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INTERNALITY OF AVERAGED GAUSS QUADRATURE RULES FOR CERTAIN MODIFICATION OF JACOBI MEASURES

D.LJ. DJUKIĆ[1], R.M. MUTAVDŽIĆ DJUKIĆ[1], L. REICHEL[2], M.M. SPALEVIĆ[1]

Abstract. The internality of quadrature rules, i.e., the property that all nodes lie in the interior of the convex hull of the support of the measure, is important in applications, because this allows the application of these quadrature rules to the approximation of integrals with integrands that are defined in the convex hull of the support of the measure only. It is known (see [1,6]) that the averaged Gauss and optimal averaged Gauss quadrature rules with respect to the four Chebyshev measures modified by a linear divisor are internal. This paper investigates the internality of similarly modified Jacobi measures, namely measures defined by weight functions of the forms

$$w(x) = \frac{1}{z-x}(1-x)^a(1+x)^b \quad \text{or} \quad w(x) = (z-x)(1-x)^a(1+x)^b$$

with $a, b > -1$ and $z \in \mathbb{R}$, $|z| > 1$. We will show that in some cases, depending on the exponents a and b, the averaged and optimal averaged Gauss rules for these measures are internal if the number of nodes is large enough.

Keywords: Gauss quadrature, generalized averaged Gauss quadrature, truncated generalized averaged Gauss quadrature, internality of quadrature rule, modified Jacobi measure.

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1. Introduction

Let $w(x)$ be a nonnegative weight function on the real interval $[-1, 1]$ with infinitely many points of support and with all of its moments $\mu_k = \int_{-1}^{1} x^k w(x)dx$ for $k = 0, 1, \ldots$ well defined. We are concerned with the approximation of integrals of the form

$$\mathcal{I}(f) = \int_a^b f(x)w(x)dx \tag{1}$$

by a quadrature rule. Gauss quadrature rules are well suited for this purpose. These rules are closely related to the sequence of monic orthogonal polynomials $P_0, P_1, \ldots$ associated with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)w(x)dx,$$

where $\deg P_k = k$. Thus, the polynomials P_j have leading coefficient one and satisfy $\langle P_j, P_k\rangle = 0$ whenever $j \neq k$ and $\langle P_k, P_k\rangle > 0$.

Clearly $P_0(x) \equiv 1$, and we define $P_{-1}(x) \equiv 0$. It is well known that the polynomials P_k satisfy a three-term recurrence relation of the form

$$P_{k+1}(x) = (x - \alpha_k)P_k(x) - \beta_k P_{k-1}(x), \quad k \geqslant 0, \tag{2}$$

where the coefficients $(\alpha_k)_{k\geqslant 0}$ and $(\beta_k)_{k\geqslant 1}$ are given by $\alpha_k = \frac{\langle xP_k, P_k\rangle}{\langle P_k, P_k\rangle}$ and $\beta_k = \frac{\langle P_k, P_k\rangle}{\langle P_{k-1}, P_{k-1}\rangle} > 0$. It is convenient to define $\beta_0 = \|P_0\|^2 = \int_a^b w(x)dx$. The orthogonal polynomial P_n has n distinct real zeros, all of which are in the open interval $(-1, 1)$. For proofs of these propeties as well as for other properties of orthogonal polynomials see, e.g., [8, 15].

[1]Department of Mathematics, University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade 35, Serbia, [2]Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA
e-mail: ddjukic@mas.bg.ac.rs

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Among all interpolatory quadrature rules with n nodes for approximating the integral (1), the rule with maximal algebraic degree of precision is the n-node Gauss quadrature rule with respect to the weight function w,

$$\mathcal{G}_n(f) = \sum_{i=1}^{n} w_i f(\xi_i). \tag{3}$$

The nodes $\xi_1, \ldots, \xi_n$ are the zeros of the orthogonal polynomial P_n and the weights $w_1, \ldots, w_n$ are positive; see, e.g., [8,15] for proofs. The degree of precision of the Gauss rule $\mathcal{G}_n$ is $2n-1$, i.e., $\mathcal{G}_n(p) = \mathcal{I}(p)$ whenever p is a polynomial of degree not exceeding $2n-1$. The nodes and weights of the Gauss rule (3) can be computed efficiently (in $\mathcal{O}(n^2)$ arithmetic floating point operations) by applying the Golub-Welsch algorithm [9] to the symmetric tridiagonal *Jacobi matrix*:

$$J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{\beta_{n-2}} & \alpha_{n-2} & \sqrt{\beta_{n-1}} \\ 0 & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}$$

The nodes $\xi_1, \ldots, \xi_n$ are the eigenvalues of the Jacobi matrix, whereas the weights $w_1, \ldots, w_n$ are the squares of the first components of suitably normalized eigenvectors.

The Gauss rule (3) is said to be *internal*, because all its nodes are in the in the support of the measure $[-1, 1]$. This property is important because it allows the application of the Gauss rule to the approximation of the integral (1) for any integrand that is defined in the interval $(-1, 1)$.

It is important to be able to estimate the magnitude of the quadrature error,

$$\varepsilon_n(f) = |(\mathcal{I} - \mathcal{G}_n)(f)|, \tag{4}$$

because this makes it possible to determine a suitable value of n when applying the Gauss rule (3) to approximate the integral (1) to a prescribed accuracy. An unnecessarily large value of n requires the computation of needlessly many nodes and weights, as well as the evaluation of the integrand f at excessively many nodes, while a too small value of n does not yield an approximation of desired accuracy. The development of methods for estimating the error (4) therefore has received considerable attention over many years.

A popular approach to estimate the error (4) is to use another quadrature rule $\mathcal{Q}$ with $\ell > n$ nodes and with degree of precision higher than $2n - 1$. One then can use the difference

$$|(\mathcal{Q} - \mathcal{G}_n)(f)|$$

as an estimate of (4). Ideally, the nodes of the rule $\mathcal{Q}$ should include the nodes of $\mathcal{G}_n$, because then one can use the values of f needed to evaluate the Gauss rule (3) also when evaluating $\mathcal{Q}(f)$.

A classical choice for the rule $\mathcal{Q}$ is the Gauss-Kronrod rule associated with $\mathcal{G}_n$. The Gauss-Kronrod rule uses the n nodes of $\mathcal{G}_n$ and $n + 1$ additional nodes chosen to achieve the maximal algebraic degree of precision, which is at least $3n + 1$ in this case. However, the $n + 1$ extra nodes are not guaranteed to be in the the interval $(-1, 1)$ and, in fact, they might not be real; see [12] for a recent discussion on Gauss-Kronrod rules.

An alternative approach is to choose a quadrature rule $\mathcal{U}_{n+1}(f)$ with $n + 1$ new nodes and suitable weights such that the stratified rule

$$\mathcal{Q}_{2n+1}(f) = \theta \cdot \mathcal{G}_n(f) + (1 - \theta) \cdot \mathcal{U}_{n+1}(f) \tag{5}$$

has degree of precision at least $2n+1$. For any fixed $\theta > 0$, this quadrature rule is unique. Then the $(2n + 1)$-node rule $\mathcal{Q}_{2n+1}$ can be used as the rule $\mathcal{Q}$ to estimate the quadrature error (4); see [10, 13] for discussions of this approach. Thus, the rule $\mathcal{Q}_{2n+1}$ uses the n Gauss nodes ξ_i of

(3) and the $n+1$ nodes of $\mathcal{U}_{n+1}$. The nodes of $\mathcal{U}_{n+1}$ are the zeros of a polynomial P^* that is orthogonal to $P_0, P_1, \ldots, P_{n-2}$ and P_n and has the form

$$P^* = P_{n+1} - \beta^*_{n+1} P_{n-1} \tag{6}$$

for some constant β^*_{n+1} depending on θ in (5).

Two choices of θ in (5) are common:

- For $\theta = \frac{1}{2}$, the resulting rule $\mathcal{Q}^L_{2n+1} = \mathcal{Q}_{2n+1}$ is the *averaged Gauss rule* introduced by Laurie [11]. Its degree of precision is at least $2n+1$.
- For $\theta = \frac{\beta_n}{\beta_n + \beta_{n+1}}$, the resulting rule $\mathcal{Q}^S_{2n+1} = \mathcal{Q}_{2n+1}$ is the *optimal averaged Gauss rule* described by Spalević [14]. Its degree of precision is at least $2n+2$. The choice of θ leading to this property is unique. Special cases of optimal averaged rules had previosuly been described by Ehrich [7].

Spalević [14] showed that the averaged and optimal averaged rules $\mathcal{Q}^L_{2n+1}$ and $\mathcal{Q}^S_{2n+1}$ can be represented by a symmetric tridiagonal Jacobi matrix of the form

$$\widehat{J}_{2n+1} = \left[\begin{array}{cccccc|cccc}
\alpha_0 & \sqrt{\beta_1} & & & & & & & & \\
\sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & & & & & \\
 & \ddots & \ddots & \ddots & & & & & & \\
 & & \sqrt{\beta_{n-2}} & \alpha_{n-2} & \sqrt{\beta_{n-1}} & & & & & \\
 & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} & \sqrt{\beta_n} & & & & \\
 & & & & \sqrt{\beta_n} & \alpha_n & \sqrt{\beta^*_{n+1}} & & & \\
\hline
 & & & & & \sqrt{\beta^*_{n+1}} & \alpha_{n-1} & \sqrt{\beta_{n-1}} & & \\
 & & & & & & \ddots & \ddots & \ddots & \\
 & & & & & & & \sqrt{\beta_2} & \alpha_1 & \sqrt{\beta_1} \\
 & & & & & & & & \sqrt{\beta_1} & \alpha_0
\end{array}\right]$$

where $\beta^*_{n+1} = \beta_n$ in the averaged rule $\mathcal{Q}^L_{2n+1}$ and $\beta^*_{n+1} = \beta_{n+1}$ in the the optimal averaged rule $\mathcal{Q}^S_{2n+1}$. Their nodes and weights can be determined by applying the Golub-Welsch algorithm to the matrix $\widehat{J}_{2n+1}$.

Differently from Gauss-Kronrod rules, these averaged and optimal averaged Gauss rules are guaranteed to have real nodes that interlace the Gauss nodes, and to have positive weights. However, their outermost nodes might be outside of the interval $[-1, 1]$. As can be seen from (6), the smallest and largest nodes are internal (i.e., lie in the interval $[-1, 1]$) if and only if

$$\frac{P_{n+1}(-1)}{P_{n-1}(-1)} \geqslant \beta^*_{n+1} \quad \text{and} \quad \frac{P_{n+1}(1)}{P_{n-1}(1)} \geqslant \beta^*_{n+1}, \tag{7}$$

respectively.

A possible way to avoid external nodes is to reduce the Jacobi matrix associated with the optimal averaged rule by deleting a number of its last rows and columns; see [4]. By the interlacing property, the two outermost nodes move towards the middle, while the degree of precision does not fall below $2n+2$. Thus, the highest chance to obtain an internal quadrature rule in this manner is to delete the $n-1$ last rows and columns. The order of the resulting Jacobi matrix is $n+2$. We denote the quadrature rule so obtained by $\mathcal{Q}^t_{n+2}$. Its $n+2$ nodes are the zeros of the polynomial

$$P^t_{n+2}(x) = (x - \alpha_{n-1}) P_{n+1}(x) - \beta_{n+1} P_n(x).$$

We refer to $\mathcal{Q}_{n+2}^t$ as a *truncated* quadrature rule. Since only the two outermost nodes are not guaranteed to be internal, the rule $\mathcal{Q}_{n+2}^t$ is internal if and only if

$$x^{n+2}P_{n+2}^t(x) \geqslant 0 \quad \text{for} \quad x = \pm 1. \tag{8}$$

This paper investigates internality of quadrature rules with weight functions of the forms

$$w(x) = \frac{1}{z-x}(1-x)^a(1+x)^b \quad \text{or} \quad w(x) = (z-x)(1-x)^a(1+x)^b$$

with $a, b > -1$ and $z \in \mathbb{R}$, $|z| > 1$. Our analysis shows that some averaged, optimal averaged Gauss rules, and truncated quadrature rules for these measures are internal for suitable exponents a and b if the number of nodes is large enough. Analogous results on the internality of Gauss-Kronrod rules are discussed, e.g., in [12]. Our analysis complements and extends results reported in [1–3, 5, 6] on the internality of modified Chebyshev measures of the forms

$$\frac{1}{z-x}(1-x)^{\pm\frac{1}{2}}(1+x)^{\pm\frac{1}{2}} \quad \text{and} \quad \frac{u-x}{z-x}(1-x)^{\pm\frac{1}{2}}(1+x)^{\pm\frac{1}{2}},$$

where $u = u(z)$ is a function of $z \in \mathbb{R}$, $|z| > 1$. Our interest in the modification of Jacobi weight functions stems from the attention that modification methods and their applications, e.g., to computing the Hilbert transform, have received in the literature; see Gautschi [8, Section 2.4] for a thorough discussion of modification algorithms and some applications.

This paper is organized as follows. In Section 2 we outline properties of Jacobi weight functions considered as generalizations of the four Chebyshev weight functions. Section 3 investigates asymptotic properties of recurrence coefficients (2) for Jacobi weight functions modified by a linear divisor, and then discusses conditions under which the averaged and optimal averaged rules are internal. Some computed illustrations also are provided. In Section 4 we do the same for Jacobi weight functions that are modified by a linear factor. Section 5 illustrates the performance of the quadrature rules considered. Concluding remarks can be found in Section 6.

2. Jacobi weight function

The functions

$$w(x) = w_{a,b}(x) = (1-x)^a(1+x)^b, \quad -1 < x < 1, \tag{9}$$

where $a, b \in (-1, +\infty)$ are some constants are known as Jacobi weight functions. This section reviews available results on the internality of averaged and optimal averaged Gauss quadrature rules associated with these weight functions.

Given the exponents a and b, the recurrence coefficients (2) are, see, e.g., [8],

$$\begin{aligned}
\alpha_n &= \frac{b^2-a^2}{(2n+a+b)(2n+a+b+2)} \quad \text{for} \quad n \geqslant 0,\\
\beta_n &= \frac{4n(n+a)(n+b)(n+a+b)}{(2n+a+b)^2(2n+a+b+1)(2n+a+b-1)} \quad \text{for} \quad n \geqslant 1,\\
\beta_0 &= \frac{2^{a+b+1}\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}.
\end{aligned} \tag{10}$$

For $n > 0$, these coefficients can be expanded as power series in $\frac{1}{n}$:

$$\alpha_n = \frac{A_2}{n^2} + \frac{A_3}{n^3} + \cdots, \qquad \beta_n = \frac{1}{4}\Big(1 + \frac{B_2}{n^2} + \frac{B_3}{n^3} + \cdots\Big) \quad \text{as } n \to \infty. \tag{11}$$

These power series converge for $n > \frac{a+b+2}{2}$ and $n > \frac{a+b+1}{2}$, respectively. In particular,

$$\begin{aligned}
A_2 &= \frac{b^2-a^2}{4}, & A_3 &= \frac{(a^2-b^2)(a+b+1)}{4},\\
B_2 &= \frac{1-2a^2-2b^2}{4}, & B_3 &= \frac{(a+b)(2a^2+2b^2-1)}{4}.
\end{aligned} \tag{12}$$

The monic orthogonal polynomials P_k take the following values at $x = \pm 1$:

$$P_k(1) = 2^k \frac{\binom{a+k}{k}}{\binom{a+b+2k}{k}}, \qquad P_k(-1) = (-2)^k \frac{\binom{b+k}{k}}{\binom{a+b+2k}{k}}. \tag{13}$$

Laurie [11] showed that the anti-Gaussian formulas, and consequently also the averaged Gauss formulas, associated with the Jacobi weight function (9) for $n \geqslant 1$ are internal if and only if the following two conditions hold:

$$\begin{aligned}(2a{+}1)n^2 + (2a{+}1)(a{+}b{+}1)n + \tfrac{1}{2}(a{+}1)(a{+}b)(a{+}b{+}1) \geqslant 0,\\ (2b{+}1)n^2 + (2b{+}1)(a{+}b{+}1)n + \tfrac{1}{2}(b{+}1)(a{+}b)(a{+}b{+}1) \geqslant 0.\end{aligned}$$

Similarly, the optimal averaged Gauss formulas are internal if and only if the following conditions are satisfied (see [14]):

$$\begin{aligned}(2a{+}1)n^2 + (2a{+}1)(a{+}b{+}1)n + \tfrac{1}{2}(a{+}b)[(a{+}1)(a{+}b{+}1){+}2(a{-}b)] \geqslant 0,\\ (2b{+}1)n^2 + (2b{+}1)(a{+}b{+}1)n + \tfrac{1}{2}(a{+}b)[(b{+}1)(a{+}b{+}1){+}2(b{-}a)] \geqslant 0.\end{aligned}$$

Hence, when n is large enough,

- if $a, b > -\frac{1}{2}$ or $|a| = |b| = \frac{1}{2}$, then both the averaged and optimal averaged formulas are internal;
- if $a < -\frac{1}{2}$ or $b < -\frac{1}{2}$, then both formulas are external;
- if $a = -\frac{1}{2}$, then for $b \in (-\frac{1}{2}, \frac{1}{2})$ only the optimal averaged formulas are internal, and for $b > \frac{1}{2}$ only the averaged formulas are internal.

3. Modifications by a linear divisor

This section considers weight functions of the form

$$\tilde{w}(x) = \frac{(1-x)^a (1+x)^b}{z - x} \quad \text{for} \quad -1 < x < 1, \tag{14}$$

where z is a given real constant with $|z| > 1$. The latter condition is conveniently secured by setting

$$z = \tfrac{1}{2}(c + \tfrac{1}{c}) \quad \text{with} \quad -1 < c < 1. \tag{15}$$

We first discuss asymptotic properties of orthogonal polynomials associated with these weight functions, and subsequently discuss properties of averaged and optimal averaged quadrature rules.

3.1. **Orthogonal polynomials.** It follows from [8, Theorem 2.52 (Uvarov)]) that the monic orthogonal polynomials $\tilde{P}_n$ with respect to the weight function (14) satisfy

$$\tilde{P}_n = P_n - r_{n-1} P_{n-1}, \tag{16}$$

where

$$r_n = \frac{\int_{-1}^{1} P_{n+1}(x)\tilde{w}(x)\, dx}{\int_{-1}^{1} P_n(x)\tilde{w}(x)\, dx}.$$

The sequence r_n can be computed recursively by using the relations

$$r_{-1} = \int_{-1}^{1} \frac{w(t)}{z-t}\, dt \quad \text{and} \quad r_n = z - \alpha_n - \frac{\beta_n}{r_{n-1}}, \quad n \geqslant 0. \tag{17}$$

The orthogonal polynomials $\tilde{P}_n$ satisfy a three-term recursion relation of the form

$$\tilde{P}_{k+1}(x) = (x - \tilde{\alpha}_k)\tilde{P}_k(x) - \tilde{\beta}_k \tilde{P}_{k-1}(x), \quad k \geqslant 0,$$

analogously to the polynomials P_k; cf. (2). An algorithm described by Gautschi [8, eqs. (2.4.26-27)] gives the recurrence coefficients $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ in terms of the sequence r_n:

$$\tilde{\alpha}_n = \alpha_n + r_n - r_{n-1} \quad \text{and} \quad \tilde{\beta}_n = \beta_{n-1} \cdot \frac{r_{n-1}}{r_{n-2}}. \tag{18}$$

However, there is apparently no closed formula for the r_n for $n \geqslant 0$, and r_{-1} can be expressed in the form of a hypergeometric function

$$r_{-1} = {}_2F_1(1, b{+}1; a{+}b{+}2; \tfrac{2}{z+1}).$$

Therefore, only asymptotic expressions as $n \to \infty$ can be determined for the r_n and for the coefficients $\tilde{\alpha}_n$ and $\tilde{\beta}_n$.

Since the terms r_n approach $\frac{c}{2}$, where c is given by (15), we write $r_n = \frac{c}{2} + \delta_n$, with $\delta_n \to 0$ as $n \to \infty$. Then (17) becomes

$$\delta_n = \frac{1}{2c} - \alpha_n - \frac{\beta_n}{r_{n-1}} = \frac{1-4\beta_n}{2c} - \alpha_n - \Big(\frac{\beta_n}{r_{n-1}} - \frac{2\beta_n}{c}\Big) = p_{n-1} + q_{n-1}\delta_{n-1},$$

where

$$p_n = \frac{1-4\beta_{n+1}}{2c} - \alpha_{n+1}, \qquad q_n = \frac{2\beta_{n+1}}{c r_n}. \tag{19}$$

We observe that p_n is explicitly computable by (10), whereas q_n is related to r_n. Thus,

$$n^2 p_n \to \frac{2(1+c)a^2 + 2(1-c)b^2 - 1}{8c} \quad \text{and} \quad q_n \to \frac{1}{c^2} \quad \text{as } n \to \infty.$$

Therefore,

$$\begin{aligned} r_n - \frac{c}{2} = \delta_n &= \frac{1}{q_n}\delta_{n+1} - \frac{p_n}{q_n} = \frac{1}{q_n q_{n+1}}\delta_{n+2} - \frac{p_n}{q_n} - \frac{p_{n+1}}{q_n q_{n+1}} = \cdots \\ &= \frac{1}{q_n q_{n+1}\cdots q_{n+k}}\delta_{n+k+1} - \sum_{i=0}^{k} \frac{p_{n+i}}{q_n q_{n+1}\cdots q_{n+i}}. \end{aligned} \tag{20}$$

As k increases, the terms $\delta_{n+k+1}/(q_n q_{n+1}\cdots q_{n+k})$ approach zero exponentially fast (recall that $\lim_{n\to\infty} q_n > 1$). Analyzing the remaining sum over i gives us the following result.

Lemma 1. *When $n \to \infty$, we have*

$$r_n = \frac{c}{2}\big(1 + R_2 n^{-2} + o(n^{-2})\big), \quad \textit{where} \quad R_2 = \frac{1 - 2(1+c)a^2 - 2(1-c)b^2}{4(1-c^2)}. \quad \square$$

Using (18) and taking (10) into account establishes the following theorem.

Theorem 1. *When $n \to \infty$, we have*

$$\begin{aligned} \tilde{\alpha}_n &= \alpha_n + o(n^{-2}) = A_2 n^{-2} + o(n^{-2}), \\ \tilde{\beta}_n &= \beta_n + o(n^{-2}) = \frac{1}{4}\big(1 + B_2 n^{-2} + o(n^{-2})\big), \end{aligned}$$

where A_2 and B_2 are given by (11). $\square$

3.2. **Internality of averaged quadrature rules.** It follows from (7) and (16) that the averaged Gauss formulas with n nodes have no node larger than $x = 1$ if and only if

$$\frac{P_{n+1}(1) - r_n P_n(1)}{P_{n-1}(1) - r_{n-2} P_{n-2}(1)} \geqslant \beta^*_{n+1},$$

which reduces to

$$L = d_{n-1} d_{n-2} \cdot \frac{d_n - r_n}{d_{n-2} - r_{n-2}} \geqslant \beta^*_{n+1}, \quad \text{where} \quad d_k = \frac{P_{k+1}(1)}{P_k(1)}. \tag{21}$$

The values of the $P_k(1)$ are known (13). We obtain

$$d_k = \frac{2(k+a+1)(k+a+b+1)}{(2k+a+b+1)(2k+a+b+2)} = \frac{1}{2}\Big(1 + \frac{2a+1}{2k}\Big) + o(k^{-1}). \tag{22}$$

Straightforward computations yield that the left-hand side of (21) satisfies

$$L = \frac{1}{4}\Big(1 + \frac{1+2a}{n} + o(n^{-1})\Big).$$

On the other hand, the particular values of β^*_{n+1} in the $(2n+1)$-node averaged and optimal averaged Gauss formulas, which are $\tilde{\beta}_n$ and $\tilde{\beta}_{n+1}$, respectively, are both asymptotically equal to

$$\beta^*_{n+1} = \beta_n + o(n^{-2}) = \frac{1}{4} + \frac{1-2a^2-2b^2}{16n^2} + o(n^{-2}).$$

Thus, for $a \neq -\frac{1}{2}$ and n sufficiently large, $L - \beta^*_{n+1}$ and $1+2a$ have the same sign. It follows that the largest node in both the $(2n+1)$-node averaged and optimal averaged quadrature rules is internal if $a > -\frac{1}{2}$, and external if $a < -\frac{1}{2}$.

Internality and externality of the smallest node can be established similarly by switching the exponents a and b in (9). This gives the following results.

Theorem 2. *Let the weight function be defined by* (14). *Then for n large enough:*

- *The largest node in the averaged and optimal averaged Gauss rules is internal if $a > -\frac{1}{2}$, and external if $a < -\frac{1}{2}$.*
- *The smallest node in the averaged and optimal averaged Gauss rules is internal if $b > -\frac{1}{2}$, and external if $b < -\frac{1}{2}$.* □

We conclude that both the averaged and optimal averaged quadrature rules are internal if $\min\{a,b\} > -\frac{1}{2}$, and both are external if $\min\{a,b\} < -\frac{1}{2}$.

As for the truncated quadrature rule $\mathcal{Q}^t_{n+2}$, all its nodes being bounded above by one is by (8) equivalent to $(1-\tilde{\alpha}_{n-1})\tilde{P}_{n+1}(1)/\tilde{P}_n(1) \geqslant \tilde{\beta}_{n+1}$, which can be expressed as

$$(1-\tilde{\alpha}_{n-1})d_{n-1} \cdot \frac{d_n - r_n}{d_{n-1} - r_{n-1}} \geqslant \tilde{\beta}_{n+1}.$$

When $n \to \infty$, the left-hand side and the right-hand side approach $\frac{1}{2}$ and $\frac{1}{4}$, respectively. This gives the following result.

Theorem 3. *The truncated formula $\mathcal{Q}^t_{n+2}$ associated with the weight function* (14) *is internal when n is large enough.* □

Numerical experiments suggest that, generally, n does not have to be very large in order for the rule $\mathcal{Q}^t_{n+2}$ to be internal.

3.3. **Borderline cases.** When $a = -\frac{1}{2}$, the asymptotic expansions of the two sides of (21) agree up to the n^{-2}-term. Consequently, Theorem 2 is not helpful for establishing internality if $\min\{a,b\} = -\frac{1}{2}$. This section investigates this situation. Due to symmetry, it suffices to consider the case $a = -\frac{1}{2}$.

By Lemma 1, we can determine the quantity q_n, given by (19), to $o(n^{-2})$ accuracy. Substituting this expression into (20) allows us to improve Theorem 1. The next term in the asymptotic expansions for r_n is of the form $O(n^{-3})$ and can be determined by comparing the coefficients in the second relation of (17). We obtain that

$$r_n = \frac{c}{2}\big(1 + R_2 n^{-2} + R_3 n^{-3} + o(n^{-3})\big), \tag{23}$$

where the coefficient R_2 is defined in Lemma 1 and R_3 is a suitable coefficient to be determined below. Using the identity $(n-1)^{-2} = \sum_{i=2}^{\infty}(i-1)n^{-i}$, we obtain

$$r_{n-1} = \frac{c}{2}\big(1 + R_2 n^{-2} + (R_3 + 2R_2)n^{-3} + o(n^{-3})\big). \tag{24}$$

The quantities (12) can for $a = -\frac{1}{2}$ be expressed as

$$A_2 = \frac{4b^2-1}{16}, \quad A_3 = \frac{(1+2b)(1-4b^2)}{32}, \quad B_2 = \frac{1-4b^2}{8}, \quad B_3 = \frac{(1-2b)(1-4b^2)}{16}. \tag{25}$$

Moreover, the value of R_2 defined in Lemma 1 can be expressed as

$$R_2 = \frac{1-4b^2}{8(1+c)}.$$

Writing the relation (17) in the form

$$(z - \alpha_n - r_n)r_{n-1} = \beta_n$$

and substituting the expressions (23) and (24), as well as the formulas (15) and (11), into the above formula yields

$$\frac{c}{2}\Big(1 + \frac{R_2}{n^2} + \frac{2R_2+R_3}{n^3}\Big)\Big(\frac{1}{2c} - \frac{A_2}{n^2} - \frac{A_3}{n^3} - \frac{c}{2}\Big(\frac{R_2}{n^2} + \frac{R_3}{n^3}\Big)\Big) + o(n^{-3}) = \frac{1}{4}\Big(1 + \frac{B_2}{n^2} + \frac{B_3}{n^3}\Big).$$

Equating coefficients for n^{-3} in this expression gives

$$R_3 = \frac{2cA_3 + B_3 - 2R_2}{1-c^2} = (4b^2-1)\frac{2+(1+c)(1+2b)}{16(1+c)^2}.$$

Using the same notation as in (21), we obtain

$$d_n = \frac{1}{2}\Big(1 + \frac{1-4b^2}{16n^2} - \frac{(1+b)(1-4b^2)}{16n^3} + o(n^{-3})\Big). \tag{26}$$

This makes the left-hand side of condition (21) easy to evaluate:

$$L = d_{n-1}d_{n-2}\cdot\frac{d_n - r_n}{d_{n-2}-r_{n-2}} = \frac{1}{4}\Big(1 + \frac{1-4b^2}{8n^2} + \frac{1-4b^2}{8n^3}\Big(2 - b - \frac{2}{1+c}\Big) + o(n^{-3})\Big).$$

On the other hand, we obtain from (18) that

$$\tilde{\beta}_n = \frac{1}{4}\Big(1 + \frac{B_2}{n^2} + \frac{\tilde{B}_3}{n^3} + o(n^{-3})\Big),$$

where B_2 is the same as in (25) and

$$\tilde{B}_3 = B_3 - \frac{2c}{1-c^2}(2A_2 + cB_2) = \frac{1-4b^2}{8}\Big(\frac{5}{2} - b - \frac{2}{1+c}\Big).$$

Also, $\tilde{\beta}_{n+1} = \tilde{\beta}_n - \frac{1}{2}B_2 n^{-3} + o(n^{-3})$. This leads to

$$\tilde{\beta}_n = L + \frac{1-4b^2}{64n^3} \quad \text{and} \quad \tilde{\beta}_{n+1} = L - \frac{3(1-4b^2)}{64n^3}.$$

We have shown the following result.

Theorem 4. *Let the weight function be given by (14) with $a = -\frac{1}{2}$. Then for n large enough:*
- *The largest node of the averaged quadrature rule is internal if $|b| > \frac{1}{2}$ and external if $|b| < \frac{1}{2}$.*
- *The largest node of the optimal averaged quadrature rule is internal if $|b| < \frac{1}{2}$ and external if $|b| > \frac{1}{2}$.*

Note that internality of the smallest node is determined by Theorem 2. For instance, if $b < a = -\frac{1}{2}$, then neither the averaged Gauss rules nor the optimal averaged Gauss rules are internal, because the smallest node is not in the interval $[-1, 1]$.

We turn to the case $b = \pm\frac{1}{2}$. Then the Jacobi weight function reduces to the Chebyshev weight functions of the first and third kind. The following results are shown in [1, 2] for these weight functions.

Theorem 5. *Let $a = -\frac{1}{2}$ and $|b| = \frac{1}{2}$. Then both the averaged Gauss rules and the optimal averaged Gauss rules for the weight function* (14) *have the largest node equal to* 1. □

3.4. **Numerical experiments.** This section shows the smallest and largest nodes of averaged Gauss rules $\mathcal{Q}^L_{2n+1}$, optimal averaged Gauss rules $\mathcal{Q}^S_{2n+1}$, and truncated rules $\mathcal{Q}^t_{n+2}$ for several values of the parameters a, b, and z in (14). These examples provide some insight into how large n has to be in order for Theorems 2-4 to hold. We will use following notation:

- x^L_1, x^L_{2n+1} – the smallest and largest node of the averaged Gauss rule $\mathcal{Q}^L_{2n+1}$;
- x^S_1, x^S_{2n+1} – the smallest and largest node of the optimal averaged Gauss rule $\mathcal{Q}^S_{2n+1}$;
- x^t_1, x^t_{n+2} – the smallest and largest node of the truncated rule $\mathcal{Q}^t_{n+2}$.

All computations reported in this paper are carried out using high-precision arithmetic with 250 significant decimal digits.

Example 1. We first consider two weight functions with the pole z clear from the interval of integration. For both weight functions, the conclusions of Theorems 2-3 hold already for fairly small values of n. Letting the exponents a and b be close to 1/2 does not affect the results significantly.

Let $a = b = -0.49$ and $z = -2$. Then both averaged quadrature rules are internal in agreement with the theory developed. This is illustrated by Table 1.

n	x^L_1	x^L_{2n+1}	x^S_1	x^S_{2n+1}
5	−0.9998247263	0.9997788824	−0.9998172516	0.9997705095
10	−0.9999528072	0.9999470482	−0.9999523297	0.9999465425
20	−0.9999877576	0.9999870321	−0.9999877271	0.9999870007
40	−0.9999968820	0.9999967908	−0.9999968800	0.9999967888

TABLE 1: $a = b = -0.49$ and $z = -2$. The outermost nodes of $\mathcal{Q}^L_{2n+1}$ and $\mathcal{Q}^S_{2n+1}$.

Tables 2 and 3 illustrate the situation when $a = -0.4$, $b = -0.6$, and $z = -2$. Since $b < -\frac{1}{2}$, the two averaged rules are external on the left, and only the truncated rules are internal.

n	x^L_1	x^L_{2n+1}	x^S_1	x^S_{2n+1}
5	−1.0014527400	0.9974394481	−1.0014472497	0.9974314399
10	−1.0003868438	0.9993914472	−1.0003868936	0.9993915161
20	−1.0001000472	0.9998513623	−1.0001000621	0.9998513822
40	−1.0000254560	0.9999632512	−1.0000254573	0.9999632529

TABLE 2: $a = -0.4$, $b = -0.6$, and $z = -2$. The outermost nodes of $\mathcal{Q}^L_{2n+1}$ and $\mathcal{Q}^S_{2n+1}$.

n	x^t_1	x^t_{n+2}
5	−0.9824832306	0.9669509440
10	−0.9937839070	0.9889895506
20	−0.9980988847	0.9967726483
40	−0.9994700459	0.9991222850

TABLE 3: $a = -0.4$, $b = -0.6$, and $z = -2$. The outermost nodes of $\mathcal{Q}^t_{n+2}$.

Example 2. In this example, we let z be very close to the interval of integration. Then n is required to be fairly large in order for Theorems 1-4 to apply.

We first consider the situation when $a = -0.49$, $b = -0.51$, and $z = -1.0001$. Table 4 shows the leftmost node of the two averaged rules to be outside the interval $[-1, 1]$ for $n \in \{40, 80\}$.

n	x_1^L	x_{2n+1}^L	x_1^S	x_{2n+1}^S
5	−0.9999924559	0.9997492426	−0.9999917740	0.9997371106
10	−0.9999980692	0.9999437037	−0.9999980066	0.9999431487
20	−0.9999998919	0.9999866353	−0.9999998861	0.9999866073
40	−1.0000002646	0.9999967425	−1.0000002641	0.9999967411
80	−1.0000001990	0.9999991958	−1.0000001990	0.9999991957

TABLE 4: $a = -0.49$, $b = -0.51$, and $z = -1.0001$. The outermost nodes of $\mathcal{Q}_{2n+1}^L$ and $\mathcal{Q}_{2n+1}^S$.

Table 5 shows the extreme nodes for the truncated rules. These rules can be seen to be internal.

n	x_1^t	x_{n+2}^t
5	−0.9989970694	0.9702919604
10	−0.9994415796	0.9904720093
20	−0.9997123781	0.9972701298
40	−0.9998624948	0.9992669443

TABLE 5: $a = -0.49$, $b = -0.51$, and $z = -1.0001$. The outermost nodes of $\mathcal{Q}_{n+2}^t$.

We turn to the situation when $a = 1.25$, $b = -0.25$, and $z = -1.0001$. Table 6 shows all the nodes of the averaged and optimal averaged quadrature rules to be internal for large enough values of n. Although $|a|$ and $|b|$ are fairly far away from $1/2$, a quite large value of n is needed for these rules to be internal.

n	x_1^L	x_{2n+1}^L	x_1^S	x_{2n+1}^S
5	−1.0003182001	0.9017842279	−1.0004069381	0.9028177507
10	−1.0000701380	0.9727616281	−1.0000788231	0.9728393495
20	−0.9999998281	0.9928311164	−1.0000006315	0.9928363352
40	−0.9999891268	0.9981614842	−0.9999891969	0.9981618140
80	−0.9999927041	0.9995345048	−0.9999927099	0.9995345250

TABLE 6: $a = -1.25$, $b = -0.25$, and $z = -1.0001$. The outermost nodes of $\mathcal{Q}_{2n+1}^L$ and $\mathcal{Q}_{2n+1}^S$.

Example 3. This example considers the borderline case $a = -0.5$ and $b = 0.1$. We first let $z = -5$. Table 7 shows, as expected, the averaged Gauss quadrature rules to be external on the right, whereas the optimal averaged quadrature rules are internal already for small values of n.

n	x_1^L	x_{2n+1}^L	x_1^S	x_{2n+1}^S
5	−0.9804450981	1.0000182366	−0.9803287379	0.9999434585
10	−0.9947080011	1.0000013035	−0.9946995406	0.9999960545
20	−0.9986223673	1.0000000873	−0.9986217953	0.9999997375
40	−0.9996484831	1.0000000057	−0.9996484459	0.9999999830

TABLE 7: $a = -0.5$, $b = 0.1$, and $z = -5$. The outermost nodes of $\mathcal{Q}_{2n+1}^L$ and $\mathcal{Q}_{2n+1}^S$.

When, instead, $z = -1.0001$, we must let n reach 100 in order for the optimal averaged Gauss rule $\mathcal{Q}_{2n+1}^S$ to be internal. This is illustrated by Table 8.

n	x_1^L	x_{2n+1}^L	x_1^S	x_{2n+1}^S
5	−1.00159499240735	0.99996309034376	−1.00167693059885	1.00011671645370
10	−1.00019143001357	0.99999828051999	−1.00019556972029	1.00000522571056
20	−0.99995952217561	0.99999992479868	−0.99995972978294	1.00000022638554
40	−0.99995213324585	0.99999999736607	−0.99995214187641	1.00000000791011
80	−0.99997375844002	0.99999999996768	−0.99997375856818	1.00000000009703
100	−0.99997971296502	1.00000000000379	−0.99997971294907	0.99999999998865

TABLE 8: $a = -0.5$, $b = 0.1$, and $z = -1.0001$. The outermost nodes of $\mathcal{Q}_{2n+1}^L$ and $\mathcal{Q}_{2n+1}^S$.

4. Modifications by a linear factor

This section considers weight functions that are obtained by modifying the Jacobi measure by a linear factor. Thus, we consider weight functions of the form

$$\tilde{w}(x) = (z-x)(1-x)^a(1+x)^b \quad \text{for} \quad -1 < x < 1, \tag{27}$$

where z is a given real constant with $|z| > 1$. Similarly as above, we have $z = \frac{1}{2}(c + \frac{1}{c})$ for some $c \in (-1, 1)$.

4.1. **Orthogonal polynomials.** It follows from [8, Theorem 2.52 (Uvarov)]) that the monic orthogonal polynomials $\tilde{P}_n$ with respect to the weight function (27) satisfy

$$(x-z)\tilde{P}_n(x) = P_{n+1}(x) - s_n P_n(x), \tag{28}$$

where

$$s_n = \frac{P_{n+1}(z)}{P_n(z)}.$$

The scalars s_n can be computed recursively by using the relations

$$s_0 = z - \alpha_0 \quad \text{and} \quad s_n = z - \alpha_n - \frac{\beta_n}{s_{n-1}}, \quad n \geqslant 1. \tag{29}$$

The orthogonal polynomials $\tilde{P}_n$ satisfy a recursion relation analogous to (2),

$$\tilde{P}_{k+1}(x) = (x - \tilde{\alpha}_k)\tilde{P}_k(x) - \tilde{\beta}_k \tilde{P}_{k-1}(x), \quad k \geqslant 0.$$

Gautschi [8, eqs. (2.4.12-13)] describes an algorithm for computing the recursion coefficients $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ by using the relations

$$\tilde{\alpha}_n = \alpha_{n+1} + s_{n+1} - s_n \quad \text{and} \quad \tilde{\beta}_n = \beta_n \cdot \frac{s_n}{s_{n-1}}. \tag{30}$$

Similarly as above, we determine asymptotic expressions for s_n, $\tilde{\alpha}_n$, and $\tilde{\beta}_n$ as $n \to \infty$. The scalars s_n converge to $\frac{1}{2c}$ as $n \to \infty$. We therefore have

$$s_n = \frac{1}{2c}\big(1 + S_2 n^{-2} + S_3 n^{-3} + o(n^{-3})\big). \tag{31}$$

Relation (29) implies that

$$z s_{n-1} - s_n s_{n-1} - \alpha_n s_{n-1} = \beta_n.$$

Equating the coefficients for n^{-2} and n^{-3} yields

$$S_2 = \frac{c}{c^2-1}(2A_2 + cB_2), \quad S_3 = \frac{c}{c^2-1}(2A_3 + cB_3) - \frac{2c^3}{(c^2-1)^2}(2A_2 + cB_2),$$

where A_2, A_3, B_2, B_3 are the same as in (11). Now (30) leads to

$$\tilde{\alpha}_n = \alpha_{n+1} - 2S_2 n^{-3} + o(n^{-3}), \quad \tilde{\beta}_n = \beta_n(1 - 2S_2 n^{-3}) + o(n^{-3}).$$

We obtain analogously to Theorem 1 the following result.

Theorem 6. *When $n \to \infty$, we have*

$$\tilde{\alpha}_n = \alpha_n + o(n^{-2}), \quad \tilde{\beta}_n = \beta_n + o(n^{-2}).$$

4.2. **Internality of averaged quadrature rules.** It follows from (7) and (28) that averaged Gauss formulas with n nodes have no node larger than $x = 1$ if and only if

$$\frac{P_{n+2}(1) - s_{n+1}P_{n+1}(1)}{P_n(1) - s_{n-1}P_{n-1}(1)} \geqslant \beta^*_{n+1}.$$

This expression reduces to

$$L = d_n d_{n-1} \cdot \frac{d_{n+1} - s_{n+1}}{d_{n-1} - s_{n-1}} \geqslant \beta^*_{n+1}, \tag{32}$$

where the terms s_n are the same as in (29) and the d_k are given by (22). Thus, similarly as in Section 3, the left-hand side of (32) is

$$L = \frac{1}{4}\Big(1 + \frac{1+2a}{n} + o(n^{-1})\Big).$$

This leads to a result analogous to Theorem 2.

Theorem 7. *Let the weight function be given by* (27). *Then for n large enough:*
• *The largest node in the averaged and optimal averaged Gauss rules is internal if $a > -\frac{1}{2}$, and external if $a < -\frac{1}{2}$.*
• *The smallest node in the averaged and optimal averaged Gauss rules is internal if $b > -\frac{1}{2}$, and external if $b < -\frac{1}{2}$.* □

The internality condition for the truncated quadrature $\mathcal{Q}^t_{n+2}$ at $x = 1$ is equivalent to

$$(1 - \tilde{\alpha}_{n-1})d_{n-1} \cdot \frac{d_n - s_n}{d_{n-1} - s_{n-1}} \geqslant \tilde{\beta}_{n+1};$$

see the analogous discussion in Section 3. When $n \to \infty$, the left-hand side and the right-hand side approach $\frac{1}{2}$ and $\frac{1}{4}$, respectively. This leads to the following result which is analogous to Theorem 3.

Theorem 8. *The truncated formula $\mathcal{Q}^t_{n+2}$ corresponding to the weight function* (27) *is internal when n is large enough.* □

4.3. **Borderline cases.** Let $a = -\frac{1}{2}$. Then the two sides of (32) coincide up to order n^{-2}. We therefore have to compare the n^{-3}-term. With the quantities A_2, A_3 and B_2, B_3 defined as in (25), the coefficients S_2, S_3 in (31) achieve the values

$$S_2 = \frac{c}{8(1+c)}(1-4b^2), \quad S_3 = \frac{c}{16(1+c)^2}(1-4b^2)\big(2-(1+c)(3+2b)\big).$$

Using (26), it is easy to compute the left-hand side of (32):

$$L = \frac{1}{4}\Big(1 + \frac{1-4b^2}{8n^2} + \frac{1-4b^2}{8n^3}\Big(-b-2+\frac{2}{1+c}\Big) + o(n^{-3})\Big).$$

On the other hand, (30) gives

$$\tilde{\beta}_n = \frac{1}{4}\Big(1 + \frac{B_2}{n^2} + \frac{\tilde{B}_3}{n^3} + o(n^{-3})\Big),$$

where the B_2, B_3 are as in (11) and

$$\tilde{B}_3 = B_3 - 2S_2 = \frac{1-4b^2}{16}\Big(-b-\frac{3}{2}+\frac{2}{1+c}\Big).$$

Since $\tilde{\beta}_{n+1} = \tilde{\beta}_n - \frac{1}{2}B_2 n^{-3} + o(n^{-3})$, we have

$$\tilde{\beta}_n = L + \frac{1-4b^2}{64n^3} + o(n^{-3}) \quad \text{and} \quad \tilde{\beta}_{n+1} = L - \frac{3(1-4b^2)}{64n^3} + o(n^{-3}).$$

A discussion on internality at $x = -1$ is completely analogous. This leads to the following result, which is analogous to Theorem 4.

Theorem 9. *Let the weight function be given by* (27) *with* $a = -\frac{1}{2}$. *Then for* n *large enough:*
- *The largest node of the averaged Gauss rules is internal if* $|b| > \frac{1}{2}$ *and external if* $|b| < \frac{1}{2}$.
- *The largest node of the optimal averaged Gauss rules is internal if* $|b| < \frac{1}{2}$ *and external if* $|b| > \frac{1}{2}$.

4.4. **Numerical experiments.** The following examples display the two outermost nodes for several values of the parameters a, b, and z in (27). We use the same notation as in Examples 1-3, i.e.,

- x_1^L, x_{2n+1}^L – the smallest and largest nodes of the averaged Gauss rule $\mathcal{Q}_{2n+1}^L$;
- x_1^S, x_{2n+1}^S – the smallest and largest nodes of the optimal averaged rule $\mathcal{Q}_{2n+1}^S$;
- x_1^t, x_{n+2}^t – the smallest and largest nodes of the truncated rule $\mathcal{Q}_{n+2}^t$.

Example 4. Let $a = b = -0.49$ and $z = -2$. Then both the averaged and optimal averaged quadrature rules are internal also for small values of n. This is illustrated by Table 9.

n	x_1^L	x_{2n+1}^L	x_1^S	x_{2n+1}^S
5	−0.9997627594	0.9998141741	−0.9997518675	0.9998044845
10	−0.9999453234	0.9999513602	−0.9999447838	0.9999508511
20	−0.9999868267	0.9999875693	−0.9999867944	0.9999875378
40	−0.9999967657	0.9999968579	−0.9999967637	0.9999968560

TABLE 9: $a = b = -0.49$ and $z = -2$. The outermost nodes of $\mathcal{Q}_{2n+1}^L$ and $\mathcal{Q}_{2n+1}^S$.

Example 5. Let $a = -0.5$, $b = 0.3$, and $z = -1.01$. Then, as can be expected, the optimal averaged Gauss rules are internal and the averaged Gauss rules have one node larger than 1, but only for $n \geqslant 20$; see Table 10.

n	x_1^L	x_{2n+1}^L	x_1^S	x_{2n+1}^S
5	−0.9170823706	0.9999279798	−0.9177926205	1.0002244899
10	−0.9790156490	0.9999957207	−0.9790590130	1.0000130275
20	−0.9959004096	1.0000000185	−0.9959002360	0.9999999466
40	−0.9992391377	1.0000000065	−0.9992390807	0.9999999804

TABLE 10: $a = -0.5$, $b = 0.3$, and $z = -1.01$. The outermost nodes of $\mathcal{Q}_{2n+1}^L$ and $\mathcal{Q}_{2n+1}^S$.

In a seemingly very similar situation when $a = -0.5$, $b = 0.7$, and $z = -1.01$, the averaged and optimal averaged quadrature rules exchange their properties already for small values of n, i.e., the averaged Gauss rules are internal while the optimal averaged Gauss rules are not. This is illustrated by Table 11.

n	x_1^L	x_{2n+1}^L	x_1^S	x_{2n+1}^S
5	−0.8884614682	0.9998798570	−0.8897461069	1.0003754707
10	−0.9689191262	0.9999903625	−0.9690265012	1.0000292821
20	−0.9930610853	0.9999995928	−0.9930655522	1.0000012275
40	−0.9986089985	0.9999999899	−0.9986091010	1.0000000303

TABLE 11: $a = -0.5$, $b = 0.7$, and $z = -1.01$. The outermost nodes of $\mathcal{Q}_{2n+1}^L$ and $\mathcal{Q}_{2n+1}^S$.

5. Performance of the quadrature rules

We apply the two kinds of averaged quadrature rules and the truncated rules to estimate the quadrature error in the Gauss rule (3) for a few integrands f and weight functions w of the types (14) and (27). The results for several values of n are reported in the following examples. We use the notation:

- $\mathcal{E}^L$ – the estimate $|\mathcal{Q}_{2n+1}^L(f) - \mathcal{G}_n(f)|$ of the magnitude of the quadrature error obtained with the averaged Gauss rule $\mathcal{Q}_{2n+1}^L$;

- $\mathcal{E}^S$ – the estimate $|\mathcal{Q}^S_{2n+1}(f) - \mathcal{G}_n(f)|$ of the magnitude of the quadrature error obtained with the optimal averaged Gauss rule $\mathcal{Q}^S_{2n+1}$;
- $\mathcal{E}^t$ – the estimate $|\mathcal{Q}^t_{n+2}(f) - \mathcal{G}_n(f)|$ of the magnitude of the quadrature error obtained with the truncated rule $\mathcal{Q}^t_{n+2}$;
- *Error* – an accurate approximation of the magnitude of the quadrature error $|\mathcal{I}(f) - \mathcal{G}_n(f)|$ obtained by evaluating $|\mathcal{G}_m(f) - \mathcal{G}_n(f)|$ for some large $m \gg n$.

In practice, one typically would use the values $\mathcal{Q}^L_{2n+1}(f)$, $\mathcal{Q}^S_{2n+1}(f)$, or $\mathcal{Q}^t_{n+2}(f)$ not only to estimate the error in $\mathcal{G}_n(f)$, but also as approximations of $\mathcal{I}(f)$, but use the computed error estimates.

Example 6. We first consider an example with an integrand and weight function such that the integral (1) can be approximated to high accuracy. Specifically, we let

$$f(x) = \cos(\pi x), \qquad w(x) = \frac{1}{2+x}(1-x)^{0.45}(1+x)^{-0.45}. \tag{33}$$

Table 12 shows the averaged and optimal averaged Gauss rules to approximate the integral very accurately. The actual value of the integral $\mathcal{I}(f)$ is about 0.835728. The table shows the averaged Gauss rules, the optimal averaged Gauss rules, as well as the truncated rules to yield very accurate estimates of the quadrature error.

n	$\mathcal{E}^L$	$\mathcal{E}^S$	$\mathcal{E}^t$	*Error*
5	3.2385×10^{-5}	3.2385×10^{-5}	3.2376×10^{-5}	3.2385×10^{-5}
10	5.0719×10^{-15}	5.0719×10^{-15}	5.0718×10^{-15}	5.0719×10^{-15}
20	1.3524×10^{-40}	1.3524×10^{-40}	1.3524×10^{-40}	1.3524×10^{-40}
40	1.1145×10^{-103}	1.1145×10^{-103}	1.1145×10^{-103}	1.1145×10^{-103}

TABLE 12: Estimates of the quadrature error $|\mathcal{I}(f) - \mathcal{G}_n(f)|$ in (33) determined with the rules $\mathcal{Q}^L_{2n+1}$, $\mathcal{Q}^S_{2n+1}$, and $\mathcal{Q}^t_{n+2}$, as well as the quadrature error.

Our next example involves an integrand with a singularity at $x = 1$ and a "borderline" case for the weight function,

$$f(x) = \sin^3(1-x)\ln(1-x), \qquad w(x) = \frac{1}{1.25+x}(1-x)^{-0.5}(1+x)^{0.45}. \tag{34}$$

For this weight function, the rightmost node for the averaged Gauss rules is larger than 1, and we note that the integrand f is not defined at this node. The actual value of the integral $\mathcal{I}(f)$ is about -0.220893.

Table 13 displays the computed results. The table shows the optimal averaged Gauss rules to yield very accurate estimates of the quadrature error. The estimates determined by the truncated rules are less accurate, but still within a factor 4 of the actual error.

n	$\mathcal{E}^L$	$\mathcal{E}^S$	$\mathcal{E}^t$	*Error*
5	n/a	6.0108×10^{-5}	5.5562×10^{-5}	5.9785×10^{-5}
10	n/a	3.5557×10^{-7}	2.5369×10^{-7}	3.5285×10^{-7}
20	n/a	2.8474×10^{-9}	1.3712×10^{-9}	2.8252×10^{-9}
40	n/a	2.2648×10^{-11}	6.4857×10^{-12}	2.2470×10^{-11}

TABLE 13: Estimates of the quadrature error $|\mathcal{I}(f) - \mathcal{G}_n(f)|$ in (34) determined with the rules $\mathcal{Q}^L_{2n+1}$, $\mathcal{Q}^S_{2n+1}$ and $\mathcal{Q}^t_{n+2}$, as well as the actual quadrature error. The rule $\mathcal{Q}^L_{2n+1}$ cannot be evaluated.

Our last integrand and weight function illustrate a situation when z is close to -1 and neither the averaged rules nor the optimal averaged rules can be applied for large values of n. Let

$$f(x) = 999.1^{\log_{10}(1+x)}, \qquad w(x) = (1.001+x)(1-x)^{-0.75}(1+x)^{-0.75}. \tag{35}$$

The smallest node of both the averaged Gauss rules and the optimal averaged Gauss rules is external for n large as is illustrated by Table 5. Thus, the averaged Gauss rules and the optimal averaged Gauss rules can be used for $n = 5, 10, 20$, but not for $n = 40$ and larger. Note that the external rightmost nodes do not cause difficulties, since the integrand is well defined for $x > 1$. The actual value of the integral $\mathcal{I}(f)$ is about 29.2258.

Table 14 shows the averaged Gauss and optimal averaged Gauss rules to furnish accurate estimates of the quadrature error when they are applicable. The error estimates achieved with the truncated quadrature rules are within a factor 6 of the quadrature error.

n	x_1^L	x_{2n+1}^L	x_1^S	x_{2n+1}^S
5	−0.9747819187	1.0025469501	−0.9748406050	1.0025679849
10	−0.9955647026	1.0006637461	−0.9955602512	1.0006621830
20	−0.9999198797	1.0001696633	−0.9999178565	1.0001688989
40	−1.0001191621	1.0000428475	−1.0001191382	1.0000428356
80	−1.0000210001	1.0000107716	−1.0000210052	1.0000107752

Table 14: The outermost nodes of $\mathcal{Q}_{2n+1}^L$ and $\mathcal{Q}_{2n+1}^S$ for the weight function (35).

Table 15 shows computed error estimates as well as the actual error.

n	$\mathcal{E}^L$	$\mathcal{E}^S$	$\mathcal{E}^t$	*Error*
5	5.5276×10^{-9}	5.5288×10^{-9}	5.1902×10^{-9}	5.5227×10^{-9}
10	1.8259×10^{-11}	1.8257×10^{-11}	1.4011×10^{-11}	1.8183×10^{-11}
20	7.4403×10^{-14}	7.4382×10^{-14}	3.8282×10^{-14}	7.2916×10^{-14}
40	n/a	n/a	1.0638×10^{-16}	3.4033×10^{-16}
80	n/a	n/a	2.9548×10^{-19}	1.7538×10^{-18}

Table 15: Estimates of the quadrature error $|\mathcal{I}(f) - \mathcal{G}_n(f)|$ in (35) determined with the rules $\mathcal{Q}_{2n+1}^L$, $\mathcal{Q}_{2n+1}^S$ and $\mathcal{Q}_{n+2}^t$, as well as the actual quadrature error. The rules $\mathcal{Q}_{2n+1}^L$ and $\mathcal{Q}_{2n+1}^S$ cannot be evaluated for large values of n.

6. Conclusion

This paper considers two modifications of Jacobi weight functions, namely

$$\tilde{w}_1(x) = (z-x)^{-1}(1-x)^a(1+x)^b \quad \text{and} \quad \tilde{w}_2(x) = (z-x)(1-x)^a(1+x)^b$$

for z real and of magnitude strictly larger than 1. We derived asymptotic properties of the recurrence coefficients for orthogonal polynomials associated with these weight functions. This allowed us to discuss conditions under which the averaged Gauss rules and optimal averaged Gauss rules are internal. When a or b are equal to $-\frac{1}{2}$, only one of these averaged rules is internal for large values of n. We observed that when z is close to the interval $[-1, 1]$, the value of n may have to be large to achieve internality for certain averaged quadrature rules. The estimates of quadrature errors determined with the aid of averaged Gauss rules and optimal averaged Gauss rules are very accurate, but these rules are not applicable for all integrands and weight functions. The truncated quadrature rules are applicable to more integrands and weight functions and deliver useful estimates of the quadrature error.

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