LAVRENTIEV-TYPE REGULARIZATION METHODS FOR HERMITIAN PROBLEMS*

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Abstract. Lavrentiev regularization is a popular approach to the solution of linear discrete ill-posed problems with a Hermitian positive semidefinite matrix. This paper describes Lavrentiev-type regularization methods that can be applied to the solution of linear discrete ill-posed problems with a general Hermitian matrix. Fractional Lavrentiev-type methods as well as modifications suggested by the solution of certain matrix nearness problems are described. Computed examples illustrate the competitiveness of modified fractional Lavrentiev-type methods.

Key words. Discrete ill-posed problem, matrix nearness problem, Lavrentiev regularization, fractional Lavrentiev regularization

1. Introduction.

We consider the solution of linear systems of equations

$$Ax = b, \qquad A \in \mathbb{C}^{n\times n}, \qquad x, b \in \mathbb{C}^n, \tag{1.1}$$

with a Hermitian matrix, whose eigenvalues "cluster" at the origin. In particular, the matrix A is severely ill-conditioned and may be singular. Matrices of this kind are said to be of ill-determined rank. Linear systems of equations (1.1) with a matrix of ill-determined rank are commonly referred to as linear discrete ill-posed problems. Such systems arise from the discretization of Hermitian linear ill-posed problems; see, e.g., [1] for discussions. Inconsistent systems (1.1) are treated as least-squares problems.

The right-hand side b of linear discrete ill-posed problems that arise in applications typically represents measured data and is contaminated by an error $e \in \mathbb{C}^n$, which we refer to as "noise." Thus,

$$b = \widehat{b} + e, \tag{1.2}$$

where $\widehat{b}$ denotes the unknown error-free vector associated with b. The unavailable error-free linear system of equations

$$Ax = \widehat{b} \tag{1.3}$$

is assumed to be consistent, and we denote its solution of minimal Euclidean norm by $\widehat{x}$. We would like to determine an accurate approximation of $\widehat{x}$.

Due to the error e in b and the severe ill-conditioning of A, the least-squares solution of minimal Euclidean norm of (1.1) typically does not furnish a meaningful approximation of $\widehat{x}$. We therefore replace (1.1) by a nearby linear system of equations, whose solution is less sensitive to the error e. This replacement is commonly referred to as regularization.

Lavrentiev regularization is a popular regularization method that can be applied when the Hermitian matrix A is positive semidefinite. This method replaces (1.1) by the system of equations

$$(A + \mu I)x = b, \tag{1.4}$$

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where I denotes the $n \times n$ identity matrix and $\mu \geq 0$ is a regularization parameter. This parameter determines how close the system (1.4) is to (1.1) and how sensitive the solution x_μ of (1.4) is to the error e in b. We refer to the matrix μI in (1.4) as the regularization matrix. Note that the matrix $A + \mu I$ has no eigenvalue smaller than μ.

Let $\mathcal{R}(A)$ denote the range of A and $P_{\mathcal{R}(A)}$ the orthogonal projector onto $\mathcal{R}(A)$. Hochstenbach et al. [7] proposed the fractional Lavrentiev regularization method

$$(A^{1+\alpha} + \mu I)x = A^\alpha P_{\mathcal{R}(A)} b, \tag{1.5}$$

where $0 \leq \alpha \leq 1$ is a user-specified parameter. The purpose of the projector $P_{\mathcal{R}(A)}$ is to make equation (1.5) consistent for all $\mu \geq 0$ when $\alpha = 0$. This value of α yields Lavrentiev regularization, while $\alpha = 1$ corresponds to Tikhonov regularization and allows A to be Hermitian indefinite. Numerical results reported in [7] show that the solution $x_{\mu,\alpha}$ of (1.5) typically is a more accurate approximation of $\widehat{x}$ for a suitable $0 < \alpha < 1$ than when $\alpha = 0$ or $\alpha = 1$. The fractional Lavrentiev regularization method described in [7] is related to fractional Tikhonov regularization, which is discussed in [8, 10].

This paper proposes to replace the regularization matrix μI in (1.5) by a matrix $M_{\mu,\alpha} \in \mathbb{C}^{n\times n}$, such that $\|M_{\mu,\alpha}\|_F < \|\mu I\|_F$ and $A^{1+\alpha} + M_{\mu,\alpha}$ has no eigenvalue smaller than μ. Here $\|\cdot\|_F$ denotes the Frobenius matrix norm. The matrix $M_{\mu,\alpha}$ is the solution of a matrix nearness problem, which is described in Section 2. We refer to methods obtained by replacing μI in (1.5) by some matrix $M_{\mu,\alpha}$ as modified fractional Lavrentiev methods. Section 2 describes two such methods.

Section 3 considers the situation when the matrix A in (1.1) is a general Hermitian matrix. We present two modified fractional Lavrentiev-type regularization methods that can be applied in this situation. Numerical examples reported in Section 4 illustrate the performance of the methods discussed and Section 5 contains concluding remarks.

In addition to the Frobenius norm, we also will use the Euclidean vector norm and the associated induced matrix norm. They are both denoted by $\|\cdot\|$.

2. Hermitian positive semidefinite problems. Throughout this section the matrix $A \in \mathbb{C}^{n\times n}$ in (1.1) is assumed to be Hermitian and positive semidefinite. We first describe a matrix nearness problem and then discuss its application to the construction of the regularization matrix $M_{\mu,\alpha}$ to be used in (1.5). Let $A \in \mathbb{C}^{n\times n}$ have the spectral factorization

$$A = V\Lambda V^*, \qquad \Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n] \in \mathbb{R}^{n\times n}, \tag{2.1}$$

where the matrix $V = [v_1, v_2, \ldots, v_n] \in \mathbb{C}^{n\times n}$ is unitary and the eigenvalues are ordered according to

$$\lambda_1 \geq \ldots \geq \lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0. \tag{2.2}$$

The superscript * denotes transposition and complex conjugation, and r is the rank of A.

PROPOSITION 2.1. *Let the nontrivial entries of the matrix*

$$\Gamma = \mathrm{diag}[\gamma_1, \gamma_2, \ldots, \gamma_n] \in \mathbb{R}^{n\times n}$$

be ordered according to $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$ and let $\mathcal{M}$ denote the subset of Hermitian matrices of $\mathbb{C}^{n\times n}$. Let $\alpha \geq 0$ be a parameter and denote the spectrum of $A^{1+\alpha} + M$

by $\lambda(A^{1+\alpha}+M)$, where $M \in \mathcal{M}$ and the eigenvalues of A are ordered according to (2.2). Then the solution of the matrix nearness problem

$$\min_{M\in\mathcal{M}} \{\|M\|_F \,:\, \lambda(A^{1+\alpha}+M) = \{\gamma_j\}_{j=1}^n\} \tag{2.3}$$

is furnished by the matrix

$$M = V(\Gamma - \Lambda^{1+\alpha})V^*. \tag{2.4}$$

Proof. The proposition is a consequence of the Wielandt–Hoffman theorem; see, e.g., [12, pp. 104–108]. By this theorem, any Hermitian matrix B such that $A^{1+\alpha}+B$ has spectral factorization $W\Gamma W^*$, with W a unitary matrix, satisfies

$$\|B\|_F^2 \geq \sum_{j=1}^n (\gamma_j - \lambda_j^{1+\alpha})^2;$$

see [12, eq. (48.1), p. 104]. □

We turn to the application of the minimization problem (2.3) to the determination of a new regularization matrix for a fractional Lavrentiev-type method.

COROLLARY 2.2. *Define the regularization matrix*

$$M_{\mu,\alpha} = V D_{\mu,\alpha} V^*$$

with

$$D_{\mu,\alpha} = -\operatorname{diag}[\min\{\lambda_1^{1+\alpha}-\mu,0\}, \min\{\lambda_2^{1+\alpha}-\mu,0\}, \ldots, \min\{\lambda_n^{1+\alpha}-\mu,0\}].$$

Then $A^{1+\alpha}+M_{\mu,\alpha}$ is the closest matrix to $A^{1+\alpha}$ in the Frobenius norm such that the smallest eigenvalue of $A^{1+\alpha}+M_{\mu,\alpha}$ is larger than or equal to μ.

Proof. The result follows from Proposition 2.1. □

The above corollary suggests that $M_{\mu,\alpha}$ be applied as a regularization matrix instead of μI. Thus, we propose to solve

$$(A^{1+\alpha}+M_{\mu,\alpha})x = A^\alpha P_{\mathcal{R}(A)} b \tag{2.5}$$

instead of (1.5). We refer to this regularization method as modified fractional Lavrentiev regularization and denote the solution by $\widetilde{x}_{\mu,\alpha}$. An analogue of this method for Tikhonov regularization is described in [2].

The regularization properties of the fractional and modified fractional Lavrentiev methods can be investigated with the aid of filter factors. Let $\mu > 0$. Then the solution of (1.5) can be expressed as

$$x_{\mu,\alpha} = \sum_{j=1}^r \phi_j^{(\mu,\alpha)} \frac{v_j^*(P_{\mathcal{R}(A)}b)}{\lambda_j} v_j,$$

with the filter factors

$$\phi_j^{(\mu,\alpha)} = \frac{\lambda_j^{1+\alpha}}{\lambda_j^{1+\alpha}+\mu}. \tag{2.6}$$

Similarly, the solution of (2.5) can be written as

$$\widetilde{x}_{\mu,\alpha} = \sum_{j=1}^{r} \widetilde{\phi}_j^{(\mu,\alpha)} \frac{v_j^*(P_{\mathcal{R}(A)}b)}{\lambda_j} v_j$$

with the filter factors

$$\widetilde{\phi}_j^{(\mu,\alpha)} = \begin{cases} 1, & 1 \le j \le k_\alpha(\mu), \\ \frac{\lambda_j^{1+\alpha}}{\mu}, & k_\alpha(\mu) < j \le r, \end{cases} \tag{2.7}$$

where $k_\alpha(\mu)$ denotes the number of eigenvalues of $A^{1+\alpha}$ larger than or equal to μ. Clearly, $\widetilde{\phi}_j^{(\mu,\alpha)} \ge \phi_j^{(\mu,\alpha)}$ for $1 \le j \le r$.

Another modified fractional Lavrentiev method can be defined by choosing the nontrivial entries of $\Gamma = \operatorname{diag}[\gamma_1, \gamma_2, \ldots, \gamma_n] \in \mathbb{R}^{n\times n}$ according to

$$\gamma_j = \begin{cases} \lambda_j^{1+\alpha}, & \lambda_j \ge \mu, \\ \mu + \lambda_j(\lambda_j^\alpha - 1), & \lambda_j < \mu. \end{cases} \tag{2.8}$$

Substituting Γ into (2.4) yields the regularization matrix

$$\breve{M}_{\mu,\alpha} = V \breve{D}_{\mu,\alpha} V^*$$

with

$$\breve{D}_{\mu,\alpha} = -\operatorname{diag}[\min\{\lambda_1 - \mu, 0\}, \min\{\lambda_2 - \mu, 0\}, \ldots, \min\{\lambda_n - \mu, 0\}].$$

The matrix $\breve{M}_{\mu,\alpha}$ defines the modified fractional Lavrentiev regularization method

$$(A^{1+\alpha} + \breve{M}_{\mu,\alpha})x = A^\alpha P_{\mathcal{R}(A)} b, \tag{2.9}$$

whose solution $\breve{x}_{\mu,\alpha}$ can be written as

$$\breve{x}_{\mu,\alpha} = \sum_{j=1}^{r} \breve{\phi}_j^{(\mu,\alpha)} \frac{v_j^*(P_{\mathcal{R}(A)}b)}{\lambda_j} v_j$$

with the filter factors

$$\breve{\phi}_j^{(\mu,\alpha)} = \begin{cases} 1, & 1 \le j \le \breve{k}(\mu), \\ \frac{\lambda_j^{1+\alpha}}{\mu + \lambda_j(\lambda_j^\alpha - 1)}, & \breve{k}(\mu) < j \le r, \end{cases} \tag{2.10}$$

where $\breve{k}(\mu)$ denotes the number of eigenvalues of A larger than or equal to μ.

In the computed examples of Section 4, we scale A so that $\|A\| \le 1$. Then the following relations between the filter factors hold.

PROPOSITION 2.3. *Let the Hermitian positive semidefinite matrix A be scaled so that $\|A\| \le 1$. Then for $0 \le \alpha \le 1$ and $\mu > 0$, we have*

$$\breve{\phi}_j^{(\mu,\alpha)} \ge \widetilde{\phi}_j^{(\mu,\alpha)} \ge \phi_j^{(\mu,\alpha)}, \qquad 1 \le j \le r.$$

Moreover, $\breve{\phi}_j^{(\mu,0)} = \widetilde{\phi}_j^{(\mu,0)}$.

Proof. The inequalities follow from the observation that $0 \le k_\alpha(\mu) \le \breve{k}(\mu) \le r$. ∎

3. Hermitian indefinite problems. This section allows A in (1.1) to be Hermitian indefinite. Consider the spectral factorization (2.1) and let the eigenvalues be ordered according to

$$|\lambda_1| \geq \ldots \geq |\lambda_r| > |\lambda_{r+1}| = \ldots = |\lambda_n| = 0. \tag{3.1}$$

Introduce the diagonal matrix

$$\Sigma = \mathrm{diag}[\mathrm{sign}(\lambda_1), \mathrm{sign}(\lambda_2), \ldots, \mathrm{sign}(\lambda_n)],$$

where

$$\mathrm{sign}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

Then $\Sigma\Lambda = |\Lambda| \geq 0$. Letting $y = V^*x$, we can transform (1.1) into the consistent linear system of equations

$$\Sigma\Lambda y = \Sigma V^* b$$

with a Hermitian positive semidefinite matrix. This system can be regularized by the methods of Section 2. Using the original vector x and matrices $\widetilde{A}^\beta = V\Sigma|\Lambda|^\beta V^*$ for $\beta \geq 0$, we are lead to the following fractional Lavrentiev-type regularization method,

$$(\widetilde{A}^{1+\alpha} + \mu V\Sigma V^*)x = \widetilde{A}^\alpha V\Sigma V^* b, \tag{3.2}$$

which is analogous to the method (1.5). Notice that $\widetilde{A}^1 = A$.

We turn to the determination of the regularization matrix for a modified fractional Lavrentiev-type regularization method associated with the method (3.2).

PROPOSITION 3.1. *Let the entries γ_j of the matrix $\Gamma = \mathrm{diag}[\gamma_1, \gamma_2, \ldots, \gamma_n] \in \mathbb{R}^{n\times n}$ be ordered similarly as in Proposition 2.1, let $\mathcal{M}$ denote the subset of Hermitian matrices of $\mathbb{C}^{n\times n}$, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A ordered according to (3.1). Then the solution of the matrix nearness problem*

$$\min_{M\in\mathcal{M}} \{\|M\|_F \,:\, \lambda(\widetilde{A}^{1+\alpha} + M) = \{\mathrm{sign}(\lambda_j)\gamma_j\}_{j=1}^n\}$$

is furnished by the matrix

$$M = V\Sigma(\Gamma - |\Lambda|^{1+\alpha})V^*. \tag{3.3}$$

Proof. The matrix $\widetilde{A}^{1+\alpha}$ has the eigenvalues $\{\mathrm{sign}(\lambda_j)|\lambda_j|^{1+\alpha}\}_{j=1}^n$. Any Hermitian matrix B such that $\widetilde{A}^{1+\alpha} + B$ has spectral factorization $W\Sigma\Gamma W^*$, with W a unitary matrix, satisfies

$$\|B\|_F^2 \geq \sum_{j=1}^r (\mathrm{sign}(\lambda_j)\gamma_j - \mathrm{sign}(\lambda_j)|\lambda_j|^{1+\alpha})^2;$$

see [12, eq. (48.1), p. 104]. The terms in the above sum are the squares of the eigenvalues of the matrix (3.3). □

COROLLARY 3.2. *Define the regularization matrix*

$$M_{\mu,\alpha,\Sigma} = V\Sigma D_{\mu,\alpha,\Sigma}V^*$$

with

$$D_{\mu,\alpha,\Sigma} = -\operatorname{diag}[\min\{|\lambda_1|^{1+\alpha}-\mu,0\},\min\{|\lambda_2|^{1+\alpha}-\mu,0\},\ldots,\min\{|\lambda_n|^{1+\alpha}-\mu,0\}].$$

Then $\widetilde{A}^{1+\alpha}+M_{\mu,\alpha,\Sigma}$ *is the closest matrix to* $\widetilde{A}^{1+\alpha}$ *in the Frobenius norm such that the smallest magnitude of the eigenvalues of* $\widetilde{A}^{1+\alpha}+M_{\mu,\alpha,\Sigma}$ *is larger than or equal to* μ.

Proof. The result follows from Proposition 3.1. □

The above corollary suggests that we solve

$$(\widetilde{A}^{1+\alpha}+M_{\mu,\alpha,\Sigma})x = \widetilde{A}^{\alpha}V\Sigma V^*b \tag{3.4}$$

instead of (3.2). We note that the solution of (3.2) can be expressed as

$$x_{\mu,\alpha,\Sigma} = \sum_{j=1}^{r}\phi_j^{(\mu,\alpha,\Sigma)}\frac{v_j^*b}{\lambda_j}v_j,$$

with the filter factors

$$\phi_j^{(\mu,\alpha,\Sigma)} = \frac{|\lambda_j|^{1+\alpha}}{|\lambda_j|^{1+\alpha}+\mu}, \tag{3.5}$$

while the solution of (3.4) can be written as

$$\widetilde{x}_{\mu,\alpha,\Sigma} = \sum_{j=1}^{r}\widetilde{\phi}_j^{(\mu,\alpha,\Sigma)}\frac{v_j^*b}{\lambda_j}v_j$$

with the filter factors

$$\widetilde{\phi}_j^{(\mu,\alpha,\Sigma)} = \begin{cases} 1, & 1\le j\le k_\alpha(\mu), \\ \frac{|\lambda_j|^{1+\alpha}}{\mu}, & k_\alpha(\mu)<j\le r, \end{cases} \tag{3.6}$$

where $k_\alpha(\mu)$ denotes the number of eigenvalues of $\widetilde{A}^{1+\alpha}$ of absolute value larger than or equal to μ. We have $\widetilde{\phi}_j^{(\mu,\alpha,\Sigma)} \ge \phi_j^{(\mu,\alpha,\Sigma)}$. This method is an extension of the method (2.5).

We now describe a regularization method that is analogous to (2.9). Introduce the regularization matrix

$$\breve{M}_{\mu,\alpha,\Sigma} = V\Sigma\breve{D}_{\mu,\alpha,\Sigma}V^*$$

with

$$\breve{D}_{\mu,\alpha,\Sigma} = -\operatorname{diag}[\min\{|\lambda_1|-\mu,0\},\min\{|\lambda_2|-\mu,0\},\ldots,\min\{|\lambda_n|-\mu,0\}].$$

We obtain the modified fractional Lavrentiev method

$$(\widetilde{A}^{1+\alpha}+\breve{M}_{\mu,\alpha,\Sigma})x = \widetilde{A}^{\alpha}V\Sigma V^*b, \tag{3.7}$$

whose solution $\breve{x}_{\mu,\alpha,\Sigma}$ can be expressed as

$$\breve{x}_{\mu,\alpha,\Sigma} = \sum_{j=1}^{r} \breve{\phi}_j^{(\mu,\alpha,\Sigma)} \frac{v_j^* b}{\lambda_j} v_j$$

with the filter factors

$$\breve{\phi}_j^{(\mu,\alpha,\Sigma)} = \begin{cases} 1, & 1 \le j \le \breve{k}(\mu), \\ \frac{|\lambda_j|^{1+\alpha}}{\mu+|\lambda_j|(|\lambda_j|^{\alpha}-1)}, & \breve{k}(\mu) < j \le r, \end{cases} \tag{3.8}$$

where $\breve{k}(\mu)$ denotes the number of eigenvalues of $\widetilde{A}$ of absolute value larger than or equal to μ. The following result is analogous to Proposition 2.3 and can be shown in the same manner.

PROPOSITION 3.3. *Let the Hermitian matrix A be scaled so that $\|A\| \le 1$. Then for $0 \le \alpha \le 1$ and $\mu > 0$, we have*

$$\breve{\phi}_j^{(\mu,\alpha,\Sigma)} \ge \widetilde{\phi}_j^{(\mu,\alpha,\Sigma)} \ge \phi_j^{(\mu,\alpha,\Sigma)}, \qquad 1 \le j \le r.$$

Moreover, $\breve{\phi}_j^{(\mu,0,\Sigma)} = \widetilde{\phi}_j^{(\mu,0,\Sigma)}$.

4. Computed examples. We would like to determine an accurate approximation of the minimal-norm solution $\widehat{x}$ of the consistent linear system of equations (1.3) with the unknown error-free right-hand side $\widehat{b}$. This section compares the quality of the approximations x_μ, $x_{\mu,\alpha}$, $\widetilde{x}_{\mu,\alpha}$, and $\breve{x}_{\mu,\alpha}$ of $\widehat{x}$ determined by Lavrentiev's method, i.e., by (1.5) with $\alpha = 0$, the fractional Lavrentiev method (1.5) with $0 < \alpha \le 1$, and the modified fractional Lavrentiev methods (2.5) and (2.9) with $0 \le \alpha \le 1$, respectively, applied to systems of equations (1.1) with a Hermitian positive semidefinite matrix. We also report the quality of the approximations $x_{\mu,\alpha,\Sigma}$, $\widetilde{x}_{\mu,\alpha,\Sigma}$, and $\breve{x}_{\mu,\alpha,\Sigma}$ obtained with the fractional Lavrentiev-type (3.2) and modified fractional Lavrentiev-type (3.4) and (3.7) regularization methods, respectively, for linear systems (1.1) with a Hermitian indefinite matrix. The dominant computational effort for all methods of this paper is the evaluation of the spectral factorization of A.

Large-scale problems, for which it is too expensive to compute the spectral factorization of A, can be handled by first reducing A by carrying out a few steps of the symmetric Lanczos process with initial vector $b/\|b\|$ and then applying a modified fractional Lavrentiev-type method of the present paper to the reduced problem so obtained. The feasibility of this approach for fractional Tikhonov and Lavrentiev methods is illustrated in [7, 8] and, therefore, will not be discussed further in this paper.

The regularization parameter μ determines how sensitive the computed approximations of $\widehat{x}$ are to the error e in b and to round-off errors introduced during the computations; the larger μ, the less sensitive are the computed solutions to these errors. However, a large value of μ implies that the matrix of the regularized problem solved differs considerably from the matrix A in (1.1), and this may cause the computed solution to deviate significantly from the desired solution $\widehat{x}$. In the computed examples of the present section, we determine μ with the aid of the discrepancy principle, which requires that a bound ε for the relative error in b be available. This error is referred to as the noise level. Thus,

$$\frac{\|e\|}{\|\widehat{b}\|} \le \varepsilon.$$

Groetsch and Guacamene [4] propose that $\mu > 0$ be chosen so that the solution x_μ of (1.4) satisfies

$$\|b - Ax_\mu\| = \varepsilon \frac{\|\widehat{b}\|}{\sqrt{\mu}}, \tag{4.1}$$

and show that $\mu \to \widehat{x}$ as $\varepsilon \searrow 0$. A different approach to determining the regularization parameter is discussed by Morigi et al. [11]. Let $0 < s < 1$ be fixed. The parameter $\mu > 0$ then is chosen so that the solution x_μ of (1.4) satisfies

$$\|b - Ax_\mu\| = (\varepsilon\|\widehat{b}\|)^s. \tag{4.2}$$

It is shown in [11] that $x_\mu \to \widehat{x}$ as $\varepsilon \searrow 0$ for this choice of μ. Groetsch and Guacamene [4] prove that x_μ is not guaranteed to converge to $\widehat{x}$ as $\varepsilon \searrow 0$ when $s = 1$.

We determine the regularization parameter μ for the fractional and modified Lavrentiev-type methods of Sections 2 and 3 using (4.1) and (4.2). All computations are carried out in MATLAB with about 15 significant decimal digits.

Example 4.1. Consider the integral equation of the first kind

$$\int_0^1 \kappa(u,v)\, x(v)\, dv = g(u), \qquad 0 \le u \le 1, \tag{4.3}$$

where the kernel and right-hand side function g are given by

$$\kappa(u,v) = \begin{cases} u\,(v-1), & u < v, \\ v\,(u-1), & u \ge v, \end{cases} \qquad g(u) = \frac{u^3 - u}{6}.$$

The discretization of (4.3) computed by the MATLAB function deriv2 from [6] determines a symmetric negative definite matrix $B \in \mathbb{R}^{500\times 500}$ and a scaled discrete approximation $\widehat{x} \in \mathbb{R}^{500}$ of the solution $x(v) = v$. We apply Lavrentiev regularization to the system (1.1) with the matrix

$$A = -\frac{1}{2}\frac{B}{\|B\|}.$$

Let $\widehat{b} = A\widehat{x}$. An error vector e with normally distributed random entries with zero mean, scaled to correspond to a specified noise level $\varepsilon = \|e\|/\|\widehat{b}\|$, is added to $\widehat{b}$ to yield the vector b; cf. (1.2).

Table 4.1 shows relative errors for approximations of $\widehat{x}$ computed with the methods of Sections 1 and 2 for the noise levels 10^{-2}, 10^{-3}, and 10^{-4} using (4.2) for the determination of the regularization parameter μ. For the fractional methods, $\mu = \mu(\alpha)$ depends on α and $x_{\mu,\alpha}$ depends on both μ and α. We compute for the fractional methods for each $s \in \{0.1, 0.2, \ldots, 0.9, 1\}$ a value of the coefficient $\alpha \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ that gives an approximate solution $x_{\mu,\alpha}$ with the smallest relative error. In our examples, the coefficient α with this property is unique and is denoted by α^*. Table 4.1 displays the error in the computed solution obtained for $\alpha = \alpha^*$. For instance, $\alpha^*_{\mathrm{fL}} = 0$ indicates that the best result with the fractional Lavrentiev method (1.5) is obtained for $\alpha = 0$. This method is the standard Lavrentiev method with the right-hand side projected into $\mathcal{R}(A)$; it is defined by (1.5) with $\alpha = 0$. The projection improves the quality of the computed solution. Table 4.1 shows the modified fractional Lavrentiev methods (2.5) and (2.9) to determine the

ε	s	L	mL	fL	$\alpha^*_{\rm fL}$	mfL1	α^*_1	mfL2	α^*_2
10^{-2}	0.1	0.7868	0.7048	0.7868	0	0.7048	0	0.7048	0
	0.2	0.7868	0.7048	0.7868	0	0.7048	0	0.7048	0
	0.3	0.7868	0.7048	0.7868	0	0.7048	0	0.6534	1
	0.4	0.6084	0.5056	0.6084	0	0.5056	0	0.5056	0
	0.5	0.4893	0.4014	0.4893	0	0.4014	0	0.4014	0
	0.6	0.4092	0.3383	0.4092	0	0.3383	0	0.3383	0
	0.7	0.3628	0.3080	0.3623	0.1	0.3064	0.1	0.2794	1
	0.8	0.3880	0.3547	0.3196	0.1	0.2752	0.1	0.2359	0.9
	0.9	0.6164	0.6091	0.2860	0.2	0.2507	0.2	0.2245	0.3
	1	3.1287	3.2180	0.2282	0.7	<u>0.2225</u>	0.7	0.8911	0.3
10^{-3}	0.1	0.7868	0.7047	0.7868	0	0.7047	0	0.7047	0
	0.2	0.7868	0.7047	0.7868	0	0.7047	0	0.7047	0
	0.3	0.5889	0.4907	0.5889	0	0.4907	0	0.4870	1
	0.4	0.4426	0.3658	0.4426	0	0.3658	0	0.3658	0
	0.5	0.3467	0.2855	0.3467	0	0.2855	0	0.2825	1
	0.6	0.2769	0.2285	0.2769	0	0.2285	0	0.2045	1
	0.7	0.2309	0.1943	0.2309	0	0.1943	0	0.1572	1
	0.8	0.2446	0.2263	0.2061	0.1	0.1776	0.1	0.1549	0.6
	0.9	0.4344	0.4398	0.1858	0.2	0.1658	0.2	<u>0.1534</u>	0.3
	1	1.7520	1.8983	0.1666	0.4	0.1551	0.4	0.8029	1
10^{-4}	0.1	0.7868	0.7047	0.7868	0	0.7047	0	0.7047	0
	0.2	0.6919	0.5547	0.6916	0	0.5547	0	0.5547	0
	0.3	0.4636	0.3828	0.4636	0	0.3828	0	0.3828	0
	0.4	0.3388	0.2793	0.3388	0	0.2793	0	0.2782	1
	0.5	0.2537	0.2090	0.2537	0	0.2090	0	0.1809	1
	0.6	0.1923	0.1585	0.1923	0	0.1585	0	0.1227	1
	0.7	0.1497	0.1250	0.1497	0	0.1250	0	0.1106	0.6
	0.8	0.1590	0.1527	0.1342	0.1	0.1149	0.1	<u>0.1012</u>	0.2
	0.9	0.3422	0.3803	0.1247	0.2	0.1088	0.2	0.1215	0.4
	1	0.3422	0.3803	0.1247	0.2	0.1088	0.2	0.1215	0.4

TABLE 4.1

Example 4.1: Relative error in the computed solutions of a discretization of the integral equation (4.3). L stands for the Lavrentiev method (1.5) with $\alpha = 0$, mL for the modified Lavrentiev method (2.5) with $\alpha = 0$, fL for the fractional Lavrentiev method (1.5), mfL1 for the modified fractional Lavrentiev method (2.5), and mfL2 for the modified fractional Lavrentiev method (2.9). The regularization parameter is determined by (4.2) for each s and α. The smallest relative error for each noise level is underlined.

best approximations of $\widehat{x}$ for all noise levels; the smallest error for each noise level is underlined.

The large errors obtained with the standard Lavrentiev method for some noise levels when s is close to one are not surprising in view of the theory presented in [4]. Table 4.1 shows that the modified fractional Lavrentiev methods may perform well for $0.8 \leq s \leq 1$.

Table 4.2 reports, for the same noise levels as in Table 4.1, relative errors in the approximations of $\widehat{x}$ computed by Lavrentiev, fractional Lavrentiev, and modified fractional Lavrentiev regularization when the regularization parameter μ is determined by (4.1) with $\alpha \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$. Similarly as in Table 4.1, $\mu = \mu(\alpha)$ depends on α and $x_{\mu,\alpha}$ on both μ and α. The results for $\alpha = 1$ in the column "fL" correspond to (standard) Tikhonov regularization for the situation of a Hermitian matrix A. Similarly, the results for $\alpha = 1$ reported in the column labeled "mfL1"

ε	α	μ	fL	mfL1	mfL2
10^{-2}	0	$2.79 \cdot 10^{-2}$	0.3616	0.3080	0.3080
	0.1	$2.61 \cdot 10^{-2}$	0.3618	0.3059	<u>0.2937</u>
	0.2	$2.45 \cdot 10^{-2}$	0.3810	0.3281	0.3015
	0.3	$2.28 \cdot 10^{-2}$	0.3998	0.3484	0.3084
	0.4	$2.13 \cdot 10^{-2}$	0.4174	0.3678	0.3128
	0.5	$1.98 \cdot 10^{-2}$	0.4332	0.3875	0.3153
	0.6	$1.84 \cdot 10^{-2}$	0.4476	0.4054	0.3184
	0.7	$1.71 \cdot 10^{-2}$	0.4609	0.4209	0.3203
	0.8	$1.59 \cdot 10^{-2}$	0.4733	0.4340	0.3209
	0.9	$1.47 \cdot 10^{-2}$	0.4844	0.4446	0.3181
	1	$1.37 \cdot 10^{-2}$	0.4953	0.4535	0.3041
10^{-3}	0	$5.60 \cdot 10^{-3}$	0.2333	0.1957	0.1957
	0.1	$5.20 \cdot 10^{-3}$	0.2492	0.2102	0.1978
	0.2	$4.70 \cdot 10^{-3}$	0.2696	0.2315	0.2049
	0.3	$4.20 \cdot 10^{-3}$	0.2885	0.2516	0.2089
	0.4	$3.80 \cdot 10^{-3}$	0.3066	0.2711	0.2118
	0.5	$3.40 \cdot 10^{-3}$	0.3228	0.2881	0.2121
	0.6	$3.00 \cdot 10^{-3}$	0.3371	0.3040	0.2093
	0.7	$2.60 \cdot 10^{-3}$	0.3493	0.3171	0.2056
	0.8	$2.30 \cdot 10^{-3}$	0.3618	0.3298	0.2017
	0.9	$2.00 \cdot 10^{-3}$	0.3725	0.3425	0.1963
	1	$1.80 \cdot 10^{-3}$	0.3841	0.3560	<u>0.1917</u>
10^{-4}	0	$1.20 \cdot 10^{-3}$	0.1551	0.1289	0.1289
	0.1	$1.10 \cdot 10^{-3}$	0.1747	0.1475	0.1362
	0.2	$9.00 \cdot 10^{-4}$	0.1910	0.1641	0.1395
	0.3	$8.00 \cdot 10^{-4}$	0.2097	0.1829	0.1426
	0.4	$7.00 \cdot 10^{-4}$	0.2266	0.2002	0.1429
	0.5	$6.00 \cdot 10^{-4}$	0.2417	0.2158	0.1406
	0.6	$5.00 \cdot 10^{-4}$	0.2547	0.2294	0.1367
	0.7	$4.00 \cdot 10^{-4}$	0.2652	0.2408	0.1302
	0.8	$3.00 \cdot 10^{-4}$	0.2726	0.2490	0.1223
	0.9	$3.00 \cdot 10^{-4}$	0.2902	0.2666	0.1226
	1	$2.00 \cdot 10^{-4}$	0.2918	0.2694	<u>0.1119</u>

TABLE 4.2

Example 4.1: Relative error in the computed solutions of a discretization of the integral equation (4.3). fL stands for the fractional Lavrentiev method (1.5), mfL1 for the modified fractional Lavrentiev method (2.5), and mfL2 for the modified fractional Lavrentiev method (2.9). The regularization parameter μ is determined by (4.1). The smallest relative error for each noise level is underlined.

correspond to the modified Tikhonov regularization method described in [2] in the special case when A is Hermitian. Table 4.2 shows the modified fractional Lavrentiev method mfL2 to determine the best approximations of $\widehat{x}$ for all noise levels and all choices of α. The smallest error for each noise level is underlined. However, these approximations are not as accurate as the best approximations reported in Table 4.1.

Table 4.3 displays the performance of the methods (1.5), (2.5), and (2.9) when the fractional parameter α is set to unity. Then the methods (1.5) and (2.5) simplify to the standard and modified Tikhonov regularization methods, respectively; see [2] for the latter. The regularization parameter is determined by (4.2) with $s = 1$ (and $\alpha = 1$). This is the natural way of determining the regularization parameter for Tikhonov regularization. It is interesting to compare this table to the entries of Table

ε	μ	fL	mfL1	mfL2
10^{-2}	$1.00 \cdot 10^{-4}$	0.2691	0.2481	7.9761
10^{-3}	$1.00 \cdot 10^{-4}$	0.2676	0.2469	0.8029
10^{-4}	$1.00 \cdot 10^{-4}$	0.2676	0.2472	0.1218

TABLE 4.3

Example 4.1: Relative error in the computed solutions of a discretization of the integral equation (4.3). fL stands for the fractional Lavrentiev method (1.5), mfL1 for the modified fractional Lavrentiev method (2.5), and mfL2 for the modified fractional Lavrentiev method (2.9). The fractional parameter α is set to one and the regularization parameter μ is determined by (4.2) with $s = 1$.

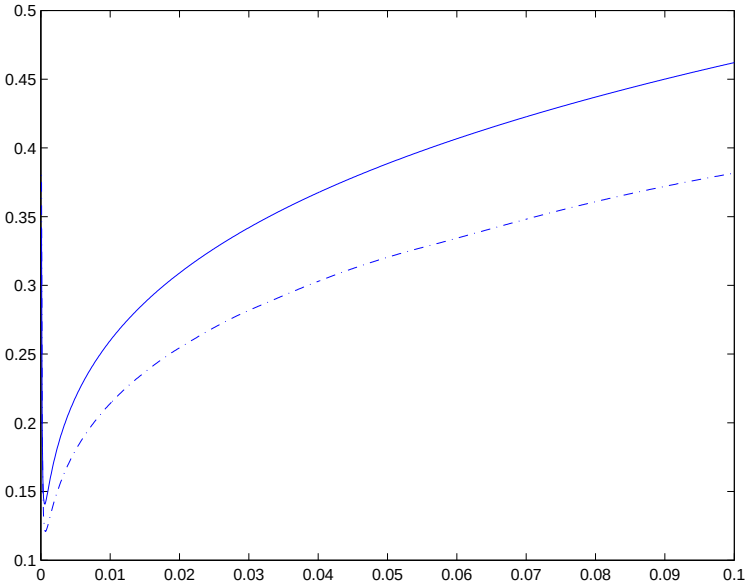

FIG. 4.1. *Example 4.1: Relative errors $\|x_{\mu,0} - \hat{x}\|/\|\hat{x}\|$ (solid curve) and $\|\widetilde{x}_{\mu,0} - \hat{x}\|/\|\hat{x}\|$ (dash-dotted curve) as functions of $0 < \mu < 0.1$. The noise level is $\varepsilon = 10^{-4}$.*

4.1 in rows for $s = 1$. The errors for the methods (1.5), (2.5), and (2.9) in Table 4.1 are generally smaller than the corresponding entries for the same noise levels in Table 4.3. This illustrates that letting α be strictly smaller than one and letting $\mu = \mu(\alpha)$ be a function of α, typically yields approximations $x_{\mu,\alpha}$ of $\widehat{x}$ of higher quality than fixing $s = 1$ and $\alpha = 1$.

Figure 4.1 displays, for increasing values of $0 < \mu < 0.1$ and $\varepsilon = 10^{-4}$, the relative errors of the computed approximate solutions $x_{\mu,0}$ and $\widetilde{x}_{\mu,0}$ determined by the Lavrentiev method (1.5) with $\alpha = 0$ and the modified Lavrentiev (2.5) method with $\alpha = 0$. The latter method is seen to give the smallest error for all values of μ. We remark that the method defined by (2.9) agrees with the method (2.5) when $\alpha = 0$. Finally, we show in Figure 4.2, for the same values of μ and ε, the relative errors of the computed approximate solutions $x_{\mu,1}$, $\widetilde{x}_{\mu,1}$, and $\breve{x}_{\mu,1}$ determined by the standard Tikhonov method (1.5) with $\alpha = 1$, the modified Tikhonov method (2.5) with $\alpha = 1$, and the modified Tikhonov method (2.9) with $\alpha = 1$. The method (2.9)

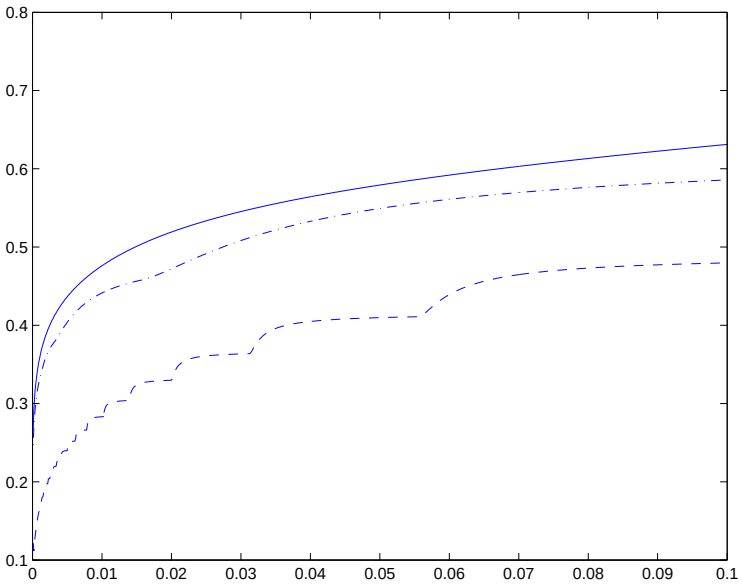

FIG. 4.2. *Example 4.1: Relative errors* $\|x_{\mu,1} - \hat{x}\|/\|\hat{x}\|$ *(solid curve),* $\|\widetilde{x}_{\mu,1} - \hat{x}\|/\|\hat{x}\|$ *(dash-dotted curve) and* $\|\breve{x}_{\mu,1} - \hat{x}\|/\|\hat{x}\|$ *(dashed curve) as functions of* $0 < \mu < 0.1$. *The noise level is* $\varepsilon = 10^{-4}$.

is seen to give the smallest error for all values of μ. □

Example 4.2. Consider the integral equation of the first kind

$$\int_{-6}^{6} \kappa(u,v)\, x(v)\, dv = g(u), \qquad -6 \le u \le 6, \tag{4.4}$$

with kernel κ and right-hand side g given by

$$\begin{aligned} \kappa(u,v) &= \phi(u-v), \\ g(u) &= (6-|u|)(1+\frac{1}{2}\cos(\frac{\pi u}{3})) + \frac{9}{2\pi}\sin(\frac{\pi|u|}{3}), \end{aligned}$$

where

$$\phi(y) = \begin{cases} 1+\cos(\frac{\pi y}{3}), & |y| < 3, \\ 0, & |y| \ge 3. \end{cases}$$

The discretization of (4.4) determined by the function phillips from [6] gives a symmetric indefinite matrix $A \in \mathbb{R}^{500\times 500}$ and a scaled discrete approximation $\widehat{x} \in \mathbb{R}^{500}$ of the solution $x(v) = \phi(v)$. We scale A so that $\|A\| = 0.5$. Let $\widehat{b} = A\widehat{x}$. An error vector e with normally distributed random entries with zero mean, scaled to correspond to a specified noise level $\varepsilon = \|e\|/\|\widehat{b}\|$ is added to $\widehat{b}$ to yield the vector b.

Table 4.4 is analogous to Table 4.1 and displays for three noise levels the relative errors in the approximate solutions $x_{\mu,0,\Sigma}$, $\widetilde{x}_{\mu,0,\Sigma}$, $x_{\mu,\alpha,\Sigma}$, $\widetilde{x}_{\mu,\alpha,\Sigma}$, and $\breve{x}_{\mu,\alpha,\Sigma}$ computed with the Lavrentiev-type methods (3.2) with $\alpha = 0$, (3.4) with $\alpha = 0$, (3.2), (3.4), and

ε	s	L	mL	fL	$\alpha^*_{\rm fL}$	mfL1	α^*_1	mfL2	α^*_2
10^{-2}	0.1	0.5215	0.2032	0.5215	0	0.2032	0	0.1044	1
	0.2	0.3508	0.0738	0.3508	0	0.0738	0	0.0738	0
	0.3	0.2378	0.0560	0.2378	0	0.0532	0.1	0.0254	0.6
	0.4	0.1693	0.0670	0.1616	0.1	0.0291	0.1	0.0247	0.3
	0.5	0.1470	0.1056	0.1117	0.1	0.0244	0.3	0.0261	0.3
	0.6	0.1800	0.1676	0.0769	0.2	0.0243	0.4	0.0179	1
	0.7	0.2726	0.2703	0.0541	0.3	0.0243	0.5	<u>0.0162</u>	0.6
	0.8	0.4464	0.4481	0.0390	0.4	0.0245	0.6	0.0307	0.8
	0.9	0.8131	0.8192	0.0291	0.5	0.0244	0.6	0.0764	0.3
	1	3.8767	3.9177	0.0215	1	0.0212	1	0.4582	0.4
10^{-3}	0.1	0.4222	0.1193	0.4222	0	0.1193	0	0.0923	0.4
	0.2	0.2298	0.0429	0.2298	0	0.0429	0	0.0245	0.5
	0.3	0.1253	0.0220	0.1253	0	0.0219	0.1	0.0216	0.1
	0.4	0.0703	0.0234	0.0703	0	0.0192	0.1	0.0126	1
	0.5	0.0510	0.0357	0.0408	0.1	0.0159	0.1	0.0084	0.4
	0.6	0.0715	0.0691	0.0258	0.2	0.0113	0.2	0.0087	0.5
	0.7	0.1348	0.1356	0.0179	0.3	0.0094	0.3	0.0133	0.6
	0.8	0.2674	0.2703	0.0141	0.4	0.0083	0.5	0.0338	0.6
	0.9	0.6068	0.6142	0.0120	0.6	<u>0.0082</u>	0.7	0.0545	0.3
	1	4.1297	4.2448	0.0104	0.7	0.0083	1	0.3405	1
10^{-4}	0.1	0.3418	0.0714	0.3418	0	0.0714	0	0.0714	0
	0.2	0.1506	0.0213	0.1506	0	0.0213	0	0.0213	0
	0.3	0.0667	0.0156	0.0667	0	0.0156	0	0.0125	1
	0.4	0.0304	0.0076	0.0304	0	0.0076	0	0.0068	0.1
	0.5	0.0178	0.0115	0.0158	0.1	0.0057	0.1	0.0037	0.2
	0.6	0.0275	0.0271	0.0096	0.2	0.0042	0.2	0.0040	0.6
	0.7	0.0708	0.0716	0.0068	0.3	0.0036	0.4	0.0070	0.6
	0.8	0.1403	0.1427	0.0054	0.4	<u>0.0033</u>	0.6	0.0207	0.5
	0.9	0.4130	0.4245	0.0048	0.5	<u>0.0033</u>	0.6	0.0340	0.8
	1	0.4130	0.4245	0.0048	0.5	<u>0.0033</u>	0.6	0.0340	0.8

TABLE 4.4

Example 4.2: Relative error in the computed solutions of a discretization of the integral equation (4.4). L stands for the Lavrentiev-type method (3.2) with $\alpha = 0$, mL for the modified Lavrentiev-type method (3.4) with $\alpha = 0$, fL for the fractional Lavrentiev-type method (3.2), mfL1 for the modified fractional Lavrentiev method (3.4), and mfL2 for the modified fractional Lavrentiev method (3.7). The regularization parameter is defined by (4.2) for each s and α. The smallest relative error for each noise level is underlined.

(3.7), respectively, of Section 3. The regularization parameter μ is determined by (4.2). Similarly as for Table 4.1, μ is a function of α for the fractional methods, and we report for each s-value the smallest relative error obtained for $\alpha \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ as well as the value α^* that gives the best approximation of $\widehat{x}$. The smallest relative error for each noise level is underlined. The most accurate approximations of $\widehat{x}$ are determined by the modified fractional Lavrentiev-type methods (3.4) and (3.7) for each noise level.

Table 4.5 shows relative errors obtained when the regularization parameter μ is determined by (4.1) with $\alpha \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$. This table is analogous to Table 4.2. Comparing Tables 4.4 and 4.5 shows that the parameter choice rule (4.2) can give more accurate approximations of $\widehat{x}$ for all noise levels. Though, this rule requires the choice of a suitable value of the parameter s. The smallest errors are achieved for $0.7 \leq s \leq 1$.

ε	α	μ	fL	mfL1	mfL2
10^{-2}	0	$2.78 \cdot 10^{-2}$	0.1723	0.1576	0.1576
	0.1	$2.63 \cdot 10^{-2}$	0.0888	0.0476	0.0478
	0.2	$2.48 \cdot 10^{-2}$	0.0845	0.0271	0.0277
	0.3	$2.35 \cdot 10^{-2}$	0.0893	0.0244	0.0254
	0.4	$2.21 \cdot 10^{-2}$	0.0948	0.0244	0.0257
	0.5	$2.09 \cdot 10^{-2}$	0.1011	<u>0.0240</u>	0.0262
	0.6	$1.96 \cdot 10^{-2}$	0.1071	0.0297	0.0266
	0.7	$1.85 \cdot 10^{-2}$	0.1138	0.0414	0.0270
	0.8	$1.74 \cdot 10^{-2}$	0.1203	0.0526	0.0272
	0.9	$1.64 \cdot 10^{-2}$	0.1272	0.0625	0.0274
	1	$1.54 \cdot 10^{-2}$	0.1337	0.0706	0.0276
10^{-3}	0	$5.80 \cdot 10^{-3}$	0.0768	0.0750	0.0750
	0.1	$5.50 \cdot 10^{-3}$	0.0295	0.0212	0.0217
	0.2	$5.20 \cdot 10^{-3}$	0.0255	0.0112	0.0108
	0.3	$4.90 \cdot 10^{-3}$	0.0277	0.0138	0.0089
	0.4	$4.60 \cdot 10^{-3}$	0.0305	0.0171	<u>0.0086</u>
	0.5	$4.30 \cdot 10^{-3}$	0.0334	0.0196	<u>0.0086</u>
	0.6	$4.00 \cdot 10^{-3}$	0.0363	0.0213	<u>0.0086</u>
	0.7	$3.70 \cdot 10^{-3}$	0.0394	0.0224	0.0087
	0.8	$3.50 \cdot 10^{-3}$	0.0432	0.0232	0.0087
	0.9	$3.20 \cdot 10^{-3}$	0.0464	0.0237	0.0130
	1	$3.00 \cdot 10^{-3}$	0.0506	0.0241	0.0130
10^{-4}	0	$1.20 \cdot 10^{-3}$	0.0360	0.0360	0.0360
	0.1	$1.20 \cdot 10^{-3}$	0.0111	0.0093	0.0098
	0.2	$1.10 \cdot 10^{-3}$	0.0088	0.0042	0.0051
	0.3	$1.00 \cdot 10^{-3}$	0.0100	0.0047	0.0042
	0.4	$1.00 \cdot 10^{-3}$	0.0125	0.0062	0.0041
	0.5	$9.00 \cdot 10^{-4}$	0.0145	0.0071	<u>0.0040</u>
	0.6	$8.00 \cdot 10^{-4}$	0.0164	0.0091	0.0043
	0.7	$8.00 \cdot 10^{-4}$	0.0192	0.0140	0.0043
	0.8	$7.00 \cdot 10^{-4}$	0.0208	0.0169	0.0042
	0.9	$7.00 \cdot 10^{-4}$	0.0233	0.0197	0.0042
	1	$6.00 \cdot 10^{-4}$	0.0245	0.0211	0.0042

TABLE 4.5

Example 4.2: Relative error in the computed solutions of a discretization of the integral equation (4.4). fL stands for the fractional Lavrentiev-type method (3.2), mfL1 for the modified fractional Lavrentiev-type method (3.4), and mfL2 for the modified fractional Lavrentiev-type method (3.7). The regularization parameter μ is determined by (4.1). The smallest relative error for each noise level is underlined.

ε	μ	fL	mfL1	mfL2
10^{-2}	$2.00 \cdot 10^{-4}$	0.0215	0.0212	2.0619
10^{-3}	$1.00 \cdot 10^{-4}$	0.0140	0.0083	0.3405
10^{-4}	$1.00 \cdot 10^{-4}$	0.0141	0.0082	0.0340

TABLE 4.6

Example 4.2: Relative error in the computed solutions of a discretization of the integral equation (4.4). fL stands for the fractional Lavrentiev-type method (3.2), mfL1 for the modified fractional Lavrentiev-type method (3.4), and mfL2 for the modified fractional Lavrentiev-type method (3.7). The fractional parameter α is set to one and the regularization parameter μ is determined by (4.2) with $s = 1$.

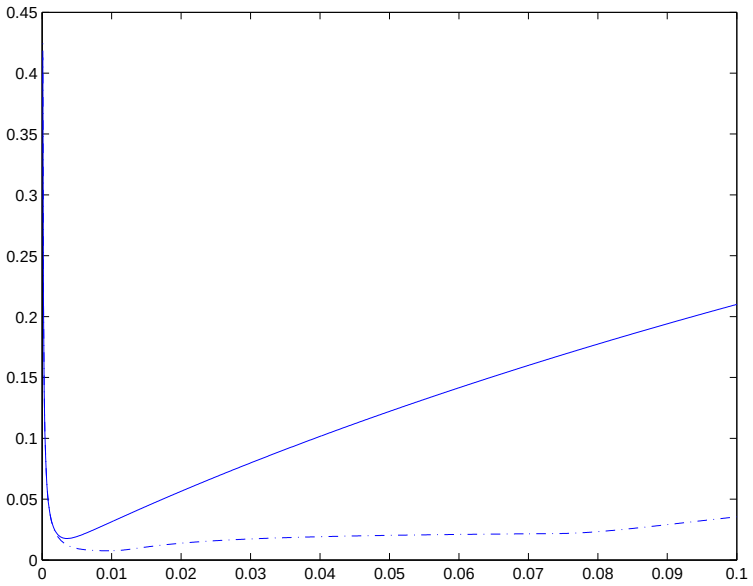

FIG. 4.3. *Example 4.2: Relative errors* $\|x_{\mu,0,\Sigma} - \hat{x}\|/\|\hat{x}\|$ *(solid curve) and* $\|\widetilde{x}_{\mu,0,\Sigma} - \hat{x}\|/\|\hat{x}\|$ *(dash-dotted curve) as functions of* $0 < \mu < 0.1$. *The noise level is* $\varepsilon = 10^{-4}$.

In our last table, Table 4.6, the fractional parameter α is set to one and the regularization parameter is determined by (4.2) with $s = 1$. The results achieved with the methods (3.2), (3.4), and (3.7) with μ and α chosen as in Table 4.4 are seen to be competitive with the results of Table 4.6.

Figure 4.3 displays, for increasing values of $0 < \mu < 0.1$ and $\varepsilon = 10^{-4}$, the relative errors of the computed approximate solutions $x_{\mu,0,\Sigma}$ and $\widetilde{x}_{\mu,0,\Sigma}$ determined by the Lavrentiev and modified Lavrentiev-type methods (3.2) and (3.4), respectively. The latter method is seen to give the smallest error for all μ. Note that the methods (3.4) and (3.7) agree when $\alpha = 0$. Our last figure, Figure 4.4, shows for the same values of μ and ε, the relative errors of the computed approximate solutions $x_{\mu,1,\Sigma}$, $\widetilde{x}_{\mu,1,\Sigma}$, and $\breve{x}_{\mu,1,\Sigma}$, determined by the methods (3.2), (3.4), and (3.7), respectively. □

In summary, the computed examples of this section illustrate that modified fractional Lavrentiev-type methods are able to determine approximate solutions of higher quality than available methods.

5. Conclusion. We discuss several variants of the standard Lavrentiev regularization method. New regularization matrices, obtained by solving a matrix nearness problem, are described both for the standard and fractional Lavrentiev methods. Lavrentiev-type methods that can be applied to Hermitian indefinite problems also are presented. Computed results show that fractional modified Lavrentiev-type methods can perform much better than available methods. The proper choice of the fractional parameter deserves further study.

Acknowledgments. We would like to thank Esther Klann and Ronny Ramlau for pointing out reference [10], and the referee for comments that lead to improvements

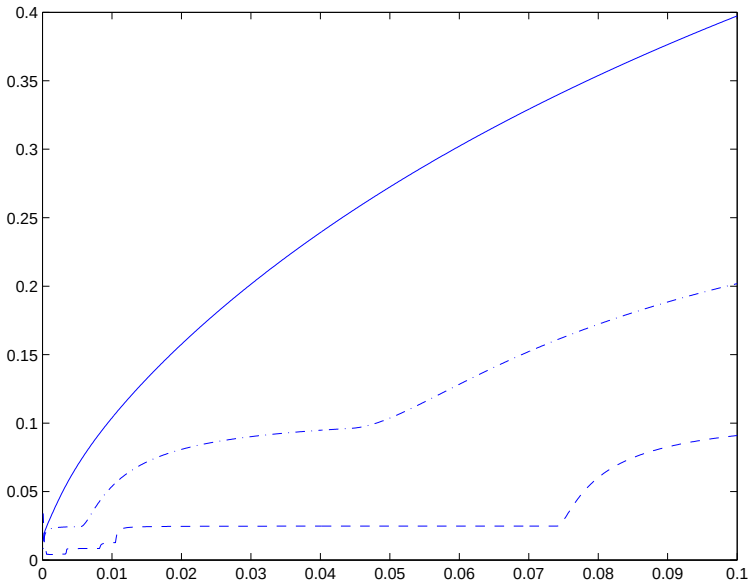

FIG. 4.4. *Example 4.2: Relative errors* $\|x_{\mu,1,\Sigma} - \hat{x}\|/\|\hat{x}\|$ *(solid curve),* $\|\widetilde{x}_{\mu,1,\Sigma} - \hat{x}\|/\|\hat{x}\|$ *(dash-dotted curve) and* $\|\breve{x}_{\mu,1,\Sigma} - \hat{x}\|/\|\hat{x}\|$ *(dashed curve) as functions of* $0 < \mu < 0.1$*. The noise level is* $\varepsilon = 10^{-4}$.

of the presentation.

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