Network analysis with the aid of the path length matrix

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Abstract Let a network be represented by a simple graph $\mathcal{G}$ with n vertices. A common approach to investigate properties of a network is to use the adjacency matrix $A = [a_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{n \times n}$ associated with the graph $\mathcal{G}$, where $a_{ij} > 0$ if there is an edge pointing from vertex v_i to vertex v_j, and $a_{ij} = 0$ otherwise. Both A and its positive integer powers reveal important properties of the graph. This paper proposes to study properties of a graph $\mathcal{G}$ by also using the path length matrix for the graph. The $(ij)^{th}$ entry of the path length matrix is the length of the shortest path from vertex v_i to vertex v_j; if there is no path between these vertices, then the value of the entry is ∞. Powers of the path length matrix are formed by using min-plus matrix multiplication and are important for exhibiting properties of $\mathcal{G}$. We show how several known measures of communication such as closeness centrality, harmonic centrality, and eccentricity are related to the path length matrix, and we introduce new measures of communication, such as the harmonic K-centrality and global K-efficiency, where only (short) paths made up of at most K edges are taken into account. The sensitivity of the global K-efficiency to changes of the entries of the adjacency matrix also is considered.

Keywords network analysis, path length matrix, harmonic centrality, global efficiency

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1 Introduction

An important characteristic of a network is how well communication can flow in it, i.e., how easy or difficult it is to reach one part of the network from another part by following edges. How well information flows through the whole network can be measured by the diameter of the graph that represents the network or by its global efficiency. Both these measures can be determined with the aid of the path length matrix associated with the network. We will discuss these connections and introduce new measures of communication based on the path length matrix.

Let us introduce some notation and definitions that will be used throughout this paper. A network is represented by a graph. A *weighted graph* $\mathcal{G} = \langle \mathcal{V}, \mathcal{E}, \mathcal{W} \rangle$ consists of a set of *nodes* or *vertices* $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, a set of *edges* $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ that connect the vertices, and a set of weights $\mathcal{W} =$

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$\{a_{ij}\}_{i,j=1}^n$; the weights a_{ij} are the entries of the adjacency matrix $A = [a_{ij}]_{i,j=1}^n$ associated with the graph $\mathcal{G}$; see below. An edge is said to be *directed* if it starts at a vertex v_i and ends at a vertex v_j, and is denoted by $e(v_i \rightarrow v_j)$. An edge between the vertices v_i and v_j is said to be *undirected* when the pair of vertices is unordered and the weights a_{ij} and a_{ji} are positive and equal. An undirected egde between the vertices v_i and v_j is denoted by $e(v_i \leftrightarrow v_j)$. A graph with only undirected edges is said to be *undirected*; otherwise the graph is *directed*. A *simple graph* is a graph without multiple edges or self-loops. In particular, this implies that the diagonal entries of the adjacency matrix for the graph vanish. This work considers simple graphs.

The *adjacency matrix* $A = [a_{ij}]_{i,j=1}^n \in \mathbb{R}^{n \times n}$ for a weighted graph $\mathcal{G}$ is determined by the weights a_{ij} of the graph with $a_{ij} > 0$ if there is an edge $e(v_i \rightarrow v_j)$ in $\mathcal{G}$. If there is no edge $e(v_i \rightarrow v_j)$ in $\mathcal{G}$, then $a_{ij} = 0$. For an *unweighted graph*, all positive entries a_{ij} of A equal one. A sequence of k edges (not necessarily distinct) such that $\{e(v_1 \rightarrow v_2), e(v_2 \rightarrow v_3), \ldots, e(v_k \rightarrow v_{k+1})\}$ form a *walk*. If $v_{k+1} = v_1$, then the walk is said to be *closed*. A sequence of distinct edges such that $\{e(v_1 \rightarrow v_2), e(v_2 \rightarrow v_3), \ldots, e(v_k \rightarrow v_{k+1})\}$ form a *path*. The *length* of a path is given by the sum of all weights of the edges in the path. (In the unweighted case, the sum of all weights of the edges in a path of length k is k.) For further discussions on networks and graphs; see [8,14].

To construct the path length matrix associated with the network, we will make use of *min-plus matrix multiplication*, i.e., matrix multiplication in the tropical algebra [12]:

$$C = [c_{ij}]_{i,j=1}^n = A \star B : \qquad c_{ij} = \min_{h=1,2,\ldots,n} \{a_{ih} + b_{hj}\}, \qquad 1 \le i, j \le n,$$

where $A, B, C \in \mathbb{R}^{n \times n}$. We denote by $A^{1,\star} = [a_{ij}^{(1,\star)}]_{i,j=1}^n \in \mathbb{R}^{n \times n}$ the matrix obtained by setting to ∞ the vanishing off-diagonal entries of the adjacency matrix A associated with the graph $\mathcal{G}$ under consideration. For $k > 1$, the kth *min-plus power* of $A^{1,\star}$ is given by

$$A^{k,\star} = [a_{ij}^{(k,\star)}]_{i,j=1}^n \in \mathbb{R}^{n \times n} : \qquad a_{ij}^{(k,\star)} = \min_{h=1,2,\ldots,n} \{a_{ih}^{(k-1,\star)} + a_{hj}^{(1,\star)}\}.$$

Notice that the matrix $A^{k,\star}$ gives vertex distances using paths of at most k edges. In detail, the entry $a_{ij}^{(k,\star)}$, with $i \ne j$, represents the length of the *shortest path* from v_i to v_j made up of at most k edges. The diagonal entries of $A^{k,\star}$ are zero by definition. One has $a_{ij}^{(k,\star)} = \infty$ if every path from v_i to v_j is made up of more than k edges, or if there is no path from v_i to v_j.

The *diameter* of a graph $\mathcal{G}$ is the maximal length $d_{\mathcal{G}}$ of the shortest path between any distinct vertices of the graph and provides a measure of how easy it is for the vertices of the graph to communicate. One has

$$d_{\mathcal{G}} = \max_{1 \le i,j \le n} a_{ij}^{(n-1,\star)}. \tag{1}$$

Indeed, the entry $a_{ij}^{(n-1,\star)}$ of the matrix $A^{n-1,\star} = [a_{ij}^{(n-1,\star)}]_{i,j=1}^n$ yields the length of the shortest path from v_i to v_j. We will refer to $A^{n-1,\star}$ as the *path length matrix*. Note that the triangle inequality holds for the entries of a path length matrix. Specifically,

$$a_{ij}^{(n-1,\star)} \le a_{ih}^{(n-1,\star)} + a_{hj}^{(n-1,\star)}, \qquad 1 \le i, j \le n.$$

Consider an unweighted connected graph $\mathcal{G}$ with associated adjacency matrix $A \in \mathbb{R}^{n \times n}$. Recall that the diameter of $\mathcal{G}$ is the maximal number of edges in the shortest path between all pairs of distinct vertices of the graph. Given the vertices v_i and v_j, there is an integer $\widehat{k}$, $1 \le \widehat{k} < n$, such that

$$a_{ij}^{(n-1,\star)} = \cdots = a_{ij}^{(\widehat{k}+1,\star)} = a_{ij}^{(\widehat{k},\star)} = \widehat{k}$$

since the graph is connected, whereas for $1 \le h < \widehat{k}$, one has $a_{ij}^{(h,\star)} = \infty$. Thus, information provided by the path length matrix $A^{n-1,\star}$ includes information about all powers $A^{k,\star}$ for $1 \le k < n-1$.

Let v_i and v_j be distinct vertices in a weighted graph. Then there is an integer $\widehat{k}$, $1 \leq \widehat{k} < n-1$, such that

$$a_{ij}^{(n-1,\star)} \leq \cdots \leq a_{ij}^{(\widehat{k}+1,\star)} \leq a_{ij}^{(\widehat{k},\star)} < \infty$$

and, for $1 \leq h < \widehat{k}$, $a_{ij}^{(h,\star)} = \infty$. Thus, as in the unweighted case, information provided by the path length matrix $A^{n-1,\star}$ refines information given by the powers $A^{k,\star}$ for $1 \leq k < n-1$. However, the information of the minimal number of steps required to reach vertex v_j from vertex v_i is lost.

As mentioned above, the path length matrix may be constructed by evaluating the min-plus powers of $A^{1,\star}$ $n-2$ times; here $A^{1,\star}$ is obtained from the adjacency matrix A by setting all zero off-diagonal entries to ∞. The following MATLAB function, with the adjacency matrix A for a graph and the level $= n-1$ as input arguments, returns the path length matrix associated with the adjacency matrix A. The function implements the dynamic programming Bellmann-Ford algorithm for solving the well-known "all-pairs shortest path problem". The algorithm requires $\mathcal{O}(n^2 m)$ arithmetic floating point operations (flops), where n is the number of vertices and m is the number of edges of the graph; if the graph is undirected, then the cost of the algorithm is halved. Notice that the function can be applied to determine shortest paths in a weighted graph having positive or negative weights.

```
 1: function PATHLENGTH_MATRIX(A, level)
 2:     n = size(A, 1);
 3:     for i = 1 : n do
 4:         for j = 1 : n do
 5:             if A(i, j) = 0 and i ≠ j then
 6:                 A(i, j) = inf;
 7:             end if
 8:         end for
 9:     end for
10:     B = A;
11:     C = inf(n);
12:     for count = 1 : level − 1 do
13:         for i = 1 : n do
14:             for j = 1 : n do
15:                 if j ≠ i then
16:                     for k = 1 : n do
17:                         if B(k, j) ≠ inf and k ≠ j then
18:                             C(i, j) = min(C(i, j), A(i, k) + B(k, j));
19:                         end if
20:                     end for
21:                 end if
22:             end for
23:         end for
24:         A = C;
25:         for i = 1 : n do
26:             A(i, i) = 0;
27:         end for
28:     end for
29: return Path length matrix;
30: end function
```

In line 2 of the function Pathlength_matrix, the function call $\mathsf{size}(A, 1)$ yields the order of the matrix $A \in \mathbb{R}^{n \times n}$; in line 6, the matrix entry $A(i, j)$ is set to ∞. Similarly, in line 11, C is defined as an $n \times n$ matrix with all entries equal to ∞. If the graph is unweighted and connected, then the above MATLAB function can be modified by introducing a break before the last **end for** when there is no entry ∞. The diameter of the graph then is count+1.

For both weighted and unweighted graphs, also when the graph is not connected, the above MATLAB function with argument level $= K$, where $1 < K < n$, computes $A^{K,\star}$, i.e., the matrix of the distances

between any distinct vertices of the graph using paths with at most K edges. We note that the triangle inequality might not hold for the entries of $A^{K,\star}$ and some entries of this matrix may have the value ∞.

We will see how the matrix $A^{n-1,\star}$ associated with a connected graph $\mathcal{G}$ sheds light on the communication within the network determined by the graph. In fact, as a measure of the ease of communication in the graph, we like to use the average inverse geodesic length of $\mathcal{G}$ (i.e., its *global efficiency*, cf. Section 3) instead of the maximum geodesic length of $\mathcal{G}$ (i.e., its *diameter*). To this end, we introduce the *reciprocal path length matrix* $A^{n-1,\star,-1} = [a_{ij}^{(n-1,\star,-1)}]_{i,j=1}^n$ obtained by replacing the off-diagonal entries of the path length matrix by their reciprocals, i.e.,

$$a_{ij}^{(n-1,\star,-1)} = 1/a_{ij}^{(n-1,\star)}, \qquad 1 \le i,j \le n,$$

where $1/\infty$ is identified with 0.

We are interested in determining the shortest paths that use at most K edges. We therefore also consider the *reciprocal K-path length matrix* $A^{K,\star,-1} = [a_{ij}^{(K,\star,-1)}]_{i,j=1}^n$, with $a_{ij}^{(K,\star,-1)} = 1/a_{ij}^{(K,\star)}$. Thus, the entry $a_{ij}^{(K,\star,-1)}$ vanishes if $a_{ij}^{(K,\star)} = \infty$. Note that the same would happen if $K = n-1$, in case the graph $\mathcal{G}$ that determines the adjacency matrix A is not connected. The matrix $A^{K,\star,-1}$ allows us to define the *global K-efficiency* of $\mathcal{G}$; cf. Section 3.

In order to enhance communication using paths with at most K edges, with $1 < K < n$, i.e., to increase the global K-efficiency of the graph associated with the adjacency matrix A, we select edge weights by analyzing centrality properties of the vertices of the graph and, if computationally feasible, the spectral properties of the reciprocal K-path length matrix $A^{K,\star,-1}$. In detail, if $K \ll n$ and the (sparse) non-negative reciprocal K-path length matrix $A^{K,\star,-1}$ is irreducible, then we apply the Perron-Frobenius theory by following the approach in [4,7]. In our context, the choice of weights is dictated by the analysis of the sensitivity to perturbations in A.

Applications of our approach include city planning and information transmission. As for disease propagation, a recent research study of the Zhejiang City Planning Center (China) pointed out a strong connection between the spread of the Covid-19 epidemic and the shape of the city. Cities with a radial structure (such as Milan) have good internal connections due to the capillarity of the public transportation system - buses, trams, subways, and trains. This dynamic made citizens of such cities more vulnerable to the arrival of the Covid-19 virus: the incidence of infections compared to the number of inhabitants was generally larger in cities with a radial urban structure than in cities without this structure, due to the good communication of the people in cities with radial urban structure.

This paper is organized as follows: Section 2 analyzes differences and similarities of powers and tropical powers of the adjacency matrix for undirected and unweighted graphs. Section 3 reviews well-known measures that can be easily computed by means of the path length matrix and introduces novel ones. In Section 4 we present two algorithms that determine which edge-weight should be changed in order to boost global efficiency. Changing the edge weights may entail widening streets or increasing the number of trams on a route, decreasing travel times on a highway by increasing the travel speed, or decreasing the waiting time for trams on a route. Finally, numerical tests are reported in Section 5 and concluding remarks can be found in Section 6.

2 Powers versus tropical powers

Consider an undirected and unweighted simple graph $\mathcal{G}$ with adjacency matrix $A \in \mathbb{R}^{n \times n}$. Then the entry $a_{ij}^{(k)}$ of the matrix $A^k = [a_{ij}^{(k)}]_{i,j=1}^n \in \mathbb{R}^{n \times n}$ counts the number of walks of length k between the vertices v_i and v_j. A *matrix function* based on the powers A^k that is analytic at the origin, and vanishes there, can be defined by a formal Maclaurin series

$$f(A) = \sum_{k=1}^{\infty} c_k A^k, \tag{2}$$

where we for the moment ignore the convergence properties of this series. Usually long walks are considered less important than short walks, because information flows more easily through short walks than through long ones. Therefore matrix functions applied in network analysis generally have the property that $0 \leq c_{k+1} \leq c_k$ for all $k \geq 1$. The most common matrix function used in network analysis is the matrix exponential; see [5,6,8–10] for discussions and illustrations. We prefer to use the the modified matrix exponential

$$\exp_0(A) := \exp(A) - I,$$

where I denotes the identity matrix, because the first term in the Maclaurin series of $\exp(A)$ has no natural interpretation in the context of network modeling. For the modified matrix exponential, we have $c_k = 1/k!$, and the series (2) converges for any adjacency matrix A.

The *communicability* between distinct vertices v_i and v_j, $i \neq j$, is defined by

$$[\exp_0(A)]_{ij} = \sum_{k=1}^{\infty} \frac{a_{ij}^{(k)}}{k!};$$

see [8] for the analogous definition based on $\exp(A)$. The communicability accounts for all possible routes of communication between the vertices v_i and v_j in the network defined by the adjacency matrix A, and assigns a larger weight to shorter walks than to longer ones. The larger the value of $[\exp_0(A)]_{ij}$, the better is the communicability between the vertices v_i and v_j.

Remark 1 Notice that even if there exists an integer $\widehat{k}$, $1 \leq \widehat{k} < n - 1$, such that $a_{ij}^{(\widehat{k})} > 0$, one may have $a_{ij}^{(\widehat{k}+1)} = 0$. Information provided by A^{n-1} does not include information provided by all A^h for $1 \leq h < n - 1$. This is one of the reasons for the interest in the matrix functions $\exp_0(A)$ and $\exp(A)$.

It is straightforward to show the following result.

Proposition 1 *Let $\widehat{k}$, $1 \leq \widehat{k} < n$, be the smallest integer power such that $a_{ij}^{(\widehat{k})} = p > 0$, that is to say,*

$$a_{ij}^{(h)} = 0, \quad \forall h < \widehat{k}, \quad a_{ij}^{(\widehat{k})} = p.$$

Then $\widehat{k}$ is the length of the shortest path that connects v_i and v_j, i.e.,

$$a_{ij}^{(h,\star)} = \infty, \quad \forall h < \widehat{k}, \quad a_{ij}^{(h,\star)} = \widehat{k}, \quad \forall h \geq \widehat{k},$$

and p is the number of shortest paths that connect v_i and v_j.

Example 1 In view of Proposition 1, the information provided by both the path length matrix and a suitable power of the adjacency matrix is of interest. Consider the undirected and unweighted graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ depicted in Figures 1-2. The adjacency matrices of these graphs are

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$
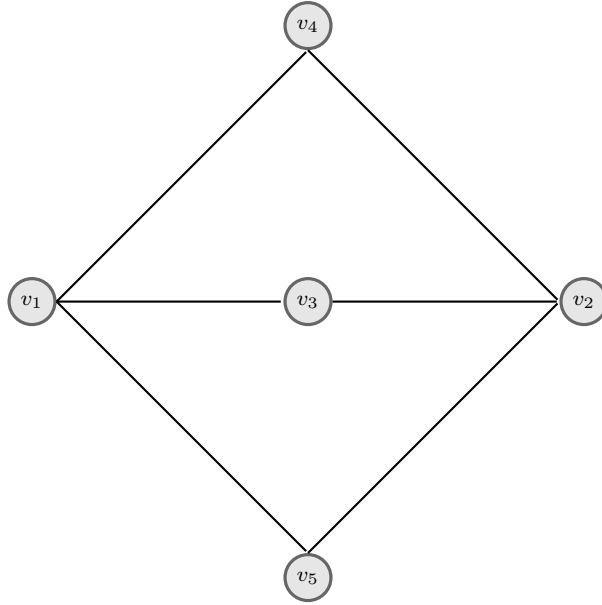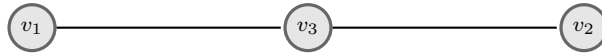
Fig. 1 Graph $\mathcal{G}_1$ in Example 1.

Fig. 2 Graph $\mathcal{G}_2$ in Example 1.

For both graphs, the shortest path between the vertices v_1 and v_2 has length 2. However, in $\mathcal{G}_1$ there are three shortest paths that connect v_1 and v_2, while there is only one in $\mathcal{G}_2$. Thus, there is surely better communication between these vertices in $\mathcal{G}_1$ than in $\mathcal{G}_2$, even though this information is not provided by the path length matrices for these graphs. The path length matrices are

$$A_1^{4,\star} = \begin{bmatrix} 0\ 2\ 1\ 1\ 1 \\ 2\ 0\ 1\ 1\ 1 \\ 1\ 1\ 0\ 2\ 2 \\ 1\ 1\ 2\ 0\ 2 \\ 1\ 1\ 2\ 2\ 0 \end{bmatrix}, \qquad A_2^{2,\star} = \begin{bmatrix} 0\ 2\ 1 \\ 2\ 0\ 1 \\ 1\ 1\ 0 \end{bmatrix}. \tag{3}$$

On the other hand, the second powers of the above adjacency matrices are

$$A_1^2 = \begin{bmatrix} 3\ 3\ 0\ 0\ 0 \\ 3\ 3\ 0\ 0\ 0 \\ 0\ 0\ 2\ 2\ 2 \\ 0\ 0\ 2\ 2\ 2 \\ 0\ 0\ 2\ 2\ 2 \end{bmatrix}, \qquad A_2^2 = \begin{bmatrix} 1\ 1\ 0 \\ 1\ 1\ 0 \\ 0\ 0\ 2 \end{bmatrix}.$$

Remark 2 For weighted graphs, the interpretation of the entry $a_{ij}^{(k)}$ of the matrix A^k has to be modified. Indeed, $a_{ij}^{(k)}$ yields the sum of all products of all weights of the edges in the walks of length k between the vertices v_i and v_j. (In the unweighted case, any product of all weights of the edges in a walk is 1.) Remark 1 holds true, while Proposition 1 does not.

3 Measures that depend on the path length matrix

Let for now the graph $\mathcal{G}$ be undirected.

3.1 Closeness centrality

Let the graph $\mathcal{G}$ be connected. Then the reciprocal of the sum of all lengths of the shortest paths starting from vertex v_i,

$$c_i = \frac{1}{\sum_{j \neq i} a_{ij}^{(n-1,\star)}} = [A^{n-1,\star} \mathbf{1}_i]^{-1},$$

where $\mathbf{1}_i \in \mathbb{R}^n$ denotes the vector with all zero entries except for the ith entry, which is one, is referred to as the *closeness centrality* of v_i in $\mathcal{G}$; see, e.g., [2]. This measure gives a large value to vertices that have small shortest path distances to the other vertices of the graph.

3.2 The radius and center of a network

The maximum length over all shortest paths starting from vertex v_i in a weighted or unweighted connected graph $\mathcal{G}$, that is to say,

$$e_i = \max_{j \neq i} a_{ij}^{(n-1,\star)},$$

is commonly referred to as the *eccentricity* of the vertex v_i. The diameter may be seen as the maximum eccentricity among the vertices of the network. The radius $r_\mathcal{G}$ of $\mathcal{G}$ is the minimum eccentricity of a vertex. One has

$$r_\mathcal{G} = \min_i \max_{j \neq i} a_{ij}^{(n-1,\star)}.$$

It is shown in [3] that

$$r_\mathcal{G} \leq d_\mathcal{G} \leq 2r_\mathcal{G}.$$

A vertex is said to be *central* if its eccentricity is equal to the radius of the graph. The *center* of the graph is the set of all central vertices; see, e.g., [8].

3.3 Average shortest path length

The average shortest path length of a weighted or unweighted connected graph $\mathcal{G}$ computed over all possible pairs of vertices in the network [2] is given by

$$a_\mathcal{G} = \frac{1}{n(n-1)} \sum_{i,j \neq i} a_{ij}^{(n-1,\star)} = \frac{1}{n(n-1)} \mathbf{1}^T A^{n-1,\star} \mathbf{1},$$

where $\mathbf{1} \in \mathbb{R}^n$ denotes the vector with all entries one. Let the graph $\mathcal{G}$ be unweighted and be formed by a path of n vertices. Then the largest average shortest path length is $a_\mathcal{G} = (n+1)/3$. If $a_\mathcal{G}$ scales logarithmically with n, then $\mathcal{G}$ displays the "small-world phenomenon"; see, e.g., [8].

3.4 Harmonic centrality and global efficiency

The *efficiency* of a path between any two vertices of a weighted or unweighted graph $\mathcal{G}$ is defined as the inverse of the length of the path. The sum of the inverses of the length of all shortest paths starting from vertex v_i, i.e., the sum of the efficiencies of all shortest paths starting from v_i,

$$h_i = \sum_{j \neq i} [a_{ij}^{(n-1,\star)}]^{-1}, \tag{4}$$

is referred to as the *harmonic centrality* of v_i; see, e.g., [2]. The latter measure gives a large centrality to vertices v_i that have small shortest path distances to the other vertices of the graph. Harmonic centrality will play a central role in our analysis. We define the *h-center* of a graph as the set of all vertices with the largest harmonic centrality.

$\mathcal{G}$	$d_{\mathcal{G}}$	$r_{\mathcal{G}}$	$a_{\mathcal{G}}$	$e_{\mathcal{G}}$
$\mathcal{G}_1$	2	2	1.40	0.80
$\mathcal{G}_2$	2	1	1.33	0.83

Table 1 Example 2. Diameter, radius, average shortest path length, and global efficiency of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ depicted in Figures 1-2. Note that the average shortest path length of $\mathcal{G}_2$ is maximal; indeed, $(n+1)/3 = 1.33$.

If the graph is connected, then the average shortest path efficiency over all possible pairs of vertices, also known as the *average inverse geodesic length*, is referred to as the *global efficiency* of the graph [2]:

$$e_{\mathcal{G}} = \frac{1}{n(n-1)} \sum_{i,j \neq i} [a_{ij}^{(n-1,\star)}]^{-1}. \tag{5}$$

Finally, we remark that in the context of molecular chemistry, the sum of reciprocals of distances between all pairs of vertices of an undirected and unweighted connected graph is known as the *Harary index* and the reciprocal path length matrix $A^{n-1,\star,-1}$ is referred to as the *Harary matrix* [17].

Remark 3 The measures (4) and (5) can be especially useful when the network has more than one connected component, because infinite distances do not contribute to these "harmonic" averages.

Example 2 As an illustration of the above measures, consider again the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ of Example 1. Table 1 reports the diameter, radius, average shortest path length, and global efficiency of these graphs. Notice that the center of $\mathcal{G}_1$ is given by the set of all vertices and the center of $\mathcal{G}_2$ is made up of the vertex v_3, only. The h-center of $\mathcal{G}_1$ is formed by the vertices v_1 and v_2, and the h-center of $\mathcal{G}_2$ is given by the vertex v_3, only. Table 2 shows the eccentricity, harmonic centrality, and closeness centrality of all the vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$. All the measures in this example are computed by using the path length matrices (3).

$v_i(\mathcal{G}_1)$	e_i	h_i	c_i
1	2	3.50	0.20
2	2	3.50	0.20
3	2	3.00	0.17
4	2	3.00	0.17
5	2	3.00	0.17

$v_i(\mathcal{G}_2)$	e_i	h_i	c_i
1	2	1.50	0.33
2	2	1.50	0.33
3	1	2.00	0.50

Table 2 Example 2. Eccentricity, harmonic centrality, and closeness centrality of the vertices of $\mathcal{G}_1$ (left table) and of $\mathcal{G}_2$ (right table).

3.5 Harmonic K-centrality and global K-efficiency

When one considers shortest paths that are made up of at most K edges, the matrix $A^{K,\star,-1}$ takes the role of the reciprocal path length matrix $A^{n-1,\star,-1}$. We are in a position to introduce the *harmonic K-centrality* of the vertex v_i. It is given by

$$h_i^K = \sum_{j \neq i} [a_{ij}^{(K,\star)}]^{-1}.$$

The *global K-efficiency* of a graph $\mathcal{G}$ is defined by

$$e_{\mathcal{G}}^K = \frac{1}{n(n-1)} \sum_{i,j \neq i} [a_{ij}^{(K,\star)}]^{-1}.$$

The set of all vertices of a graph $\mathcal{G}$ with the largest harmonic K-centrality is referred to as the h^K-*center* of the graph.

$v_i(\tilde{\mathcal{G}}_2)$	e_i	h_i	c_i
1	1.50	1.67	0.40
2	1.50	2.67	0.50
3	1.00	3.00	0.67

Table 3 Example 3. Eccentricity, harmonic centrality, and closeness centrality of the vertices of $\tilde{\mathcal{G}}_2$.

3.6 Out-centrality versus in-centrality

Let the graph $\mathcal{G}$ be directed. Then the above centrality measures (closeness, eccentricity, harmonic K-centrality, and harmonic K-centrality) determine the importance of a vertex v_j by taking into account the paths that start at v_j. These measures therefore may be considered measures of out-centrality. One also may be interested in measuring the importance of a vertex v_j by considering the paths that end at it, that is to say by measuring the in-centrality of v_j. This can be achieved by replacing the path length matrix $A^{n-1,\star}$ in the measures mentioned by its transpose. This allows us to introduce the measures *closeness in-centrality*, *in-eccentricity*, *harmonic in-centrality*, and *harmonic K_{in}-centrality*. These measures are defined as

$$c_j^{\mathrm{in}} = \frac{1}{\sum_{i \neq j} a_{ij}^{(n-1,\star)}}, \quad e_j^{\mathrm{in}} = \max_{i \neq j} a_{ij}^{(n-1,\star)}, \quad h_j^{\mathrm{in}} = \sum_{i \neq j} [a_{ij}^{(n-1,\star)}]^{-1}, \quad h_j^{K_{\mathrm{in}}} = \sum_{i \neq j} [a_{ij}^{(K,\star)}]^{-1}.$$

We also define the *in-radius* of $\mathcal{G}$,

$$r_{\mathcal{G}}^{\mathrm{in}} = \min_j \max_{i \neq j} a_{ij}^{(n-1,\star)}.$$

Finally, the vertex v_j is said to be *in-central* if its in-eccentricity equals the in-radius of the graph. The $h^{K_{\mathrm{in}}}$-*center* of a graph $\mathcal{G}$ is the set of all vertices with the largest harmonic K_{in}-centrality. The notions of $h^{K_{\mathrm{in}}}$-center and $h^{K_{\mathrm{out}}}$-center will be of interest in the sequel.

4 Enhancing network communication

The diameter of a weighted or unweighted graph provides a measure of how easy it is for the vertices of the graph to communicate. When graphs are used as models for communication networks, the diameter plays an important role in the performance analysis and cost optimization. A simple way to decrease the diameter of a graph so that information can be transmitted more easily between vertices of the graph [4] is to decrease the weight of an edge that belongs to all maximal shortest paths, if feasible.

Example 3 Consider the graph $\mathcal{G}_2$ in Example 1. The edge $e(v_2 \leftrightarrow v_3)$ belongs to the maximal shortest path and is a *bridge*, that is its removal would make the vertex v_2 unreachable and the perturbed graph $\tilde{\mathcal{G}}_2$ so-obtained disconnected. In detail, the edge $e(v_2 \leftrightarrow v_3)$ belongs to the shortest path between the vertices v_1 and v_2, which is the only shortest path of length $d_{\mathcal{G}}$. Thus, decreasing the weight of the edge $e(v_2 \leftrightarrow v_3)$ decreases both the average shortest path length and the diameter of the graph. Specifically, if one decreases the weights a_{23} and a_{32} from 1 to 0.5, one obtains the matrices

$$\tilde{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0.5 \\ 1 & 0.5 & 0 \end{bmatrix}, \qquad \tilde{A}_2^{2,\star} = \begin{bmatrix} 0 & 1.5 & 1 \\ 1.5 & 0 & 0.5 \\ 1 & 0.5 & 0 \end{bmatrix}.$$

The perturbed graph $\tilde{\mathcal{G}}_2$ has diameter $d_{\tilde{\mathcal{G}}_2} = 1.5$, radius $r_{\tilde{\mathcal{G}}_2} = 1$, average shortest path length $a_{\tilde{\mathcal{G}}_2} = 1$, and global efficiency $e_{\tilde{\mathcal{G}}_2} = 1.22$. The eccentricity, harmonic centrality, and closeness centrality of the vertices of the graph $\tilde{\mathcal{G}}_2$ are reported in Table 3.

4.1 Increasing the global K-efficiency

We propose two approaches to increase the global efficiency of a network.

4.1.1 The function eKG1

Let for now the graph $\mathcal{G}$ be directed. The first approach is based on the observation that the most important vertices with respect to the global K-efficiency live in the vertex subsets h^{K_out}-center and h^{K_in}-center of the graph. These vertices may be interpreted as important intermediaries, that quickly collect information from many vertices and quickly broadcast it to many others vertices. Indeed, strengthening an existing connection from a vertex of the h^{K_in}-center to a vertex of the h^{K_out}-center is likely to strengthen their communicability by having new shorter paths with at most K steps that exploit these connections. This is likely to increase the global K-efficiency more than strengthening an existing connection between vertices with lower harmonic K_in- and K_out-centrality. Here we consider graphs whose edge weights represent travel times or waiting times. Hence, strengthening is achieved by *decreasing* appropriate weights.

We construct the perturbed adjacency matrix

$$\tilde{A} = A + \gamma_{h_1,h_2} \mathbf{1}_{h_1} \mathbf{1}_{h_2}^T, \quad \text{with } \gamma_{h_1,h_2} = -a_{h_1,h_2}/2, \tag{6}$$

If the graph is undirected, the above approach simplifies, because the h^{K_out}- and h^{K_in}-centers coincide. Hence, the idea is to strengthen the connection between vertices with the largest harmonic K-centrality. The perturbed adjacency matrix $\tilde{A}$ will be

$$\tilde{A} = A + \gamma_{h_1,h_2}(\mathbf{1}_{h_1} \mathbf{1}_{h_2}^T + \mathbf{1}_{h_2} \mathbf{1}_{h_1}^T). \tag{7}$$

The MATLAB function eKG1 describes the necessary computations. The operator $==$ in line 5 of the function eKG1 stands for logical equal to, and the symbol $./$ in line 6 denotes element-wise division. The function call $\mathsf{sum}(M)$ for a matrix $M \in \mathbb{R}^{n \times n}$ computes a row vector $m \in \mathbb{R}^n$, whose j^{th} component is the sum of the entries of the jth column; the function call $\mathsf{sum}(M,2)$ for a matrix $M \in \mathbb{R}^{n \times n}$ computes a column vector, whose i^{th} entry is the sum of the elements of row i of M. The blip in line 8 denotes transposition, and the operator $.*$ in line 11 stands for vector-vector element-wise product.

```
 1: function EKG1(A, flagsym, K);
 2:      n = size(A, 1);
 3:      P = pathlength_matrix(A, K);
 4:      Pr = P;
 5:      Pr(Pr == inf) = 0;
 6:      Pr(Pr > 0) = 1./Pr(Pr > 0);
 7:      H_in = sum(Pr);
 8:      H = sum(Pr, 2)';
 9:      eKG = 1/(n(n − 1)) sum(H);
10:      [∼, h] = max(H_in);
11:      [∼, k] = max(H .* A(h, :));
12:      A(h, k) = A(h, k)/2;
13:      if flagsym then
14:          A(k, h) = A(h, k);
15:      end if
16: return A, eKG;
17: end function
```

4.1.2 The function eKG2

Assume for now that the graph $\mathcal{G}$ is directed. If computing the Perron root ρ_K and the unique positive left and right eigenvectors of unit norm (the Perron vectors) of the reciprocal K-path length matrix $A^{K,\star,-1}$ is not computationally feasible or if this matrix is not irreducible, then one can use the function eKG1. However, if $A^{K,\star,-1}$ is irreducible and its left and right Perron vectors $\mathbf{x}_K = (x_{K,i})$ and

$\mathbf{y}_K = (y_{K,i})$ can be computed, then these vectors determine the Wilkinson perturbation $W_K = \mathbf{y}_K \mathbf{x}_K^T$; see [20, Section 2]. Following [7], to induce the maximal perturbation in ρ_K, one chooses the indices (h_1, h_2) such that $W_K(h_1, h_2)$ is the largest entry of W_K and $A(h_1, h_2) > 0$, i.e., the indices of the largest entry of the Wilkinson perturbation "projected" onto the zero-structure of A; see, e.g. [15]. Thus,

$$(h_1, h_2) : x_{K,h_2} \, y_{K,h_1} = (\mathbf{y}_K \mathbf{x}_K^T)_{h_1,h_2} = \max_{i,j:A(i,j)>0} (W_K)_{i,j}.$$

As in function eKG1, one strengthens the edge $e(v_{h_1} \to v_{h_2})$ by halving its weight. The perturbed adjacency matrix then is given by (6).

When the graph $\mathcal{G}$ is undirected, the left and right Perron vectors coincide and the perturbed adjacency matrix is constructed as in (7). The outlined approach is implemented by the MATLAB function eKG2. We recall that the function $\mathsf{abs}(\mathsf{v})$ of a vector v, used in the function eKG2, returns a vector, whose components are the absolute value of the components of v. The expression $\mathsf{sum}(\mathsf{sum}(Pr))$ on line 7 sums all entries of the matrix Pr. The operator $.*$ on line 20 denotes the Hadamard product of two matrices; the entries of the matrix $A > 0$ are one if the corresponding entry of A is positive; they are zero otherwise. The function $\mathsf{ind2sub}$ determines the equivalent subscript values corresponding to a given single index in an array.

```
1:  function EKG2(A, flagsym, K)
2:      n = size(A, 1);
3:      P = pathlength_matrix(A, K);
4:      Pr = P;
5:      Pr(Pr == inf) = 0;
6:      Pr(Pr > 0) = 1./Pr(Pr > 0);
7:      eKG = 1/(n(n − 1)) sum(sum(Pr));
8:      if not flagsym then
9:          [X, D, Y] = eig(Pr);
10:         [ρ, ind] = max(abs(diag(D)));
11:         x = X(:, ind);
12:         y = Y(:, ind)
13:     else
14:         [X, D] = eig(Pr);
15:         [ρ, ind] = max(abs(diag(D)));
16:         x = X(:, ind);
17:         y = x;
18:     end if
19:     W = yx';
20:     W = W.* (A > 0);
21:     [∼, lin] = max(W(:));
22:     [h, k] = ind2sub([n, n], lin);
23:     A(h, k) = A(h, k)/2;
24:     if flagsym then
25:         A(k, h) = A(h, k);
26:     end if
27: return A, eKG
28: end function
```

We expect the global K-efficiency to increase the most when decreasing the edge-weight that makes the Perron root ρ_K change the most. The perturbation of the Perron root generated by the function eKG2 typically is larger than the perturbation determined by the function eKG1. The difference in these perturbations is analogous to the difference between considering the most important vertex in a graph the one with the largest degree and the one with maximal eigenvector centrality.

Remark 4 Both functions eKG1 and eKG2 maximize lower bounds for the global K-efficiency of the network. Consider for the sake of clarity the undirected case. Let $\mathbf{h}_K$ denote the vector of the harmonic

K-centralities of the vertices of the graph. Its 1-norm is the sum in the numerator of the global K-efficiency, and its ∞-norm is what function eKG1 is maximizing. Indeed, one has

$$n(n-1)\, e_{\mathcal{G}}^{K} = \|\mathbf{h}_K\|_1 \geq \|\mathbf{h}_K\|_\infty = \|A^{K,\star,-1}\|_\infty \geq \rho_K.$$

Remark 5 The Perron root is bounded from below and from above by the minimal and maximal entries of $\mathbf{h}_K$, respectively. We have

$$\min_i \sum_{j \neq i} [a_{ij}^{(K,\star)}]^{-1} \leq \rho_K \leq \max_i \sum_{j \neq i} [a_{ij}^{(K,\star)}]^{-1} = \|\mathbf{h}_K\|_\infty. \tag{8}$$

Indeed, one has

$$\mathbf{1}^T A^{K,\star,-1} \mathbf{x}_K = \mathbf{1}^T \rho_K \mathbf{x}_K = \rho_K \|\mathbf{x}_K\|_1$$

and

$$\mathbf{1}^T A^{K,\star,-1} \mathbf{x}_K = \mathbf{x}_K^T A^{K,\star,-1} \mathbf{1} = \sum_{i=1}^{n} (x_{K,i} \sum_{j \neq i} [a_{ij}^{(K,\star)}]^{-1}),$$

so that one obtains (8) by observing that

$$\min_i \sum_{j \neq i} [a_{ij}^{(K,\star)}]^{-1} \|\mathbf{x}_K\|_1 \leq \rho_K \|\mathbf{x}_K\|_1 \leq \max_i \sum_{j \neq i} [a_{ij}^{(K,\star)}]^{-1} \|\mathbf{x}_K\|_1.$$

Example 4 We apply the functions eKG1 and eKG2 to the graphs of Example 1. For neither graph, there is a unique "best choice" to report.

First consider the graph $\mathcal{G}_1$ in Example 1. Then $A_1^{2,\star} = A_1^{3,\star} = A_1^{4,\star}$. We let $K = 2$ and obtain

$$A_1^{2,\star,-1} = \begin{bmatrix} 0 & 0.5 & 1 & 1 & 1 \\ 0.5 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0.5 & 0.5 \\ 1 & 1 & 0.5 & 0 & 0.5 \\ 1 & 1 & 0.5 & 0.5 & 0 \end{bmatrix}.$$

The vector $\mathbf{h}_2$ of harmonic 2-centralities is $\mathbf{h}_2 = [3.5, 3.5, 3, 3, 3]^T$, while the Perron vector $\mathbf{x}_2$ is given by $\mathbf{x}_2 = [0.47, 0.47, 0.43, 0.43, 0.43]^T$. This tells us that the vertices v_1 and v_2 are the most important ones in the sense of both harmonic centrality and eigenvector centrality. Indeed, these vertices are the only ones that are connected by paths of minimal length with three vertices. Thus, they are well connected. However, $A_1(1,2) = 0$.

Both functions eKG1 and eKG2 give $(h_1, h_2) = (1, 3)$, and yield the matrix

$$\tilde{A}_1 = \begin{bmatrix} 0 & 0 & 0.5 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0.5 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

even though other choices of existing edges are equally valid. The global 2-efficiency $e_{\tilde{\mathcal{G}}_1}^{2}$ of $\tilde{A}_1$ is 0.95; compare with the global 2-efficiency $e_{\mathcal{G}_1}^{2} = 0.80$ of A_1. We remark that a perturbation of another existing edge would have led to the same increase of the global 2-efficiency.

Consider the graph $\mathcal{G}_2$ in Example 1. One has

$$A_2^{2,\star,-1} = \begin{bmatrix} 0 & 0.5 & 1 \\ 0.5 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The vector of harmonic 2-centralities is $\mathbf{h}_2 = [1.5, 1.5, 2]^T$, while the Perron vector is given by $\mathbf{x}_2 = [0.54, 0.54, 0.64]^T$. Both functions eKG1 and eKG2 yield $(h_1, h_2) = (3, 1)$, even though the choice $(h_1, h_2) = (3, 2)$ is equally valid and gives the matrix

$$\tilde{A}_2 = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 1 \\ 0.5 & 1 & 0 \end{bmatrix}.$$

The global 2-efficiency of $\tilde{A}_2$ is $e_{\tilde{\mathcal{G}}_2}^2 = 1.22$, while the global 2-efficiency of the matrix A_2 is $e_{\mathcal{G}_2}^2 = 0.83$. The diameter associated with the graph determined by $\tilde{A}_2$ is only 1.5, while the diameter of the graph associated with A_2 is 2. The choice $(h_1, h_2) = (3, 2)$, which was considered in Example 3, gives the same results.

5 Numerical tests

The numerical tests reported in this section have been carried out using MATLAB R2022b on a 3.2 GHz Intel Core i7 6 core iMac. The Perron root and left and right Perron vectors for small to moderately sized graphs can easily be evaluated by using the MATLAB function eig. For large-scale graphs these quantities can be computed by the MATLAB function eigs or by the two-sided Arnoldi algorithm, introduced by Ruhe [18] and improved by Zwaan and Hochstenbach [21].

Example 5 Consider the adjacency matrix for the network *Air500* in [1]. This data set describes flight connections for the top 500 airports worldwide based on total passenger volume. The flight connections between airports are for the year from 1 July 2007 to 30 June 2008. The network is represented by a directed unweighted connected graph with 500 vertices and 24009 directed edges. The vertices of the network are the airports and the edges represent direct flight routes between two airports.

The path length matrix $A^{5,\star} = A^{499,\star}$ yields the diameter and the radius of the graph 5 and 3, respectively. The information provided by the vector of the harmonic centralities and the Perron vector for the reciprocal path length matrix $A^{5,\star,-1}$ is the same as the one given by the vector of harmonic K-centralities and the Perron vector for $A^{K,\star,-1}$ with $K = 2$; cf. Table 4. Therefore, the perturbation that increases the global K-efficiency the most also will enhance the global efficiency the most. The information provided by Table 4 suggests that the number of flights from the Frankfurt FRA Airport (vertex v_{161}) to the JFK Airport in New York (vertex v_{224}) should be doubled in order to half the wait time between these flights. Doubling the number of flights corresponds to halving the weight for the corresponding edge.

K	(h_1, h_2)	$e_{\mathcal{G}}^K$	$e_{\tilde{\mathcal{G}}}^K$
5	$(161, 224)$	0.4839	0.4856
4	$(161, 224)$	0.4839	0.4855
3	$(161, 224)$	0.4791	0.4807
2	$(161, 224)$	0.3604	0.3606

Table 4 Example 5. Indices chosen by the functions eKG1 and eKG2 and the global K-efficiency of both the given graph, $\mathcal{G}$, and the perturbed graph, $\tilde{\mathcal{G}}$, for $K = 2, 3, 4, 5$.

Example 6 This example considers an undirected unweighted connected graph $\mathcal{G}$ that represents the German highway system network *Autobahn*. The graph is available at [1]. Its 1168 vertices are German locations and its 1243 edges represent highway segments that connect them.

Let A be the adjacency matrix associated with $\mathcal{G}$. The path length matrix $A^{62,\star} = A^{1167,\star}$ shows that the diameter and the radius of $\mathcal{G}$ are 62 and 34, respectively, whereas its global efficiency equals

$6.7175 \cdot 10^{-2}$. One notices that there is only one shortest path of length 62, which connects the vertices v_{116} and v_{1154}. The diameter of the graph can be decreased by halving the weight of the edges $a_{120,116}$ and $a_{116,120}$ (since the graph is undirected), because this is the unique edge that connects the vertex v_{116} to the other vertices of the graph.

Let $\hat{A}$ denote the perturbed adjacency matrix and $\hat{\mathcal{G}}$ the corresponding graph. The global efficiency of $\hat{\mathcal{G}}$ is $6.7177 \cdot 10^{-2}$ and the diameter is 61.5. We turn to the application of the functions eKG1 and eKG2 to increasing the global efficiency. Table 5 reports the global K-efficiency (for several values of K) of the graph $\tilde{\mathcal{G}}$ associated with the adjacency matrix $\tilde{A}$ obtained by halving both entries a_{h_1,h_2} and a_{h_2,h_2} of A, computed by the function eKG1. Table 6 shows the global K-efficiency (for several values of K) of the graph $\tilde{\mathcal{G}}$ associated with the adjacency matrix $\tilde{A}$, computed by the function eKG2. Also in this example, the information provided by the vector of harmonic centralities and the Perron vector for the reciprocal path length matrix $A^{62,\star,-1}$ is exactly the same information that is provided by the vector of harmonic K-centralities and the Perron vector for $A^{K,\star,-1}$ with $K \geq 4$. We note that the latter vectors are less expensive to determine than the former. The information provided by both tables suggests that one should double the width of the highway that connects the cities of Duisburg (vertex v_{219}) and Krefeld (vertex v_{565}) to half the travel time. These cities are 10 miles apart. Doubling the width corresponds to halving the weight associated with the corresponding edge.

K	(h_1, h_2)	$e_{\mathcal{G}}^K$	$e_{\tilde{\mathcal{G}}}^K$
62	$(219, 565)$	$6.7175 \cdot 10^{-2}$	$6.7559 \cdot 10^{-2}$
52	$(219, 565)$	$6.7166 \cdot 10^{-2}$	$6.7550 \cdot 10^{-2}$
42	$(219, 565)$	$6.6965 \cdot 10^{-2}$	$6.7349 \cdot 10^{-2}$
32	$(219, 565)$	$6.5105 \cdot 10^{-2}$	$6.5485 \cdot 10^{-2}$
22	$(219, 565)$	$5.5674 \cdot 10^{-2}$	$5.6024 \cdot 10^{-2}$
12	$(219, 565)$	$2.8426 \cdot 10^{-2}$	$2.8621 \cdot 10^{-2}$
5	$(219, 565)$	$7.9991 \cdot 10^{-3}$	$8.0304 \cdot 10^{-3}$
4	$(219, 565)$	$6.1823 \cdot 10^{-3}$	$6.2019 \cdot 10^{-3}$
3	$(219, 217)$	$4.6017 \cdot 10^{-3}$	$4.6112 \cdot 10^{-3}$
2	$(219, 217)$	$3.2082 \cdot 10^{-3}$	$3.2124 \cdot 10^{-3}$

Table 5 Example 6. Indices chosen by function eKG1 and global K-efficiency of both $\mathcal{G}$ and $\tilde{\mathcal{G}}$, for $K = 2, 3, 4, 5$ and for $K = 12 : 10 : 62$.

K	(h_1, h_2)	$e_{\mathcal{G}}^K$	$e_{\tilde{\mathcal{G}}}^K$
62	$(565, 219)$	$6.7175 \cdot 10^{-2}$	$6.7559 \cdot 10^{-2}$
52	$(565, 219)$	$6.7166 \cdot 10^{-2}$	$6.7550 \cdot 10^{-2}$
42	$(565, 219)$	$6.6965 \cdot 10^{-2}$	$6.7349 \cdot 10^{-2}$
32	$(565, 219)$	$6.5105 \cdot 10^{-2}$	$6.5485 \cdot 10^{-2}$
22	$(565, 219)$	$5.5674 \cdot 10^{-2}$	$5.6024 \cdot 10^{-2}$
12	$(565, 219)$	$2.8426 \cdot 10^{-2}$	$2.8621 \cdot 10^{-2}$
5	$(565, 219)$	$7.9991 \cdot 10^{-3}$	$8.0304 \cdot 10^{-3}$
4	$(565, 219)$	$6.1823 \cdot 10^{-3}$	$6.2019 \cdot 10^{-3}$
3	$(267, 219)$	$4.6017 \cdot 10^{-3}$	$4.6111 \cdot 10^{-3}$
2	$(693, 543)$	$3.2082 \cdot 10^{-3}$	$3.2136 \cdot 10^{-3}$

Table 6 Example 6. Indices chosen by function eKG2 and global K-efficiency of both $\mathcal{G}$ and $\tilde{\mathcal{G}}$, for $K = 2, 3, 4, 5$ and for $K = 12 : 10 : 62$.

6 Concluding remarks

The adjacency matrix of a graph is a well-known tool for studying properties of a network defined by the graph. The path length matrix associated with a graph also sheds light on properties of the network,

but so far has not received much attention. A review of measures that can be defined in terms of the path length matrix is provided, and new such measures are introduced. The sensitivity of the transmission of information to perturbations of the entries of the adjacency matrix is investigated.

Data availability statement

Data sharing not is applicable to this article as no new datasets were generated during the current study.

Conflict of interest

The authors declare that they have no conflict of interest.

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