$ even. The rules $\widehat{G}_{2m+1}^S$ are internal for $m \geq 1$ odd. Further, the quadrature formulas $\widehat{G}_{2m+1}^L$ are not internal for $m \geq 1$ odd, and the rules $\widehat{G}_{2m+1}^L$ are internal for $m \geq 2$ even.*

Corollary 3.4. *The truncated variants $Q_{2m-r+1}^{(m-r)}$ $(m \geq 2)$, $r = 1, 2, \ldots, m-1$, of the generalized averaged Gaussian quadrature formula $\widehat{G}_{2m+1}^S$ relative to the measure $d\hat{\lambda}_1^{[1]}$ are internal for m odd.*

Proof. See [5, Remark 4.1] for a proof. □

Theorem 3.5. *The truncated variants $Q_{m+2}^{(1)}$ $(m \geq 2)$ of the generalized averaged Gaussian quadrature formulas $\widehat{G}_{2n+1}^S$ relative to the measure $d\hat{\lambda}_1^{[1]}$ are internal for m even.*

Proof. Since the nodes of $Q_{m+2}^{(1)}$ are the zeros of

$$t_{m+2,1}^{[1]}(t) = t\,\hat{\pi}_{m+1,1}^{[1]}(t) - \hat{\beta}_{m+1,1}^{[1]}\,\hat{\pi}_{m,1}^{[1]}(t) = \hat{\pi}_{m+2,1}^{[1]}(t),$$

and the zeros of the orthogonal polynomial $\hat{\pi}_{m+2,1}^{[1]}(t)$ belong to the interval $[-1, 1]$, the desired result follows. We note that the theorem also can be shown in a different manner: in view of that $\hat{\alpha}_{m-1,1}^{[1]} = \hat{\alpha}_{m+1,1}^{[1]}$ $(= 0)$, the proof follows from [3, Theorem 3.1]. □

For the measure of this subsection, the rules $\widehat{G}_{2n+1}^S$ and $\widehat{G}_{2n+1}^L$ are exact for all polynomials in $\mathbb{P}_{2n+2}$ and $\mathbb{P}_{2n+1}$, respectively.

Example 3.2. This example illustrates the theory and shows an application of quadrature rules discussed in this subsection. Table 3.7 displays the largest and smallest nodes of some quadrature rules determined by the measure $d\hat{\lambda}_1^{[1]}$. The table illustrates the importance of the choice of quadrature rule when it is desirable that the rule be internal.

Consider the integral of a function f relative to the measure $d\hat{\lambda}_1^{[1]}$,

$$I[f] = \int_{-1}^{1} f(t)\, d\hat{\lambda}_1^{[1]}(t).$$

We estimate the magnitude of the error of the Gaussian quadrature rule, $|I[f] - G_n[f]|$, which we denote by "Error" in Table 3.8, by means of the

n	$x_1(\widehat{G}^S_{2n+1}),\ x_{2n+1}(\widehat{G}^S_{2n+1})$	$x_1(Q^{(1)}_{n+2}),\ x_{n+2}(Q^{(1)}_{n+2})$	$x_1(\widehat{G}^L_{2n+1}),\ x_{2n+1}(\widehat{G}^L_{2n+1})$
2	$\mp 1.118033989(+0)$	$\mp 9.510565163(-1)$	$\mp 9.574271078(-1)$
3	$\mp 9.772159309(-1)$	$\mp 9.639132662(-1)$	$\mp 1.047309158(+0)$
4	$\mp 1.032440198(+0)$	$\mp 9.749279128(-1)$	$\mp 9.850158587(-1)$
5	$\mp 9.897885688(-1)$	$\mp 9.801641622(-1)$	$\mp 1.020436445(+0)$
10	$\mp 1.006087743(+0)$	$\mp 9.927088741(-1)$	$\mp 9.969338516(-1)$
11	$\mp 9.974342623(+0)$	$\mp 9.936473470(-1)$	$\mp 1.005024780(+0)$

Table 3.7: First and last nodes for quadrature rules $\widehat{G}^S_{2n+1}$, $Q^{(1)}_{n+2}$, and $\widehat{G}^L_{2n+1}$ relative to $d\hat{\lambda}^{[1]}_1$ for some n.

n	$e^S_n[f_2]$	$e^{(1)}_n[f_2]$	$e^L_n[f_2]$	Error
2	n.a.	$1.155(-5)$	$1.116(-5)$	$1.159(-5)$
3	$5.552(-7)$	$5.514(-7)$	n.a.	$5.639(-7)$
4	n.a.	$4.451(-8)$	$4.031(-8)$	$4.755(-8)$
5	$1.181(-8)$	$1.126(-8)$	n.a.	$1.247(-8)$
10	n.a.	$7.175(-11)$	$7.956(-11)$	$1.022(-10)$
11	$4.965(-11)$	$3.845(-11)$	n.a.	$5.693(-11)$

Table 3.8: Estimates $e^S_n[f_2]$, $e^{(1)}_n[f_2]$, and $e^L_n[f_2]$ of the magnitude of the error, and the magnitude of the exact error $|I[f_2] - G_n[f_2]|$, denoted by "Error", for the values of n in Table 3.7.

quantities

$$\begin{aligned} e^S_n[f] &= \left|\widehat{G}^S_{2n+1}[f] - G_n[f]\right|, \\ e^{(1)}_n[f] &= \left|Q^{(1)}_{n+2}[f] - G_n[f]\right|, \\ e^L_n[f] &= \left|\widehat{G}^L_{2n+1}[f] - G_n[f]\right|. \end{aligned}$$

Table 3.8 shows error estimates for the integrand

$$f_2(t) = 999.1^{\log(1+\varepsilon+t)}, \quad \varepsilon = 10^{-6},$$

for quadrature rules from Table 3.7. The function $f_2(t)$ is not defined for $t \leq -1-\varepsilon$. Therefore, quadrature rules with a node smaller than or equal to $-1-\varepsilon$ cannot be applied. We denote these cases by "n.a." in Table 3.8.

The exact values of the integral $I[f_2]$ is 5.103...(+0). The error estimate $e_n^{(1)}[f_2]$ is applicable for all values n and can be seen to give quite accurate estimates of the magnitude of the exact error. This suggests that it may be beneficial to use $Q_{n+2}^{(1)}[f_2]$ as an approximation of $I[f_2]$ with error estimate $e_n^{(1)}[f_2]$.

3.3. The measures $d\hat{\lambda}_n^{[2]}$

This subsection discusses quadrature rules relative to the measures

$$d\hat{\lambda}_n^{[2]}(t) = \left[\overset{\circ}{U}_n\,(t)\right]^2 \sqrt{1-t^2}\,dt, \quad -1 < t < 1, \quad n \geq 1.$$

We first consider the generalized averaged Gaussian quadrature rules $\widehat{G}_{2n+1}^S$ and truncated variants $Q_{2n-r+1}^{(n-r)}$ $(n \geq 2)$, $r = 1, 2, \ldots, n-1$. It follows from Theorem 2.4 that (2.10) hold. Moreover, we obtain from (2.12) that $\hat{\beta}_{n,n}^{[2]} = 1/4$ and $\hat{\beta}_{n+1,n}^{[2]} = 1/8$. Finally, (2.11) yields that for $n \geq 2$,

$$\begin{aligned} \hat{\pi}_{n+1,n}^{[2]}(t) &= t\hat{\pi}_{n,n}^{[2]}(t) - \hat{\beta}_{n,n}^{[2]}\hat{\pi}_{n-1,n}^{[2]}(t) \\ &= t\,\overset{\circ}{T}_n\,(t) - \frac{1}{4}\,\overset{\circ}{T}_{n-1}\,(t) = \overset{\circ}{T}_{n+1}\,(t). \end{aligned}$$

Hence, the polynomial F_{n+1} given by (1.4) can be written as

$$\begin{aligned} F_{n+1,n}^{[2]S}(t) &= \hat{\pi}_{n+1,n}^{[2]}(t) - \hat{\beta}_{n+1,n}^{[2]}\hat{\pi}_{n-1,n}^{[2]}(t) \\ &= t\,\overset{\circ}{T}_n\,(t) - \frac{3}{8}\,\overset{\circ}{T}_{n-1}\,(t). \end{aligned}$$

In the special case $n = 1$, we have

$$\hat{\pi}_{2,1}^{[2]}(t) = t\hat{\pi}_{1,1}^{[2]}(t) - \hat{\beta}_{1,1}^{[2]}\hat{\pi}_{0,1}^{[2]}(t) = t\,\overset{\circ}{T}_1\,(t) - \frac{1}{2}$$

and, therefore,

$$F_{2,1}^{[2]S}(t) = t^2 - \frac{5}{8},$$

which gives

$$F_{2,1}^{[2]S}(\mp 1) = \frac{3}{8} \geq 0.$$

For $n \geq 2$, we obtain

$$F_{n+1,n}^{[2]S}(1) = \frac{1}{2^{n-1}} - \frac{3}{8}\frac{1}{2^{n-2}} = \frac{1}{2^{n+1}} \geq 0$$

and

$$(-1)^{n+1} F^{[2]S}_{n+1,n}(-1) = (-1)^{n+1} \left[\frac{(-1)^{n+1}}{2^{n-1}} - \frac{3}{8} \frac{(-1)^{n-1}}{2^{n-2}} \right] = \frac{1}{2^{n+1}} \geq 0.$$

We have shown the following result.

Proposition 3.6. *The generalized averaged Gaussian quadrature formulas $\widehat{G}^S_{2n+1}$ relative to the measure $d\hat{\lambda}^{[2]}_n(t)$ and truncated variants $Q^{(n-r)}_{2n-r+1}$ ($n \geq 2$), $r = 1, 2, \ldots, n-1$, are internal. Further, the rule $\widehat{G}^S_3$ is internal.*

We turn to the generalized averaged Gaussian rules $\widehat{G}^L_{2n+1}$. For $n \geq 2$, we have

$$\begin{aligned} F^{[2]L}_{n+1,n}(t) &= \hat{\pi}^{[2]}_{n+1,n}(t) - \hat{\beta}^{[2]}_{n,n}\, \hat{\pi}^{[2]}_{n-1,n}(t) \\ &= t\, \overset{\circ}{T}_n\,(t) - \frac{1}{2}\, \overset{\circ}{T}_{n-1}\,(t) = F^{[1]S}_{n+1,n}(t). \end{aligned}$$

This equation together with Proposition 3.1 and

$$F^{[2]L}_{2,1}(t) = t^2 - 1, \quad F^{[2]L}_{2,1}(\mp 1) \geq 0$$

give the following property.

Proposition 3.7. *The generalized averaged Gaussian quadrature formulas $\widehat{G}^L_{2n+1}$ relative to the measure $d\hat{\lambda}^{[2]}_n(t)$ are internal for all $n \geq 1$.*

It can be shown that for the measures of this subsection, the rules $\widehat{G}^S_{2n+1}$ and $\widehat{G}^L_{2n+1}$ are exact for all polynomials in $\mathbb{P}_{2n+3}$ and $\mathbb{P}_{2n+1}$, respectively. The rule $\widehat{G}^S_{2n+1}$ is exact for all polynomials in $\mathbb{P}_{2n+3}$, since $d\hat{\lambda}^{[2]}_n(-t) = d\hat{\lambda}^{[2]}_n(t)$ on $[-1, 1]$ (see, e. g., [22, Remark 2.1 on p. 1486]).

3.4. The measures $d\hat{\lambda}^{[3]}_n$

We consider the measures

$$d\hat{\lambda}^{[3]}_n(t) = \left[\overset{\circ}{V}_n\,(t) \right]^2 \sqrt{\frac{1+t}{1-t}}\, dt, \quad -1 < t < 1, \quad n \geq 1,$$

and first discuss the internality of the generalized averaged Gaussian rules $\widehat{G}^S_{2n+1}$. The relation (2.13) holds due to Theorem 2.6, and it follows from

(2.15) and (2.16) that $\hat{\alpha}_{n,n}^{[3]} = 1/4$, $\hat{\beta}_{n,n}^{[3]} = 1/4$, and $\hat{\beta}_{n+1,n}^{[3]} = 3/16$. We obtain from (2.14) that, for $n \geq 2$,

$$\begin{aligned} \hat{\pi}_{n+1,n}^{[3]}(t) &= \left(t - \hat{\alpha}_{n,n}^{[3]}\right) \hat{\pi}_{n,n}^{[3]}(t) - \hat{\beta}_{n,n}^{[3]} \hat{\pi}_{n-1,n}^{[2]}(t) \\ &= \left(t - \frac{1}{4}\right) \overset{\circ}{T}_n\,(t) - \frac{1}{4}\, \overset{\circ}{T}_{n-1}\,(t). \end{aligned}$$

The polynomial F_{n+1} in (1.4) is of the form

$$\begin{aligned} F_{n+1,n}^{[3]S}(t) &= \hat{\pi}_{n+1,n}^{[3]}(t) - \hat{\beta}_{n+1,n}^{[3]} \hat{\pi}_{n-1,n}^{[3]}(t) \\ &= \left(t - \frac{1}{4}\right) \overset{\circ}{T}_n\,(t) - \frac{7}{16}\, \overset{\circ}{T}_{n-1}\,(t). \end{aligned}$$

We have for $n = 1$,

$$\hat{\pi}_{2,1}^{[3]}(t) = \left(t - \hat{\alpha}_{1,1}^{[3]}\right) \hat{\pi}_{1,1}^{[3]}(t) - \hat{\beta}_{1,1}^{[3]} \hat{\pi}_{0,1}^{[3]}(t) = \left(t - \frac{1}{4}\right) \overset{\circ}{T}_1\,(t) - \frac{1}{2} = t^2 - \frac{1}{4}t - \frac{1}{2},$$

and it follows that

$$F_{2,1}^{[3]S}(t) = t^2 - \frac{1}{4}t - \frac{11}{16},$$

which yields

$$F_{2,1}^{[3]S}(1) = \frac{1}{16} \geq 0, \quad F_{2,1}^{[3]S}(-1) = \frac{9}{16} \geq 0.$$

When $n \geq 2$, we have

$$F_{n+1,n}^{[3]S}(1) = \frac{3}{4}\,\frac{1}{2^{n-1}} - \frac{7}{16}\frac{1}{2^{n-2}} = -\frac{1}{2^{n+2}} < 0$$

and

$$(-1)^{n+1} F_{n+1,n}^{[3]S}(-1) = (-1)^{n+1}\left[-\frac{5}{4}\,\frac{(-1)^n}{2^{n-1}} - \frac{7}{16}\,\frac{(-1)^{n-1}}{2^{n-2}}\right] = \frac{3}{2^{n+2}} \geq 0.$$

We have shown the following results.

Proposition 3.8. *The generalized averaged Gaussian quadrature formulas $\widehat{G}_{2n+1}^S$ relative to the measure $d\hat{\lambda}_n^{[3]}$ and truncated variants $Q_{2n-r+1}^{(n-r)}$ ($n \geq 2$), $r = 1, 2, \ldots, n-1$, have their first node in the interval $[-1, 1]$. The rules $\widehat{G}_{2n+1}^S$ ($n \geq 2$) have their last node outside the interval $[-1, 1]$. The quadrature rule $\widehat{G}_3^S$ is internal.*

The truncated quadrature rules $Q_{2n-r+1}^{(n-r)}$ $(n \geq 2)$, $r = 1, 2, \ldots, n-1$, relative to the measure $d\hat{\lambda}_n^{[3]}$ might not have their last node inside the interval $[-1, 1]$. The simplest of these truncated rules is $Q_{n+2}^{(1)}$. Its zeros are those of the polynomial t_{n+2}, see (1.5), which in our situation is given by

$$\begin{aligned} t_{n+2}^{[3]S}(t) &= \left(t - \hat{\alpha}_{n-1,n}^{[3]}\right) \hat{\pi}_{n+1,n}^{[3]}(t) - \hat{\beta}_{n+1,n}^{[3]} \hat{\pi}_{n,n}^{[3]}(t) \\ &= \left(t^2 - \frac{1}{4}t - \frac{3}{16}\right) \overset{\circ}{T}_n(t) - \frac{1}{4} \overset{\circ}{T}_{n-1}(t). \end{aligned}$$

Hence,

$$t_{n+2}^{[3]S}(1) = \frac{9}{16} \frac{1}{2^{n-1}} - \frac{1}{4} \frac{1}{2^{n-2}} = \frac{1}{2^{n+3}} \geq 0.$$

This yields the following statement.

Proposition 3.9. *The last node of the truncated quadrature formula $Q_{n+2}^{(1)}$ $(n \geq 2)$ relative to the measure $d\hat{\lambda}_n^{[3]}$ is inside the interval $[-1, 1]$.*

Proposition 3.9 illustrates that a truncated rule may be internal when the associated generalized averaged Gaussian rule $\widehat{G}_{2n+1}^S$ is not. We remark that the proposition also can be shown by using [3, Theorem 3.1] and the fact that $\hat{\alpha}_{n-1,n}^{[3]} = 0 > -1/4 = \hat{\alpha}_{n+1,n}^{[3]}$.

We turn to the rules $\widehat{G}_{2n+1}^L$. For $n \geq 2$, we obtain

$$\begin{aligned} F_{n+1,n}^{[3]L}(t) &= \hat{\pi}_{n+1,n}^{[3]}(t) - \hat{\beta}_{n,n}^{[3]} \hat{\pi}_{n-1,n}^{[3]}(t) \\ &= \left(t - \frac{1}{4}\right) \overset{\circ}{T}_n(t) - \frac{1}{2} \overset{\circ}{T}_{n-1}(t). \end{aligned}$$

Consequently,

$$F_{n+1,n}^{[3]L}(1) = \frac{3}{4} \frac{1}{2^{n-1}} - \frac{1}{2} \frac{1}{2^{n-2}} = -\frac{1}{2^{n+1}} < 0$$

and

$$(-1)^{n+1} F_{n+1,n}^{[3]L}(-1) = (-1)^{n+1} \left[-\frac{5}{4} \frac{(-1)^n}{2^{n-1}} - \frac{1}{2} \frac{(-1)^{n-1}}{2^{n-2}} \right] = \frac{1}{2^{n+1}} \geq 0.$$

When $n = 1$, we have $F_{2,1}^{[3]L}(t) = t^2 - t/4 - 1$. Therefore, $F_{2,1}^{[3]L}(1) < 0$ and $F_{2,1}^{[3]L}(-1) \geq 0$. The above inequalities yield the following result.

Proposition 3.10. *The generalized averaged Gaussian quadrature formulas* $\widehat{G}^L_{2n+1}$ *relative to the measure* $d\hat{\lambda}^{[3]}_n(t)$ *have their first node in the interval* $[-1,1]$ *and their last node outside the interval* $[-1,1]$ *for all* $n \geq 1$.

The rules $\widehat{G}^S_{2n+1}$ and $\widehat{G}^L_{2n+1}$ are exact for polynomials in $\mathbb{P}_{2n+2}$ and $\mathbb{P}_{2n+1}$, respectively.

n	$x_{2n+1}(\widehat{G}^S_{2n+1})$	$x_{2n+1}(\widehat{G}^L_{2n+1})$	$x_{2n+1}(H_{2n+1})$
2	1.037492289(+0)	1.073237845(+0)	1.009829348(+0)
4	1.021171800(+0)	1.044396345(+0)	1.001340181(+0)
6	1.015804941(+0)	1.036059813(+0)	1.000230559(+0)
8	1.013274576(+0)	1.032871125(+0)	1.000043900(+0)
10	1.011899090(+0)	1.031592484(+0)	1.000008857(+0)
12	1.011103092(+0)	1.031087998(+0)	1.000001856(+0)

Table 3.9: The nodes $x_{2n+1}(\widehat{G}^S_{2n+1})$, $x_{2n+1}(\widehat{G}^L_{2n+1})$, and $x_{2n+1}(H_{2n+1})$ relative to $d\hat{\lambda}^{[3]}_n$ for some even values of n.

Example 3.3. Table 3.9 displays the nodes $x_{2n+1}(\widehat{G}^S_{2n+1})$, $x_{2n+1}(\widehat{G}^L_{2n+1})$, and $x_{2n+1}(H_{2n+1})$ relative to the measure $d\hat{\lambda}^{[3]}_n$ for some even values of n.

n	$e^{(1)}_n[f_3]$	Error	$I_n[f_3]$
2	2.907(−6)	2.904(−6)	4.907...(−1)
4	6.222(−10)	6.245(−10)	3.067...(−2)
6	2.065(−12)	2.123(−12)	1.917...(−3)
8	1.648(−14)	1.753(−14)	1.198...(−4)
10	2.081(−16)	2.301(−16)	7.488...(−6)
12	3.500(−18)	4.032(−18)	4.680...(−7)

Table 3.10: Estimate $e^{(1)}_n[f_3]$ of the magnitude of the error, the magnitude of the exact error $|I_n[f_3] - G_n[f_3]|$, which is denoted by "Error", and the exact value of the integral $I_n[f_3]$ for the values of n in Table 3.9.

Consider the integral of a function f relative to the measure $d\hat{\lambda}^{[3]}_n$,

$$I_n[f] = \int_{-1}^{1} f(t)\, d\hat{\lambda}^{[3]}_n(t).$$

We would like to estimate the magnitude of the error in the Gaussian quadrature rule, $|I_n[f] - G_n[f]|$, which we denote by "Error" in Table 3.10, for the integrand

$$f_3(t) = 999.1^{\log(1+\varepsilon-t)}, \quad \varepsilon = 10^{-6},$$

for the values of n in Table 3.9. Since the function $f_3(t)$ is not defined for $t \geq 1+\varepsilon$ and the quadrature rules $\widehat{G}^S_{2n+1}$, $\widehat{G}^L_{2n+1}$, and the Gauss-Kronrod rule H_{2n+1} have their last node larger than $1+\varepsilon$ when n is even, these rules cannot be applied for the values of n in Table 3.9. Proposition 3.9 shows that we can use the quadrature formula $Q^{(1)}_{n+2}$. Table 3.10 displays the estimates of the magnitude of the errors,

$$e^{(1)}_n[f_3] = \left| Q^{(1)}_{n+2}[f_3] - G_n[f_3] \right|,$$

the magnitude of the exact errors $|I_n[f_3] - G_n[f_3]|$, and the exact values of the integral $I_n[f_3]$ for the values of n in Table 3.9. The error estimates are seen to be of the correct order of magnitude. Similarly as in the previous examples, the accuracy of the estimates $e^{(1)}_n[f_3]$ suggests that it may be beneficial to use $Q^{(1)}_{n+2}[f_3]$ as an approximation of $I[f_3]$ instead of $G_n[f_3]$.

4. Conclusion

Generalized averaged Gaussian quadrature rules associated with some measure (and truncated variants of these rules) may be attractive alternatives to Gauss-Kronrod quadrature formulas to estimate the error of a Gaussian rule. The generalized averaged Gaussian rules and truncated variants exist for all values of n, while positive Gauss-Kronrod rules might not. Moreover, generalized averaged Gaussian rules and truncated variants are easy to compute. The main disadvantage of generalized averaged Gaussian rules is that they may have nodes outside the interval of integration, which can make them impossible to apply for certain integrands. We investigated in this work whether generalized Gaussian quadratures and their truncated variants for measures induced by Chebyshev polynomials are internal, i.e., whether all quadrature nodes are in the interval of integration, and found that in many situations some of these rules have this property.

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