

# A MODIFIED TSVD METHOD FOR DISCRETE ILL-POSED PROBLEMS

SILVIA NOSCHESE\* AND LOTHAR REICHEL†

**Abstract.** Truncated singular value decomposition (TSVD) is a popular method for solving linear discrete ill-posed problems with a small to moderately sized matrix  $A$ . Regularization is achieved by replacing the matrix  $A$  by its best rank- $k$  approximant, which we denote by  $A_k$ . The rank may be determined in a variety of ways, e.g., by the discrepancy principle or the L-curve criterion. This paper describes a novel regularization approach, in which  $A$  is replaced by the closest matrix in a unitarily invariant matrix norm with the same spectral condition number as  $A_k$ . Computed examples illustrate that this regularization approach often yields approximate solutions of higher quality than the replacement of  $A$  by  $A_k$ .

**Key words.** ill-posed problem, truncated singular value decomposition, regularization

**1. Introduction.** Consider the computation of an approximate solution of the minimization problem

$$(1.1) \quad \min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2,$$

where  $A \in \mathbb{R}^{m \times n}$  is a matrix with many singular values of different size close to the origin. Throughout this paper  $\|\cdot\|_2$  denotes the Euclidean vector norm or the associated induced matrix norm. Minimization problems (1.1) with a matrix of this kind often are referred to as discrete ill-posed problems. They arise, for example, from the discretization of linear ill-posed problems, such as Fredholm integral equations of the first kind with a smooth kernel. The vector  $\mathbf{b} \in \mathbb{R}^m$  in (1.1) represents error-contaminated data. We will for notational simplicity assume that  $m \geq n$ ; however, the methods discussed also can be applied when  $m < n$ .

Let  $\mathbf{e} \in \mathbb{R}^m$  denote the (unknown) error in  $\mathbf{b}$ , and let  $\hat{\mathbf{b}} \in \mathbb{R}^m$  be the (unknown) error-free vector associated with  $\mathbf{b}$ , i.e.,

$$(1.2) \quad \mathbf{b} = \hat{\mathbf{b}} + \mathbf{e}.$$

We are interested in computing an approximation of the solution  $\hat{\mathbf{x}}$  of minimal Euclidean norm of the error-free least-squares problem

$$(1.3) \quad \min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \hat{\mathbf{b}}\|_2.$$

Let  $A^\dagger$  denote the Moore-Penrose pseudoinverse of  $A$ . Because  $A$  has many positive singular values close to the origin,  $A^\dagger$  is of very large norm and the solution of (1.1),

$$(1.4) \quad \check{\mathbf{x}} = A^\dagger \mathbf{b} = A^\dagger (\hat{\mathbf{b}} + \mathbf{e}) = \hat{\mathbf{x}} + A^\dagger \mathbf{e},$$

typically is dominated by the propagated error  $A^\dagger \mathbf{e}$  and then is meaningless. This difficulty can be mitigated by replacing the matrix  $A$  by a less ill-conditioned nearby matrix. This replacement commonly is referred to as regularization. One of the

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\*SAPIENZA Università di Roma, P.le A. Moro, 2, I-00185 Roma, Italy. E-mail: noschese@mat.uniroma1.it. Research supported by a grant from SAPIENZA Università di Roma.

†Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA. E-mail: reichel@math.kent.edu. Research supported in part by NSF grant DMS-1115385.

most popular regularization methods for discrete ill-posed problem (1.1) of small to moderate size is truncated singular value decomposition (TSVD); see, e.g., [4, 5]. Introduce the singular value decomposition

$$(1.5) \quad A = U\Sigma V^*,$$

where  $U = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m] \in \mathbb{R}^{m \times m}$  and  $V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n] \in \mathbb{R}^{n \times n}$  are orthogonal matrices, the superscript  $*$  denotes transposition, and the entries of the (possibly rectangular) diagonal matrix

$$\Sigma = \text{diag}[\sigma_1, \sigma_2, \dots, \sigma_n] \in \mathbb{R}^{m \times n}$$

are ordered according to

$$(1.6) \quad \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_\ell > \sigma_{\ell+1} = \dots = \sigma_n = 0.$$

The  $\sigma_j$  are the singular values of  $A$ , and  $\ell$  is the rank of  $A$ . Define the matrix

$$\Sigma_k = \text{diag}[\sigma_1, \sigma_2, \dots, \sigma_k, 0, \dots, 0] \in \mathbb{R}^{m \times n}$$

by setting the singular values  $\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_n$  to zero. The matrix  $A_k = U\Sigma_k V^*$  is the best rank- $k$  approximation of  $A$  in any unitarily invariant matrix norm, e.g., the spectral and Frobenius norms; see [10, Section 3.5]. We have

$$\|A_k - A\|_2 = \sigma_{k+1}, \quad \|A_k - A\|_F = \sqrt{\sum_{i=k+1}^n \sigma_i^2},$$

where  $\|\cdot\|_F$  denotes the Frobenius norm and we define  $\sigma_{n+1} = 0$ .

The TSVD method replaces the matrix  $A$  in (1.1) by  $A_k$  and determines the least-square solution of minimal Euclidean norm. We denote this solution by  $\mathbf{x}_k$ . It can be expressed with the Moore-Penrose pseudoinverse of  $A_k$ , which, for  $1 \leq k \leq \ell$ , is given by

$$A_k^\dagger = V\Sigma_k^\dagger U^*,$$

where

$$\Sigma_k^\dagger = \text{diag}[1/\sigma_1, 1/\sigma_2, \dots, 1/\sigma_k, 0, \dots, 0] \in \mathbb{R}^{n \times m}$$

is the Moore-Penrose pseudoinverse of  $\Sigma_k$ . We have

$$(1.7) \quad \mathbf{x}_k = A_k^\dagger \mathbf{b},$$

which can be expressed as

$$(1.8) \quad \mathbf{x}_k = \sum_{j=1}^{\ell} \phi_j^{(k)} \frac{\mathbf{u}_j^* \mathbf{b}}{\sigma_j} \mathbf{v}_j,$$

where  $\mathbf{u}_j$  and  $\mathbf{v}_j$  are columns of the matrices  $U$  and  $V$  in (1.5), respectively, and  $\phi_j^{(k)}$  are filter factors defined by

$$(1.9) \quad \phi_j^{(k)} = \begin{cases} 1, & 1 \leq j \leq k, \\ 0, & k < j \leq \ell. \end{cases}$$

The condition number of  $A_k$  with respect to the spectral norm is defined as

$$(1.10) \quad \kappa_2(A_k) = \frac{\sigma_1}{\sigma_k}.$$

The larger the condition number, the more sensitive can  $\mathbf{x}_k$  be to the error  $\mathbf{e}$  in  $\mathbf{b}$ ; see, e.g., [13, Lecture 18] for a detailed discussion.

The truncation index  $k$  is a regularization parameter. It determines how close  $A_k$  is to  $A$  and how sensitive the computed solution  $\mathbf{x}_k$  is to the error in  $\mathbf{b}$ . The condition number (1.10) increases and the distance between  $A_k$  and  $A$  decreases as  $k$  increases. It is important to choose a suitable value of  $k$ ; a too large value gives a computed solution  $\mathbf{x}_k$  that is severely contaminated by propagated error stemming from the error  $\mathbf{e}$  in  $\mathbf{b}$ , and a too small value of  $k$  gives an approximate solution that is an unnecessarily poor approximation of the desired solution  $\hat{\mathbf{x}}$  because  $A_k$  is far from  $A$ . There are many approaches described in the literature to determining a suitable truncation index  $k$ , including the quasi-optimality criterion, the L-curve, generalized cross validation, extrapolation, and the discrepancy principle; see, e.g., [2, 3, 4, 5, 8, 9, 12] and references therein.

The size of the truncation index  $k$  depends on the norm of the error  $\mathbf{e}$  in  $\mathbf{b}$ . Typically, the truncation index has to be chosen smaller, the larger the error  $\mathbf{e}$  is, in order to avoid severe propagation of the error into the computed approximate solution  $\mathbf{x}_k$  of (1.1). Unfortunately, the matrix  $A_k$  generally is a poor approximation of  $A$  when  $k$  is small, which may lead to that  $\mathbf{x}_k$  is a poor approximation of  $\hat{\mathbf{x}}$ .

It is the purpose of this paper to describe a modification of the TSVD method in which the matrix  $A_k$  is replaced by a matrix that is closer to  $A$  and has the same condition number. Details are presented in Section 2. Computed examples in Section 3 illustrate that the method of this paper may determine approximations of  $\hat{\mathbf{x}}$  that are more accurate than those furnished by standard TSVD. We will use the discrepancy principle to determine a suitable value of the regularization parameter  $k$ ; however, our modified TSVD method also can be applied in conjunction with other techniques for determining the regularization parameter.

In this paper we consider least-squares problems (1.1) that are small enough to allow the computation of the SVD of the matrix  $A$ . Large-scale problems, for which it is infeasible or unattractive to compute the SVD of the matrix, can be solved by a hybrid method that first reduces the problem to small size, and then applies a regularization method suitable for small problems, such as the TSVD method or the method of this paper. For instance, large-scale problems can be reduced to small ones by partial Lanczos bidiagonalization, partial Arnoldi decomposition, or partial singular value decomposition. Discussions on hybrid methods for solving large-scale linear discrete ill-posed problems and some illustrations are provided in, e.g., [1, 7, 11].

**2. A modified TSVD method.** In this section  $\|\cdot\|$  denotes any unitarily invariant matrix norm. The following result, whose proof follows from [10, Theorem 3.4.5] and the Fan dominance theorem contained in [10, Corollary 3.5.9], will be used below.

LEMMA 2.1. *Let  $A \in \mathbb{R}^{m \times n}$  have the singular value decomposition (1.5) and assume that  $\tilde{A} \in \mathbb{R}^{m \times n}$  has the singular value decomposition*

$$(2.1) \quad \tilde{A} = \tilde{U} \tilde{\Sigma} \tilde{V}^*,$$

where  $\tilde{U} \in \mathbb{R}^{m \times m}$  and  $\tilde{V} \in \mathbb{R}^{n \times n}$  are orthogonal matrices and

$$(2.2) \quad \tilde{\Sigma} = \text{diag}[\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_n] \in \mathbb{R}^{m \times n}, \quad \tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \dots \geq \tilde{\sigma}_n \geq 0.$$

Then

$$(2.3) \quad \|A - \tilde{A}\| \geq \|\Sigma - \tilde{\Sigma}\|$$

for any unitarily invariant matrix norm  $\|\cdot\|$ .

**THEOREM 2.2.** *Let the matrix  $A$  have the SVD (1.5) with the singular values ordered according to (1.6), and let  $\tilde{\Sigma}$  be of the form (2.2). Then*

$$\min_{\tilde{U}, \tilde{V}} \|A - \tilde{U}\tilde{\Sigma}\tilde{V}^*\| = \|\Sigma - \tilde{\Sigma}\|$$

for any unitarily invariant matrix norm  $\|\cdot\|$ , where the minimization is over all orthogonal matrices  $\tilde{U} \in \mathbb{R}^{m \times m}$  and  $\tilde{V} \in \mathbb{R}^{n \times n}$ .

*Proof.* Let  $U$  and  $V$  be the orthogonal matrices in the SVD of  $A$ . Then

$$\min_{\tilde{U}, \tilde{V}} \|A - \tilde{U}\tilde{\Sigma}\tilde{V}^*\| \leq \|A - U\tilde{\Sigma}V^*\| = \|\Sigma - \tilde{\Sigma}\|.$$

The theorem now follows from the lower bound (2.3).  $\square$

**THEOREM 2.3.** *Let  $A \in \mathbb{R}^{m \times n}$  have the singular value decomposition (1.5) with the singular values ordered according to (1.6). A closest matrix to  $A$  in the spectral or Frobenius norms with smallest singular value  $\sigma_k$  is given by*

$$(2.4) \quad \tilde{A}_{\tilde{k}} = U\tilde{\Sigma}_{\tilde{k}}V^*,$$

where  $U$  and  $V$  are the orthogonal matrices in the SVD (1.5) of  $A$  and  $\tilde{\Sigma}_{\tilde{k}}$  has the entries

$$(2.5) \quad \begin{aligned} \tilde{\sigma}_j &= \sigma_j, & 1 \leq j \leq \tilde{k}, \\ \tilde{\sigma}_j &= \sigma_k, & k < j \leq \tilde{k}, \\ \tilde{\sigma}_j &= 0, & \tilde{k} < j \leq n, \end{aligned}$$

where  $\tilde{k}$  is determined by the inequalities  $\sigma_{\tilde{k}} \geq \sigma_k/2$  and  $\sigma_{\tilde{k}+1} < \sigma_k/2$ .

*Proof.* We first consider the matrix  $\tilde{\Sigma}_{\tilde{k}}$ . If  $\sigma_k = 0$ , then  $\tilde{k} = n$  and  $\tilde{\Sigma}_{\tilde{k}} = \Sigma$ . When  $\sigma_k > 0$ , a closest matrix  $\tilde{\Sigma}_{\tilde{k}}$  to  $\Sigma$  in the spectral or Frobenius norms with smallest singular value  $\sigma_k$  is obtained by modifying each singular value of  $\Sigma$  that is smaller than  $\sigma_k$  as little as possible to be either  $\sigma_k$  or zero. Thus, singular values of  $\Sigma$  that are larger than  $\sigma_k/2$  are set to  $\sigma_k$ , while singular values that are smaller than  $\sigma_k/2$  are set to zero. Singular values that are equal to  $\sigma_k/2$  can be set to either  $\sigma_k$  or zero. We set them to  $\sigma_k$ .

It follows from Lemma 2.1 and Theorem 2.2 that

$$\|A - \tilde{A}_{\tilde{k}}\| \geq \|\Sigma - \tilde{\Sigma}_{\tilde{k}}\|$$

with equality for  $\tilde{A}_{\tilde{k}}$  given by (2.4). The theorem now follows from the choice of  $\tilde{\Sigma}_{\tilde{k}}$ .  $\square$

For the matrix  $\tilde{A}_{\tilde{k}}$  of Theorem 2.3, we have in the spectral norm

$$\|A - \tilde{A}_{\tilde{k}}\|_2 \leq \frac{\sigma_k}{2}, \quad \|A - A_k\|_2 = \sigma_{k+1},$$

and in the Frobenius norm

$$\|A - \tilde{A}_{\tilde{k}}\|_F \leq \frac{\sigma_k}{2} \sqrt{n-k}, \quad \|A - A_k\|_F = \sqrt{\sum_{i=k+1}^n \sigma_i^2}.$$

Moreover,

$$(2.6) \quad \|A - A_k\|_2 > \|A - \tilde{A}_{\tilde{k}}\|_2, \quad \|A - A_k\|_F > \|A - \tilde{A}_{\tilde{k}}\|_F$$

when  $\sigma_{k+1} > \sigma_k/2$ . The singular values for mildly to moderately ill-posed discrete ill-posed problems satisfy  $\sigma_{j+1} > \sigma_j$  for all  $1 \leq j < \ell$ . Therefore, the inequalities (2.6) hold for  $1 \leq k < \ell$  for this kind of problems. We conclude that for many discrete ill-posed problems, the rank- $\tilde{k}$  matrix  $\tilde{A}_{\tilde{k}}$  is a better approximation of  $A$  than the rank- $k$  matrix  $A_k$ . Moreover,

$$(2.7) \quad \kappa_2(\tilde{A}_{\tilde{k}}) = \kappa_2(A_k).$$

Because of the latter equality, we do not anticipate the propagated error in the approximation

$$(2.8) \quad \tilde{\mathbf{x}}_{\tilde{k}} = \tilde{A}_{\tilde{k}}^{\dagger} \mathbf{b}$$

of  $\hat{\mathbf{x}}$  to be larger than the propagated error in the TSVD solution  $\mathbf{x}_k$  given by (1.7). In view of (2.6) and (2.7), we expect that for many discrete ill-posed problems (1.1) the computed approximate solution (2.8) to be a more accurate approximation of  $\hat{\mathbf{x}}$  than the corresponding TSVD solution  $\mathbf{x}_k$ .

The approximate solution (2.8) can be expressed as

$$(2.9) \quad \tilde{\mathbf{x}}_{\tilde{k}} = \sum_{j=1}^{\ell} \tilde{\phi}_j^{(\tilde{k})} \frac{\mathbf{u}_j^* \mathbf{b}}{\sigma_j} \mathbf{v}_j,$$

with the filter factors

$$(2.10) \quad \tilde{\phi}_j^{(\tilde{k})} = \begin{cases} 1, & 1 \leq j \leq k, \\ \frac{\sigma_j}{\sigma_k}, & k < j \leq \tilde{k}, \\ 0, & \tilde{k} < j \leq \ell. \end{cases}$$

**PROPOSITION 2.4.** *Let the filter factors  $\phi_j^{(k)}$  and  $\tilde{\phi}_j^{(\tilde{k})}$  be defined by (1.9) and (2.10), respectively, for  $1 \leq j \leq \ell$ . Then*

$$\tilde{\phi}_j^{(\tilde{k})} \geq \phi_j^{(k)}, \quad 1 \leq j \leq \ell,$$

and

$$\frac{1}{2} \leq \tilde{\phi}_j^{(\tilde{k})} \leq 1, \quad k < j \leq \tilde{k}.$$

*Proof.* The inequalities follow from (2.5), (2.10), and the definition of  $\tilde{k} \geq k$ .  $\square$

We conclude from the above proposition that the approximation (2.8) typically is of strictly larger norm than the corresponding TSVD solution  $\mathbf{x}_k$ . The norm of the TSVD solutions  $\mathbf{x}_k$  increases with  $k$ . For discrete ill-posed problems with a severely error contaminated data vector  $\mathbf{b}$ , one may have to choose a very small value of  $k$  and this may yield an approximation  $\mathbf{x}_k$  of  $\hat{\mathbf{x}}$  that is of much smaller norm than  $\|\hat{\mathbf{x}}\|$ . This effect can be at least partially remedied by the modified TSVD method of this paper.

**3. Computed examples.** The calculations of this section were carried out in MATLAB with machine epsilon about  $2.2 \cdot 10^{-16}$ . We use examples from the MATLAB package Regularization Tools [6]. The regularization parameter  $k$  is determined by the discrepancy principle. Its application requires the (unavailable) least-squares problem (1.3) to be consistent and that a bound  $\varepsilon$  for  $\|\mathbf{e}\|_2$  be known. Specifically, we first choose the value of the regularization parameter  $k$  as small as possible so that

$$\|A\mathbf{x}_k - \mathbf{b}\|_2 \leq \varepsilon,$$

where  $\mathbf{x}_k$  is defined by (1.7). Then we determine  $\tilde{\mathbf{x}}_{\tilde{k}}$  from (2.8) with  $\tilde{A}_{\tilde{k}}$  given by (2.4). We refer to this approach to computing an approximation of  $\hat{\mathbf{x}}$  as the modified TSVD method.

The first two examples are obtained by discretizing Fredholm integral equations of the first kind

$$(3.1) \quad \int_a^b \kappa(s, t)x(t) dt = g(s), \quad c \leq s \leq d,$$

with a smooth kernel  $\kappa$ . The discretizations are carried out by Galerkin or Nyström methods and yield linear discrete ill-posed problems (1.1). MATLAB functions in [6] determine discretizations  $A \in \mathbb{R}^{n \times n}$  of the integral operators and scaled discrete approximations  $\hat{\mathbf{x}} \in \mathbb{R}^n$  of the solution  $x$  of (3.1). We add an error vector  $\mathbf{e} \in \mathbb{R}^n$  with normally distributed random entries with zero mean to  $\hat{\mathbf{b}} = A\hat{\mathbf{x}}$  to obtain the vector  $\mathbf{b}$  in (1.1); cf. (1.2). The vector  $\mathbf{e}$  is scaled to yield a specified noise level  $\|\mathbf{e}\|_2/\|\hat{\mathbf{b}}\|_2$ . In particular,  $\|\mathbf{e}\|_2$  is available and we can apply the discrepancy principle with  $\varepsilon = \|\mathbf{e}\|_2$  to determine the truncation index  $k$  in TSVD.

Let  $\mathbf{x}_k$  and  $\tilde{\mathbf{x}}_{\tilde{k}}$  denote the computed solutions determined by the TSVD and modified TSVD methods, respectively. We are interested in comparing the relative errors  $\|\mathbf{x}_k - \hat{\mathbf{x}}\|_2/\|\hat{\mathbf{x}}\|_2$  and  $\|\tilde{\mathbf{x}}_{\tilde{k}} - \hat{\mathbf{x}}\|_2/\|\hat{\mathbf{x}}\|_2$ . The errors depend on the entries of the error vector  $\mathbf{e}$ . To gain insight into the average behavior of the solution methods, we report in every example the average of the relative errors in  $\mathbf{x}_k$  and  $\tilde{\mathbf{x}}_{\tilde{k}}$  over 1000 runs with different noise vectors  $\mathbf{e}$  for each specified noise level. We let  $n = 200$  in all examples.

Example 3.1. We first consider the problem `phillips` from [6]. Let

$$\phi(t) = \begin{cases} 1 + \cos(\frac{\pi t}{3}), & |t| < 3, \\ 0, & |t| \geq 3. \end{cases}$$

The kernel, right-hand side function, and solution of the integral equation (3.1) are given by

$$\kappa(s, t) = \phi(s - t), \quad x(t) = \phi(t), \quad g(s) = (6 - |s|) \left( 1 + \frac{1}{2} \cos\left(\frac{\pi s}{3}\right) \right) + \frac{9}{2\pi} \sin\left(\frac{\pi |s|}{3}\right),$$

and  $a = c = -6$ ,  $b = d = 6$ . Let the vector  $\mathbf{b}$  be contaminated by 10% noise. Then the average relative error in  $\mathbf{x}_k$  obtained with the TSVD method over 1000 runs is  $7.9 \cdot 10^{-2}$ , while the corresponding error for the approximate solution  $\tilde{\mathbf{x}}_{\tilde{k}}$  obtained with the modified TSVD method is  $7.6 \cdot 10^{-2}$ . The largest quotient  $\|\mathbf{x}_k - \hat{\mathbf{x}}\|_2/\|\tilde{\mathbf{x}}_{\tilde{k}} - \hat{\mathbf{x}}\|_2$  over these runs is 2.1. Thus, our modified TSVD method can produce an error that is less than half the error achieved with standard TSVD. The average truncation indices for TSVD and modified TSVD are  $k_{\text{avg}} = 6.20$  and  $\tilde{k}_{\text{avg}} = 6.63$ , respectively.

Table 3.1 displays the filter factors for one of the runs. The truncation indices for standard and modified TSVD are  $k = 6$  and  $\tilde{k} = 7$ , respectively. The table shows the first 8 singular values  $\sigma_1, \sigma_2, \dots, \sigma_8$  of  $A$ , the first filter factors  $\phi_1^{(6)}, \phi_2^{(6)}, \dots, \phi_8^{(6)}$  used in the construction of the standard TSVD solution (1.8) with  $k = 6$ , and the first filter factors  $\tilde{\phi}_1^{(7)}, \tilde{\phi}_2^{(7)}, \dots, \tilde{\phi}_8^{(7)}$  that describe the modified TSVD solution (2.9) with  $\tilde{k} = 7$ . In particular,  $\tilde{\phi}_7^{(7)} = \sigma_7/\sigma_6$  and  $\tilde{\phi}_j^{(7)} = 0$  for  $j \geq 8$ .

| $j$                    | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    |
|------------------------|------|------|------|------|------|------|------|------|
| $\sigma_j$             | 5.80 | 5.24 | 4.41 | 3.43 | 2.45 | 1.56 | 0.86 | 0.37 |
| $\phi_j^{(6)}$         | 1    | 1    | 1    | 1    | 1    | 1    | 0    | 0    |
| $\tilde{\phi}_j^{(7)}$ | 1    | 1    | 1    | 1    | 1    | 1    | 0.55 | 0    |

TABLE 3.1

*Example 3.1:* The first 8 singular values and the filter factors  $\phi_1^{(6)}, \phi_2^{(6)}, \dots, \phi_8^{(6)}$  and  $\tilde{\phi}_1^{(7)}, \tilde{\phi}_2^{(7)}, \dots, \tilde{\phi}_8^{(7)}$  used in the construction of the approximate solutions  $\mathbf{x}_6$  and  $\tilde{\mathbf{x}}_7$ , respectively.

The Tikhonov regularization method is an alternative to TSVD and modified TSVD. In Tikhonov regularization, the problem (1.1) is replaced by the penalized least-squares problem

$$(3.2) \quad \min_{\mathbf{x} \in \mathbb{R}^n} \{ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \mu \|\mathbf{x}\|_2^2 \},$$

where  $\mu > 0$  is a regularization parameter. The discrepancy principle prescribes that the regularization parameter  $\mu$  be chosen so that the solution  $\mathbf{x}_\mu$  of (3.2) satisfies  $\|\mathbf{A}\mathbf{x}_\mu - \mathbf{b}\|_2 = \varepsilon$ ; see, e.g., [4, 5] for properties and details on Tikhonov regularization. Evaluating the relative error  $\|\mathbf{x}_\mu - \hat{\mathbf{x}}\|_2 / \|\hat{\mathbf{x}}\|_2$  over 1000 runs with different noise vectors  $\mathbf{e}$ , each of which corresponding to 10% noise, yields the average relative error  $1.6 \cdot 10^{-1}$ . The minimum relative error over these runs is  $1.5 \cdot 10^{-1}$ . Thus, the modified TSVD method performs better.  $\square$

| Noise level | TSVD                  | $k_{\text{avg}}$   | modified TSVD         | $\tilde{k}_{\text{avg}}$ |
|-------------|-----------------------|--------------------|-----------------------|--------------------------|
| 10.0%       | $3.959 \cdot 10^{-1}$ | $4.222 \cdot 10^0$ | $3.912 \cdot 10^{-1}$ | $5.558 \cdot 10^0$       |
| 5.0%        | $3.526 \cdot 10^{-1}$ | $5.270 \cdot 10^0$ | $3.448 \cdot 10^{-1}$ | $7.045 \cdot 10^0$       |
| 1.0%        | $2.680 \cdot 10^{-1}$ | $8.841 \cdot 10^0$ | $2.544 \cdot 10^{-1}$ | $1.198 \cdot 10^1$       |
| 0.1%        | $1.832 \cdot 10^{-1}$ | $1.865 \cdot 10^1$ | $1.696 \cdot 10^{-1}$ | $2.571 \cdot 10^1$       |

TABLE 3.2

*Example 3.2:* Average relative errors in and average truncation indices of the computed solutions for the deriv2 test problem for several errors.

Example 3.2. This test problem uses the code deriv2 from [6]. The kernel, solution, and right-hand side of (3.1) are given by

$$\kappa(s, t) = \begin{cases} s(t-1), & s < t, \\ t(s-1), & s \geq t, \end{cases}$$

$$x(t) = t,$$

$$g(s) = \frac{s^3 - s}{6},$$

and  $a = c = 0$ ,  $b = d = 1$ . Thus, the kernel  $\kappa$  is the Green's function for the second derivative. Table 3.2 shows the average relative errors in and the average truncation

indices of the computed solutions for several noise levels. The modified TSVD is seen to yield the smallest average errors for all noise levels considered.  $\square$

| Noise level | TSVD                  | $k_{\text{avg}}$   | modified TSVD         | $\tilde{k}_{\text{avg}}$ |
|-------------|-----------------------|--------------------|-----------------------|--------------------------|
| 10.0%       | $3.040 \cdot 10^{-1}$ | $9.567 \cdot 10^0$ | $2.878 \cdot 10^{-1}$ | $1.261 \cdot 10^1$       |
| 5.0%        | $2.571 \cdot 10^{-1}$ | $1.142 \cdot 10^1$ | $2.292 \cdot 10^{-1}$ | $1.482 \cdot 10^1$       |
| 1.0%        | $1.191 \cdot 10^{-1}$ | $1.614 \cdot 10^1$ | $1.038 \cdot 10^{-1}$ | $2.017 \cdot 10^1$       |
| 0.1%        | $4.604 \cdot 10^{-2}$ | $2.374 \cdot 10^1$ | $3.472 \cdot 10^{-2}$ | $2.881 \cdot 10^1$       |

TABLE 3.3

Example 3.3: Average relative errors in and average truncation indices of the computed solutions for the heat test problem for several noise levels.

Example 3.3. The test problem *heat* from [6] is a discretization of a first kind Volterra integral equation on the interval  $[0, 1]$  with a convolution kernel; see [6] for details. Table 3.3 shows the average relative errors in and the average truncation indices of the computed solutions over 1000 runs for each noise level. The modified TSVD method yields the smallest average errors for all noise levels considered.

As a further illustration of the performance of the proposed method, let us consider one of the computed tests for the noise level 0.1%. The discrepancy principle determines  $k = 27$ , and the relative error in the approximate solution  $\mathbf{x}_{27}$ , defined by (1.7), is  $\|\mathbf{x}_k - \hat{\mathbf{x}}\|_2 / \|\hat{\mathbf{x}}\|_2 = 3.795 \cdot 10^{-2}$ . Similarly, the relative error in the approximate solution  $\tilde{\mathbf{x}}_k$  defined by (2.8) is  $\|\tilde{\mathbf{x}}_k - \hat{\mathbf{x}}\|_2 / \|\hat{\mathbf{x}}\|_2 = 2.757 \cdot 10^{-2}$ . Moreover, one has  $\|A - \tilde{A}_k\|_2 / \|A - A_k\|_2 = 5.638 \cdot 10^{-1}$  and  $\|A - \tilde{A}_k\|_F / \|A - A_k\|_F = 6.807 \cdot 10^{-1}$ . The regularization parameter for modified TSVD is  $\tilde{k} = 33$ . Figures 3.1(a) and 3.1(b) display  $\mathbf{x}_{27}$  and  $\tilde{\mathbf{x}}_{33}$ , respectively. Figure 3.2 displays the error in the approximate solutions. Figures 3.3(a) and 3.3(b) display the first 33 singular values and the associated filter factors in the approximate solution  $\tilde{\mathbf{x}}_{33}$ , respectively.  $\square$

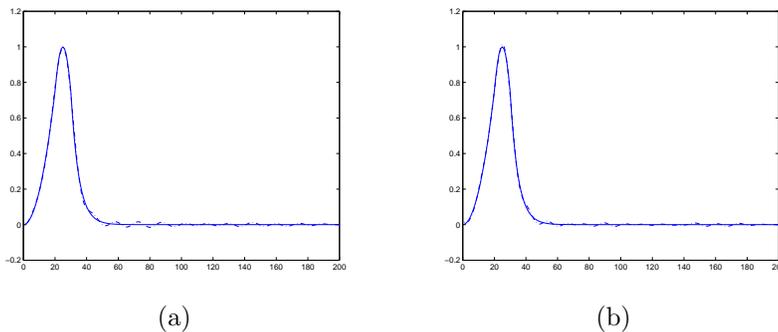


FIG. 3.1. Example 3.3. Solution  $\hat{\mathbf{x}}$  to the error-free problem *heat* (solid curves) and computed approximations (dash-dotted curves); the approximate solutions are  $\mathbf{x}_{27}$  in (a) and  $\tilde{\mathbf{x}}_{33}$  in (b).

**4. Conclusion.** The above examples and many other ones show the modified TSVD method to generally give approximations of the desired solution  $\hat{\mathbf{x}}$  of the unavailable noise-free problem (1.3) of higher or the same accuracy as the TSVD method. The computational effort for both methods is dominated by the computation of the SVD (1.5) of the matrix and, consequently, is essentially the same. The modified TSVD method therefore is an attractive alternative to the standard TSVD method.

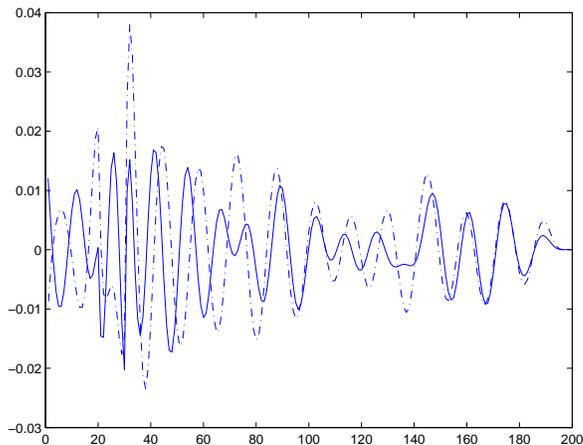


FIG. 3.2. *Example 3.3. Errors  $\tilde{\mathbf{x}}_{33} - \hat{\mathbf{x}}$  (solid curve) and  $\mathbf{x}_{27} - \hat{\mathbf{x}}$  (dash-dotted curve).*

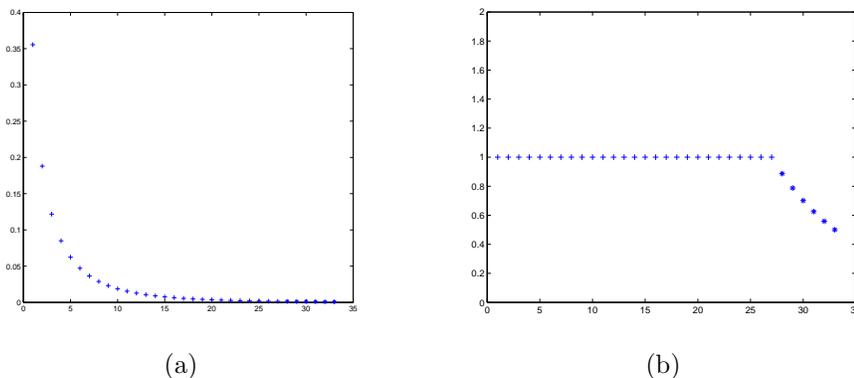


FIG. 3.3. *Example 3.3. First 33 singular values in (a) and associated filter factors in  $\tilde{\mathbf{x}}_{33}$  in (b). They are marked by a plus from 1 to 27 and by a star from 28 to 33.*

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