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# Partial Eigenvalue Assignment for Large Linear Control Systems

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Abstract. The eigenvalue assignment problem is a classical problem in Control Theory. This paper presents new algorithms for the stabilization of large single-input time-invariant control systems by partial eigenvalue assignment. Our algorithms are based on the implicitly restarted Arnoldi method, and are well suited for control systems that require the reassignment of a few eigenvalues.

## 1. Introduction

Consider the single-input time-invariant linear control system

$$\frac{d}{dt}x(t) = Ax(t) + bu(t), \qquad x(0) = x_0, \qquad t \geq 0, \tag{1.1}$$

where $A \in \mathbb{R}^{n \times n}$ is a large, nonsymmetric, possibly sparse, matrix, $b, x_0 \in \mathbb{R}^n$, $x(t)$ is a vector-valued function with values in $\mathbb{R}^n$ and $u(t)$ is a real-valued function. Let $\lambda(A) = \{\lambda_j\}_{j=1}^n$ denote the spectrum of $A$ and introduce the set $\mathbb{P} = \{\psi_j\}_{j=1}^m$ of $m \leq n$ complex numbers. We refer to the $\psi_j$ as poles. This paper is concerned with the problem of determining a vector $f \in \mathbb{R}^n$ with the property that

$$\lambda(A - bf^T) = \mathbb{P} \cup \{\lambda_j\}_{j=m+1}^n, \tag{1.2}$$

when such a vector exists. This problem is known as the *partial eigenvalue assignment problem*. The vector $f$ in (1.2) is referred to as the feedback gain vector, because substituting $u(t) = -f^T x(t)$ into (1.1) yields a closed-loop system with solution

$$x(t) = \exp((A - bf^T)t)x_0, \qquad t \geq 0. \tag{1.3}$$

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Assume that all but $m$ eigenvalues of the matrix $A$ have nonnegative real part. For definiteness, let

$$\mathrm{Re}(\lambda_j) \geq 0, \qquad 1 \leq j \leq m, \tag{1.4}$$

$$\mathrm{Re}(\lambda_j) < 0, \qquad m+1 \leq j \leq n. \tag{1.5}$$

If $f = 0$ in (1.3), then there are initial vectors $x_0$, such that the closed-loop solution $x(t)$ becomes unbounded as $t$ increases. This is generally undesirable. Instead, we would like to choose $f$ so that the closed-loop solution satisfies

$$\lim_{t\to\infty} x(t) = 0 \tag{1.6}$$

for any choice of initial vector $x_0$. In order to achieve this, let the poles $\psi_j$ have negative real part and choose a feedback gain vector $f$, so that (1.2) holds. Then all eigenvalues of the matrix $A - bf^T$ have negative real part and (1.6) holds independently of the choice of $x_0$.

To simplify our exposition, we consider throughout this paper the problem of replacing a set of a few eigenvalues $\{\lambda_j\}_{j=1}^m$ of $A$ with nonnegative real part by a set of poles $\mathbb{P}$ with negative real part. However, the methods presented can be used to replace other subsets of eigenvalues as well. The methods of this paper are designed for the solution of partial eigenvalue assignment problems (1.2) with $m$, the number of eigenvalues to be assigned, much smaller than $n$, the order of the matrix $A$.

The eigenvalue assignment problem (where $\mathbb{P}$ consists of $n$ poles) has received considerable attention in the literature. Analyses of the sensitivity of this problem to perturbations can be found in [**7, 8**] and discussions on how to choose the poles $\psi_j$ are presented in [**3, 9**]. Several numerical methods are available for small to medium-sized problems; see, e.g., [**1, 2, 5, 10, 11**] and references therein. Although these methods can be applied to the partial eigenvalue assignment problem for large systems, they are not computationally economical. These methods typically require $O(n^3)$ floating point operations and storage of up to $n$ vectors in $\mathbb{R}^n$. Numerical methods for large-scale partial eigenvalue assignment problems have, so far, received less attention. Saad [**12**] described a projection method based on the Arnoldi process. We review this algorithm in Section 2 and present a modification thereof based on the implicitly restarted Arnoldi method. Both Saad's method and our modification require the evaluation of matrix-vector products with the matrix $A^T$, but not with $A$. Section 3 describes a variant of the partial eigenvalue assignment problem that allows both the vectors $f$ and $b$ to be chosen in (1.2). The possibility to choose both $f$ and $b$ reduces the sensitivity of the partial eigenvalue assignment problem to perturbations. Two algorithms are presented, one that requires the evaluation of matrix-vector products with the matrix $A^T$ and one with the matrix $A$. Section 4 displays a few computed examples and Section 5 contains concluding remarks.

Finally, we remark that an eigenvalue assignment problem in which the vector $f$ is fixed and $b$ is to be determined so that (1.2) holds is discussed in [**4**]. The numerical method proposed is analogous to the method discussed in Section 2.

## 2. The partial eigenvalue assignment problem

Throughout this paper, we assume that the sets $\mathbb{P}$ and $\{\lambda_j\}_{j=m+1}^n$ in (1.2) are invariant under complex conjugation. It can be shown that the desired feedback gain vector $f$, if it exists, then has real-valued entries only.

Saad [**12**] solves the partial eigenvalue assignment problem (1.2) by computing the left invariant subspace of $A$ associated with the set of eigenvalues $\{\lambda_j\}_{j=1}^m$. Let the columns of the matrix $V \in \mathbb{R}^{n\times m}$ form an orthonormal basis for this subspace. Then

$$A^TV = VH, \qquad V^TV = I_m, \tag{2.1}$$

for some matrix $H \in \mathbb{R}^{m\times m}$ with spectrum

$$\lambda(H) = \{\lambda_j\}_{j=1}^m. \tag{2.2}$$

Here $I_m$ denotes the $m \times m$ identity matrix. We remark that the decomposition (2.1) is not unique; if the matrix pair $\{H, V\}$ satisfies (2.1) and (2.2), then so does the pair $\{U^THU, VU\}$ for any orthogonal matrix $U \in \mathbb{R}^{m\times m}$.

Let the matrix pair $\{H, V\}$ satisfy (2.1) and (2.2). Introduce the vector $\tilde{b} = V^Tb$ and consider the problem of determining a vector $\tilde{f} \in \mathbb{R}^m$, such that

$$\lambda(H - \tilde{f}\tilde{b}^T) = \mathbb{P}. \tag{2.3}$$

We refer to this eigenvalue assignment problem as the *projected eigenvalue assignment problem*. The matrix-vector pair $\{H^T, \tilde{b}\}$ is said to be controllable if

$$\mathrm{rank}(H^T - zI_m, \tilde{b}) = m, \qquad \forall z \in \mathbb{C}. \tag{2.4}$$

It is known that the projected eigenvalue assignment problem (2.3) has a solution $\tilde{f}$ for any set of poles $\mathbb{P}$, if and only if the matrix-vector pair $\{H^T, \tilde{b}\}$ is controllable; see, e.g., Wonham [**14**, Section 2.2].

The following result is a slight modification of Theorem 2.1 of Saad [**12**]. We present the proof because related arguments will be employed to show other results below.

Theorem 2.1. *Let $V$ and $H$ satisfy equations (2.1) and (2.2), and let $\tilde{b} = V^Tb$, where the vector $b$ is given by (1.1). The partial eigenvalue assignment problem (1.2) has a solution $f \in range(V)$ if and only if the projected eigenvalue assignment problem (2.3) has a solution $\tilde{f} \in \mathbb{R}^m$.*

Proof. Extend $V$ to an orthogonal matrix $X = (V, W) \in \mathbb{R}^{n\times n}$. The decomposition (2.1) and

$$W^TV = 0 \tag{2.5}$$

yield $V^TA^TV = H$ and $W^TA^TV = W^TVH = 0$. These identities show that the matrix $A^T$ is similar to the block-triangular matrix

$$X^TA^TX = \begin{pmatrix} H & V^TA^TW \\ 0 & W^TA^TW \end{pmatrix}.$$

It follows from (2.2) that

$$\lambda(W^TA^TW) = \{\lambda_j\}_{j=m+1}^n. \tag{2.6}$$

Assume that $f \in \mathrm{range}(V)$ satisfies the partial eigenvalue assignment problem (1.2) and introduce $\tilde{f} = V^T f$. The equations (2.1), (2.5) and $W^T f = 0$ yield

$$\begin{aligned} V^T(A - bf^T)^T V &= H - \tilde{f}\tilde{b}^T, \\ W^T(A - bf^T)^T V &= W^T A^T V - W^T f b^T V^T = 0, \\ W^T(A - bf^T)^T W &= W^T A^T W - W^T f b^T W = W^T A^T W \end{aligned}$$

and therefore

$$X^T(A - bf^T)^T X = \begin{pmatrix} H - \tilde{f}\tilde{b}^T & V^T A^T W - \tilde{f} b^T W \\ 0 & W^T A^T W \end{pmatrix}. \tag{2.7}$$

It follows from $\lambda(X^T(A-bf^T)^T X) = \mathbb{P} \cup \{\lambda_j\}_{j=m+1}^n$ and (2.6) that $\lambda(H - \tilde{f}\tilde{b}^T) = \mathbb{P}$. Thus, $\tilde{f}$ satisfies the projected eigenvalue assignment problem (2.3).

Conversely, assume that $\tilde{f} \in \mathbb{R}^m$ satisfies the projected eigenvalue assignment problem (2.3), where the matrix-pair $\{H, V\}$ satisfies (2.1) and (2.2). Define $f = V\tilde{f}$. Substituting (2.1) and $f$ into $X^T(A - bf^T)^T X$ yields the right-hand side of (2.7). Equations (2.3) and (2.6) yield (1.2). ■

Saad [**12**] proposed to use the Arnoldi process to determine a decomposition of the form (2.1). This approach is attractive for large-scale problems, because the matrix $A$ is only used to evaluate matrix-vector products with $A^T$. In particular, neither $A$ nor $A^T$ have to be factored or transformed. Application of $m$ steps of the Arnoldi process to the matrix $A^T$ with initial unit vector $v_1$ yields the Arnoldi decomposition

$$A^T V_m - V_m H_m = \eta_m v_{m+1} e_m^T, \tag{2.8}$$

where $V_m \in \mathbb{R}^{n \times m}$, $V_m^T V_m = I_m$, $V_m e_1 = v_1$ and $H_m \in \mathbb{R}^{m \times m}$ is an upper Hessenberg matrix with positive subdiagonal entries. The scalar $\eta_m$ in (2.8) is nonnegative. When $\eta_m > 0$, we require the vector $v_{m+1} \in \mathbb{R}^n$ to satisfy $v_{m+1}^T v_{m+1} = 1$ and $V_m^T v_{m+1} = 0$. Moreover, $e_m = (0, \ldots, 0, 1)^T \in \mathbb{R}^m$. We assume for now that $m$ is small enough so that the decomposition (2.8) with the stated properties exists.

Assume for the moment that the matrix $H_m$ and scalar $\eta_m$ in the Arnoldi decomposition (2.8) satisfy

$$\lambda(H_m) = \{\lambda_j\}_{j=1}^m, \qquad \eta_m = 0. \tag{2.9}$$

Then the matrix pair $\{H_m, V_m\}$ in (2.8) satisfies (2.1). Let $\tilde{b} = V_m^T b$, where $b$ is given by (1.1), and assume that $\tilde{f}$ solves the projected eigenvalue assignment problem $\lambda(H_m - \tilde{f}\tilde{b}^T) = \mathbb{P}$. It then follows from Theorem 2.1 that $f = V_m \tilde{f}$ solves the partial eigenvalue assignment problem (1.2). Thus, given an Arnoldi decomposition (2.8) such that (2.9) holds, we can solve the partial eigenvalue assignment problem (1.2), if a solution exists, by computing a solution to the projected eigenvalue assignment problem (2.3).

However, given an arbitrary initial vector $v_1$, application of $m$ steps of the Arnoldi process to the matrix $A^T$ is unlikely to produce a decomposition (2.8) such that (2.9) is satisfied. Several restarting strategies for the Arnoldi process have been proposed in the literature with the aim of iteratively determining an initial vector $v_1$ that gives an Arnoldi decomposition (2.8) with the property (2.9). The Implicitly Restarted Arnoldi (IRA) method, proposed by Sorensen [**13**], is one of the most effective restarting strategies. This method combines the Arnoldi process with the implicitly shifted QR algorithm to determine an Arnoldi decomposition (2.8) that satisfies (2.9).

We outline the IRA method. Let the eigenvalues $\lambda_j$ of $A$ satisfy (1.4) and (1.5), and assume for the moment that $m$ is known. We would like to determine the invariant subspace of $A^T$ associated with the set of eigenvalues $\{\lambda_j\}_{j=1}^m$. Apply $2m$ steps of the Arnoldi process to the matrix $A^T$ with initial unit vector $v_1$ with normally distributed randomly generated entries to determine the decomposition

$$A^T V_{2m} = V_{2m} H_{2m} + \eta_{2m} v_{2m+1} e_{2m}^T. \tag{2.10}$$

Compute the spectrum $\lambda(H_{2m}) = \{\mu_j\}_{j=1}^{2m}$ and assume that

$$\mathrm{Re}(\mu_1) \le \mathrm{Re}(\mu_2) \le \ldots \le \mathrm{Re}(\mu_\ell) < 0 \le \mathrm{Re}(\mu_{\ell+1}) \le \ldots \le \mathrm{Re}(\mu_{2m}).$$

Define $k = \min\{m, \ell\}$. Sorensen [**13**] describes how the decomposition (2.10) can be updated without evaluating any matrix-vector products with the matrix $A^T$ to give the Arnoldi decomposition

$$A^T \hat{V}_{2m-k} = \hat{V}_{2m-k} \hat{H}_{2m-k} + \hat{\eta}_{2m-k} \hat{v}_{2m-k+1} e_{2m-k}^T, \tag{2.11}$$

with initial vector

$$\hat{v}_1 = \hat{V}_{2m-k} e_1 = \frac{\prod_{j=1}^k (A^T - \mu_j I) v_1}{\| \prod_{j=1}^k (A^T - \mu_j I) v_1 \|}, \tag{2.12}$$

where $v_1$ is the initial vector in the Arnoldi decomposition (2.10). Throughout this paper $\| \cdot \|$ denotes the Euclidean vector norm or the associated induced matrix norm.

Thus, the new initial vector (2.12) is obtained by multiplying the original initial vector $v_1$ by a polynomial in $A^T$. The zeros of this polynomial are Ritz values of $A^T$ with negative real part. The purpose of multiplying $v_1$ by this polynomial is to remove, or at least reduce, eigenvector components of $A^T$ in $v_1$ associated with eigenvalues of $A^T$ with negative real part, and thereby force the new initial vector (2.12) into the invariant subspace of $A^T$ associated with the set of eigenvalues $\{\lambda_j\}_{j=1}^m$ with nonnegative real part. If $\mathrm{range}(\hat{V}_{2m-k})$ does not contain a sufficiently good approximation of this subspace, then we apply $k$ steps of the Arnoldi process to obtain the Arnoldi decomposition

$$A^T \hat{V}_{2m} = \hat{V}_{2m} \hat{H}_{2m} + \hat{\eta}_{2m} \hat{v}_{2m+1} e_{2m}^T$$

from (2.11). This decomposition is of the same form as (2.10), and we update it in the same fashion as (2.10). We proceed in this manner until a sufficiently accurate approximation of the invariant subspace of $A^T$ associated with the set of eigenvalues $\{\lambda_j\}_{j=1}^m$ has been determined. Since, in general, the number $m$ of eigenvalues with positive real part is not known a priori, we also seek to determine $m$ during the iterations with the IRA method.

A detailed discussion of the IRA method is given by Sorensen [**13**] and more recently by Lehoucq et al. [**6**]. In our numerical examples of Section 4, we used a simple implementation based on the description of the IRA method in [**13**]. The application of the IRA method for partial eigenvalue assignment is outlined in the following algorithm.

**Algorithm 1** *Partial eigenvalue assignment using the IRA method.*

1. Apply the IRA method to the matrix $A^T$ to compute an Arnoldi decomposition of the form (2.8), such that $\eta_m \geq 0$ is tiny and $\lambda(H_m)$ is a good approximation of the set $\{\lambda_j\}_{j=1}^m$ of eigenvalues of $A$. The range of the matrix $V_m$ in the Arnoldi decomposition provides an approximation of the invariant subspace of $A^T$ associated with $\{\lambda_j\}_{j=1}^m$.
2. Select a set of poles $\mathbb{P} = \{\psi_j\}_{j=1}^m$. It may be attractive to use the pole selection method presented in [**3**]. The selection of poles will be addressed further in Section 4.
3. Let $\tilde{b} = V_m^T b$ and solve the projected eigenvalue assignment problem (2.3). Denote the solution by $\tilde{f}$.
4. Let $f = V_m \tilde{f}$. The vector $f$ is the computed solution of the partial eigenvalue assignment problem (1.2).

The computation of $\tilde{f}$ in Step 3 of Algorithm 1 can be accomplished by any numerical method for eigenvalue assignment for small matrices. In the numerical examples of Section 4 we used the method of Datta [**5**].

Algorithm 1 is analogous to a projection algorithm proposed by Saad [**12**]. It differs from the latter in its use of the IRA method. The IRA method is often very effective at determining the eigenvalues $\{\lambda_j\}_{j=1}^m$ and the associated invariant subspace of $A^T$.

However, it is possible that the IRA method converges too quickly, in the sense that it determines an invariant subspace associated with some, but not all, of the eigenvalues in the set $\{\lambda_j\}_{j=1}^m$. We address this situation by restarting the IRA method with a new random initial vector, that is orthogonalized against the invariant subspace of $A^T$ already determined. In fact, in order to avoid convergence to the invariant subspace already determined, we orthogonalize all columns of the matrices $V_m$ generated after a restart against the invariant subspace of $A^T$ already determined. The IRA method is restarted repeatedly in this manner until all eigenvalues in the set $\{\lambda_j\}_{j=1}^m$, characterized by (1.4), and the associated invariant subspace of $A^T$ have been computed to sufficient accuracy.

We remark that this approach does not require the number of eigenvalues of $A$ with nonnegative real part be known a priori; we restart the IRA method repeatedly until no more eigenvalues with nonnegative real part are found. This approach has reliably determined all eigenvalues with nonnegative real part in a large number of numerical experiments.

## 3. The modified partial eigenvalue assignment problem

In our experience Algorithm 1 performs very well for many large partial eigenvalue assignment problems. Nevertheless, the approach described in Section 2 can fail because the partial eigenvalue assignment problem (1.2) is not solvable for certain vectors $b$. This section introduces a variant of the partial eigenvalue assignment problem that avoids this difficulty by allowing both vectors $f$ and $b$ to be chosen. Thus, we seek to determine vectors $f, b \in \mathbb{R}^n$, such that

$$\lambda(A - bf^T) = \mathbb{P} \cup \{\lambda_j\}_{j=m+1}^n. \tag{3.1}$$

We refer to this problem as the *modified partial eigenvalue assignment problem.* The following theorem establishes that this assignment problem has a solution.

THEOREM 3.1. *Let $V \in \mathbb{R}^{n\times m}$ and the upper Hessenberg matrix $H \in \mathbb{R}^{m\times m}$ with nonvanishing subdiagonal entries satisfy (2.1) and (2.2). Then there are vectors $f, b \in$ range($V$) that solve the modified partial eigenvalue assignment problem (3.1).*

PROOF. Theorem 2.1 established that the existence of a solution $f \in \mathrm{range}(V)$ is equivalent to the existence of a solution $\tilde{f}$ of the projected eigenvalue assignment problem (2.3) with $\tilde{b} = V^T b$. We will show that for a suitable choice of $\tilde{b}$, the projected eigenvalue assignment problem (2.3) has a solution $\tilde{f}$. The proof shows how the vectors $\tilde{f}$ and $\tilde{b}$ can be computed. The construction used in the proof to determine the vectors $\tilde{f}$ and $\tilde{b}$ has previously been employed by Datta [**5**].

Let $r_1 = e_1$ and define the vectors

$$r_{j+1} = (H - \psi_j I) r_j, \qquad 1 \le j \le m. \tag{3.2}$$

The matrix $R = (r_1, r_2, \ldots, r_m)$ is upper triangular with diagonal entries

$$e_k^T R e_k = \prod_{j=1}^{k-1} h_{j+1,j}, \qquad 1 \le k \le m,$$

where $e_1^T R e_1 = 1$. Since $H$ has nonvanishing subdiagonal entries, the matrix $R$ is nonsingular. Its inverse $R^{-1}$ is upper triangular with diagonal entries

$$e_k^T R^{-1} e_k = \prod_{j=1}^{k-1} h_{j+1,j}^{-1}, \qquad 1 \le k \le m.$$

Introduce the lower bidiagonal matrix

$$B = \begin{pmatrix} \psi_1 & 0 & 0 & \cdots & 0 \\ 1 & \psi_2 & 0 & \cdots & 0 \\ 0 & 1 & \psi_3 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \psi_m \end{pmatrix}.$$

with spectrum $\lambda(B) = \mathbb{P}$. The recurrence formula (3.2) yields

$$HR - RB = r_{m+1} e_m^T$$

and, therefore,

$$H - r_{m+1} e_m^T R^{-1} = RBR^{-1}.$$

Thus,

$$\lambda(H - r_{m+1} e_m^T R^{-1}) = \mathbb{P}. \tag{3.3}$$

Note that $e_m^T R^{-1} = (e_m^T R e_m)^{-1} e_m^T$. Letting $\tilde{f} = (e_m^T R e_m)^{-1} r_{m+1}$ and $\tilde{b} = e_m$ transforms (3.3) into the form (2.3). Finally, it follows from Theorem 2.1 that $f = V\tilde{f}$ and $b = V\tilde{b}$ satisfy (1.2). ∎

We remark that since the subdiagonal entries of the upper Hessenberg matrix $H$ in Theorem 3.1 are nonvanishing, it is easy to verify that the matrix-vector pair $\{H^T, e_m\}$ in the proof of the theorem is controllable; cf. (2.4). Also, note that the solution $\{f, b\}$ of the modified partial eigenvalue assignment problem is not unique.

THEOREM 3.2. *Let the matrices $H_m$ and $V_m$, the scalar $\eta_m$ and the vector $v_{m+1}$ make up an Arnoldi decomposition (2.8). Determine the vector $\tilde{f} \in \mathbb{R}^m$, so that $\lambda(H_m - \tilde{f}e_m^T) = \mathbb{P}$. Let $b = V_m e_m$ and $f = V_m\tilde{f} + \eta_m v_{m+1}$. Then*

$$\mathbb{P} \subset \lambda(A - bf^T). \tag{3.4}$$

PROOF. Since $H_m$ is an upper Hessenberg matrix with nonvanishing subdiagonal elements, a vector $\tilde{f}$ with the desired property can be computed as described in the proof of Theorem 3.1. Extend $V_m$ to an orthogonal matrix $X = (V_m, W) \in \mathbb{R}^{n\times n}$. We obtain with the specified choices of $f$ and $b$ that

$$\begin{aligned} V_m^T(A - bf^T)^T V_m &= H_m - \tilde{f}e_m^T \\ W^T(A - bf^T)^T V_m &= W^T(A^T V_m - fb^T V_m) \\ &= W^T(V_m H_m + \eta_m v_{m+1}e_m^T - (V_m\tilde{f} + \eta_m v_{m+1})e_m^T) = 0, \end{aligned}$$

and therefore

$$X^T(A - bf^T)X = \begin{pmatrix} H_m - \tilde{f}e_m^T & V_m^T A^T W \\ 0 & W^T A^T W \end{pmatrix}.$$

This shows (3.4). ∎

Note that the inclusion (3.4) holds regardless of the size of $\eta_m \geq 0$, although the remaining eigenvalues of $A - bf^T$ may vary as $\eta_m$ increases. This depends on that spectrum $\lambda(W^T AW)$ may depend on $\eta_m$. We recall that $\lambda(W^T AW) = \{\lambda_j\}_{j=m+1}^n$ when $\eta_m = 0$.

**Algorithm 2** *Modified partial eigenvalue assignment using the IRA method applied to $A^T$.*

1. Apply the IRA method to the matrix $A^T$ to compute an Arnoldi decomposition of the form (2.8), such that $\eta_m \geq 0$ is small and $\lambda(H_m)$ is a good approximation of the set $\{\lambda_j\}_{j=1}^m$ of eigenvalues of $A$. The range of the matrix $V_m$ in the Arnoldi decomposition provides an approximation of the invariant subspace of $A^T$ associated with $\{\lambda_j\}_{j=1}^m$.
2. Select a set of poles $\mathbb{P} = \{\psi_j\}_{j=1}^m$, e.g., by the method described in [**3**].
3. Let $b = V_m e_m$, and compute $\tilde{f} \in \mathbb{R}^m$ so that $\lambda(H_m - \tilde{f}e_m^T) = \mathbb{P}$.
4. Let $f = V_m\tilde{f} + \eta_m v_{m+1}$. The vector pair $\{f, b\}$ is the computed solution of the modified partial eigenvalue assignment problem (3.1).

The vector $\tilde{f}$ in Step 3 of Algorithm 2 can be computed by any numerical method for eigenvalue assignment for small matrices. We use the method by Datta [**5**], i.e., the method in the proof of Theorem 3.1, in the numerical examples of Section 4.

We have found that Algorithm 2 often gives higher accuracy than Algorithm 1 in the sense that $\lambda(A - bf^T)$ is closer to the set $\mathbb{P} \cup \{\lambda_j\}_{j=m+1}^n$ when $f$ and $b$ are computed by Algorithm 2 than when $f$ is computed by Algorithm 1. A reason for this may be that when $\eta_m > 0$ and the vector $f$ is computed by Algorithm 1, one cannot expect $\mathbb{P}$ to be a subset of the spectrum of $A - bf^T$. On the other hand, when both vectors $f$ and $b$ are computed by Algorithm 2, Theorem 3.2 guarantees that (3.4) holds independently of the size of $\eta_m$.

Both Algorithms 1 and 2 require the evaluation of matrix-vector products with the matrix $A^T$. However, for some problems it may be possible to evaluate matrix-vector products with the matrix $A$ much faster than with $A^T$. This situation arises, for instance, when $A$ is large and sparse, and the sparse storage scheme used to represent $A$ allows much faster evaluation of matrix-vector products with $A$ than with $A^T$.

Since Algorithm 2 determines both the vectors $f$ and $b$, the algorithm can be modified to use an Arnoldi decomposition of the matrix $A$ instead of $A^T$. The matrix-vector product evaluations with the matrix $A^T$ of Algorithm 2 can then be replaced by matrix-vector product evaluations with the matrix $A$. Before describing the algorithm, we present analogues of Theorems 3.1 and 3.2.

THEOREM 3.3. *Let $V \in \mathbb{R}^{n\times m}$ and the upper Hessenberg matrix $H \in \mathbb{R}^{m\times m}$ with nonvanishing subdiagonal entries satisfy*

$$AV = VH, \qquad V^TV = I_m,$$

*and (2.2). Then there are vectors $f, b \in$ range$(V)$ that solve the modified partial eigenvalue assignment problem (3.1).*

PROOF. The proof differs from the proof of Theorem 3.1 only in that the vectors $\tilde{f}$ and $\tilde{b}$, as well as the vectors $f$ and $b$, are interchanged. ∎

THEOREM 3.4. *Consider the Arnoldi decomposition*

$$AV_m - V_mH_m = \eta_m v_{m+1}e_m^T, \tag{3.5}$$

*where $V_m \in \mathbb{R}^{n\times m}$, $V_m^TV_m = I_m$, $H_m \in \mathbb{R}^{m\times m}$ is an upper Hessenberg matrix with positive subdiagonal entries, and $\eta_m$ is a nonnegative scalar. Determine the vector $\tilde{b} \in \mathbb{R}^m$, such that $\lambda(H_m - \tilde{b}e_m^T) = \mathbb{P}$. Let $f = V_me_m$ and $b = V_m\tilde{b} + \eta_m v_{m+1}$. Then*

$$\mathbb{P} \subset \lambda(A - bf^T). \tag{3.6}$$

PROOF. The proof differs from the proof of Theorem 3.2 only in that the vectors $\tilde{f}$ and $\tilde{b}$, as well as the vectors $f$ and $b$, are interchanged. ∎

**Algorithm 3** *Modified partial eigenvalue assignment using the IRA method applied to $A$.*

1. Apply the IRA method to the matrix $A$ to compute an Arnoldi decomposition of the form (3.5), such that $\eta_m \geq 0$ is small and $\lambda(H_m)$ is a good approximation of the set $\{\lambda_j\}_{j=1}^m$ of eigenvalues of $A$. The range of the matrix $V_m$ in the Arnoldi decomposition provides an approximation of the invariant subspace of $A$ associated with $\{\lambda_j\}_{j=1}^m$.
2. Select a set of poles $\mathbb{P} = \{\psi_j\}_{j=1}^m$, e.g., by the method described in [**3**].
3. Let $f = V_me_m$, and compute $\tilde{b} \in \mathbb{R}^m$ so that $\lambda(H_m - \tilde{b}e_m^T) = \mathbb{P}$.
4. Let $b = V_m\tilde{b} + \eta_m v_{m+1}$. The vector pair $\{f, b\}$ is the computed solution of the modified partial eigenvalue assignment problem (3.1).

## 4. Numerical experiments

The computations reported in this section were carried out on an Intel Pentium workstation using Matlab 5.3 and floating point arithmetic with 16 significant digits. Our computed examples display the performance of Algorithms 1 and 2 when applied to a matrix $A \in \mathbb{R}^{2000\times 2000}$, which we define by its spectral factorization,

$$A = S\Lambda S^{-1}, \qquad \Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_{2000}].$$

The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_5$ are distributed in the disk in the complex plane with center $\frac{1}{2}$ and radius $\frac{1}{2}$. Thus, they have nonnegative real part. The remaining eigenvalues are allocated in the open disk with center 1 and radius 1, so that their distances to the center of the disk are uniformly distributed and their angles with the positive real axis are uniformly distributed in the interval $[-\pi, \pi]$. These eigenvalues have negative real part. We require all eigenvalues to be real or appear in complex conjugate pairs. Thus, $m = 5$ in (1.4). The eigenvalues of $A$ are marked by the symbol $\times$ in the figures of this section. The eigenvector matrix $S$ of $A$ has complex conjugate columns, with entries with real and imaginary parts uniformly distributed in the interval $[0, 1]$.

In Step 1 of Algorithms 1 and 2, we determine several Arnoldi decompositions of the form (2.10) and (2.11). These decompositions enable us to compute estimates of $\max\{t : t \in \mathrm{Re}(\lambda(A)), t < 0\}$ and $\max\{t : t \in \mathrm{Im}(\lambda(A))\}$. Denote the computed estimates by $\alpha$ and $\beta$, respectively. In Step 2 of Algorithms 1 and 2, we choose a set of poles $\mathbb{P} = \{\psi_j\}_{j=1}^m$ on an interval $\mathbb{I}_{\hat{\alpha},\beta} = [\hat{\alpha} + i\beta, \hat{\alpha} - i\beta]$ in the complex plane, for some $\hat{\alpha} \le \alpha$, where $i = \sqrt{-1}$. The poles are allocated by using the formulas [**3**, (3.2)-(3.3)]. This choice of poles approximately minimizes the condition number of the eigenvector matrix of $H - \tilde{f}\tilde{b}^T$ over all sets of $m$ distinct poles in the interval $\mathbb{I}_{\alpha,\beta}$. Available bounds for the sensitivity of the projected eigenvalue assignment problem (2.3) to perturbations grow with the condition number of the eigenvector matrix of $H_m - \tilde{f}\tilde{b}^T$; see [**3, 7, 9**]. It is therefore advantageous to allocate the poles so that this condition number is not very large.

The vector $b$ in (1.1) used in our experiments has normally distributed random entries with zero mean, normalized so that $\|b\| = 1$. This vector is only required by Algorithm 1. However, we also use it as starting vector for the initial Arnoldi decomposition in Step 1 of Algorithm 2. Then Step 1 of Algorithm 1 and Step 1 of Algorithm 2 give the same Arnoldi decomposition.

We wish to determine how close the eigenvalues of the matrix $A - bf^T$ are to the set $\mathbb{P} \cup \{\lambda_j\}_{j=m+1}^n$. The difference between the sets $\lambda(A - bf^T)$ and $\mathbb{P} \cup \{\lambda_j\}_{j=m+1}^n$ is measured by the following metric, defined for compact sets $\mathbb{F}$ and $\mathbb{G}$ in the complex plane,

$$d(\mathbb{F}, \mathbb{G}) = \max \left\{ \max_{z \in \mathbb{F}} \mathrm{dist}(z, \mathbb{G}), \max_{\zeta \in \mathbb{G}} \mathrm{dist}(\zeta, \mathbb{F}) \right\},$$

where $\mathrm{dist}(z, \mathbb{G}) = \min_{\zeta \in \mathbb{G}} |z - \zeta|$.

Figure 1 illustrates the performance of Algorithm 2, which computes both of the vectors $f$ and $b$. After two restarts in the IRA method, the set of poles $\mathbb{P}$ is a subset of $\lambda(A - bf^T)$ to graphing accuracy. Figure 1 shows that the eigenvalues of $A - bf^T$ with negative real part agree well with the eigenvalues of $A$ with negative real part.

Figure 2 shows the eigenvalues of the matrix $A - bf^T$ computed by Algorithm 1 when the IRA method is restarted twice in Step 1. Figure 2 displays that the partial eigenvalue assignment problem is not solved successfully, in the sense that the computed vector $f$ does not produce a matrix $A - bf^T$ with spectrum close to $\mathbb{P} \cup \{\lambda_j\}_{j=m+1}^n$. Moreover, the computed matrix $A - bf^T$ has eigenvalues with positive real part. Algorithm 1 required four restarts in the IRA method to determine a vector $f$, such that all eigenvalues of $A - bf^T$ have negative real part; see Figure 3. Comparing Figures 3 and 1 shows that four restarts in the IRA method in Step 1 of Algorithm 1 give a matrix $A - bf^T$ with a spectrum that is not as close to the set

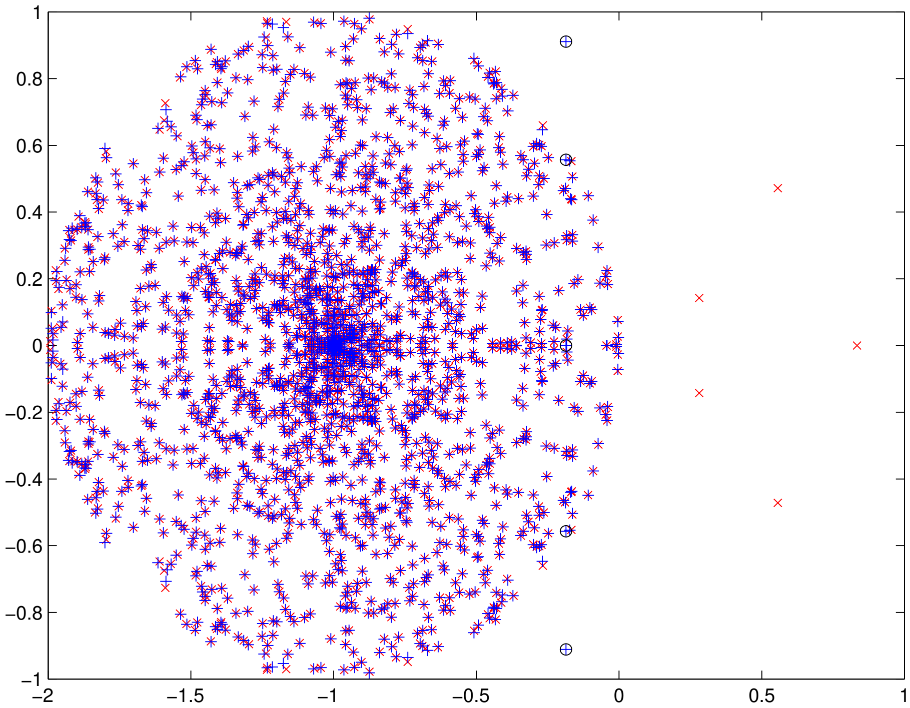


FIGURE 1. Solution determined by Algorithm 2 after two restarts in Step 1. The eigenvalues $\lambda_j$ of $A$ are marked by $\times$, the poles $\psi_j$ by $\circ$, and the eigenvalues of $A - bf^T$ by $+$.

$\mathbb{P} \cup \{\lambda_j\}_{j=m+1}^{n}$ as the spectrum of the analogous matrix determined by Algorithm 2 with only two restarts in the IRA method in Step 1. Table 1 summarizes the numerical results.

For clarity of the presentation, we omitted a few details in the above description of the experiments, such as the determination of the constants $\alpha$, $\hat{\alpha}$ and $\beta$. How these details are carried out may depend on the size of the matrix $A$. The matrix used for the computations reported in this section is small enough to allow storage of over 30 $n$-vectors in fast memory. We therefore in Step 1 of Algorithms 1 and 2 carried out 30 steps of the Arnoldi process to obtain an Arnoldi decomposition (2.8) with $m = 30$. The upper Hessenberg matrix $H_{30}$ so determined had 25 eigenvalues $\mu_1, \mu_2, \ldots, \mu_{25}$ with negative real part and five eigenvalues $\mu_{26}, \mu_{27}, \ldots, \mu_{30}$ with positive real part, and we let

$$\hat{\alpha} < \alpha = \max_{1 \leq j \leq 25} \mathrm{Re}(\mu_j), \qquad \beta = \max_{1 \leq j \leq 30} \mathrm{Im}(\mu_j).$$

The constant $\hat{\alpha}$, which together with $\beta$ determines the interval $I_{\hat{\alpha},\beta}$ on which the poles were allocated, was chosen slightly smaller that $\alpha$; see Figures 1-3. The value of $\hat{\alpha}$ was the same in all experiments.

We applied the eigenvalues $\mu_1, \mu_2, \ldots, \mu_{25}$ as shifts in the IRA method, as described in Section 2, to obtain an Arnoldi decomposition (2.8) with $m = 5$. The columns of the matrix $V_5$ in this Arnoldi decomposition did not span an invariant

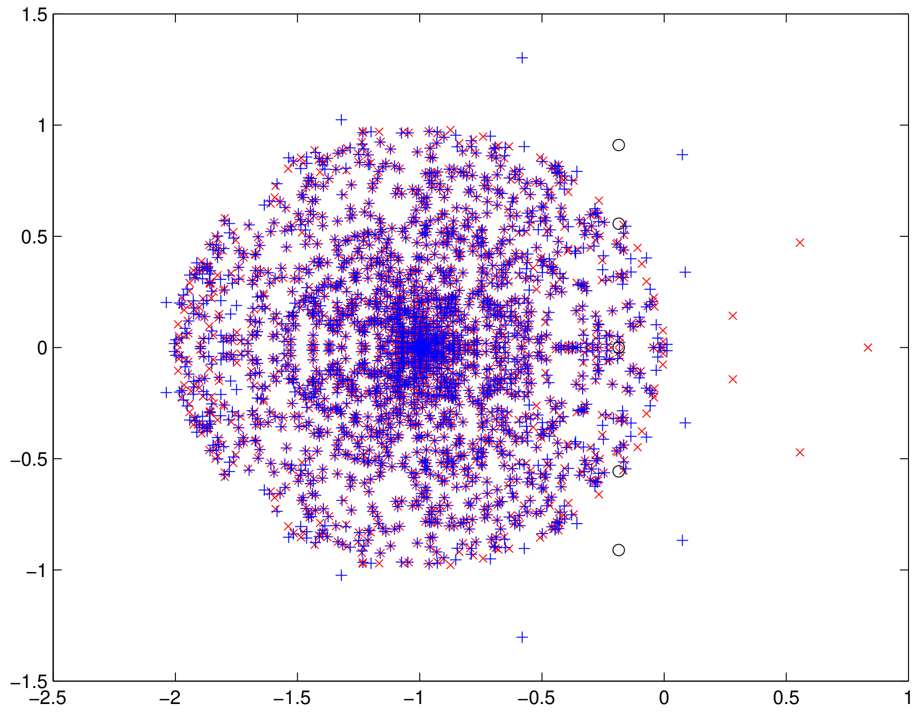


FIGURE 2. Solution determined by Algorithm 1 after two restarts in Step 1. The eigenvalues $\lambda_j$ of $A$ are marked by $\times$, the poles $\psi_j$ by $\circ$, and the eigenvalues of $A - bf^T$ by $+$.

subspace of $A$ associated with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_5$ with positive real part to desired accuracy. We therefore restarted the Arnoldi process as described in Section 2 several times until an invariant subspace associated with these eigenvalues had been determined sufficiently accurately. In each restart, we carried out five steps of the Arnoldi process to obtain an Arnoldi decomposition of the form (2.10) with $m = 5$ and then applied the eigenvalues with negative real part of the Hessenberg matrix in this decomposition as shifts.

The computations with Algorithms 1 and 2 reported in this section required a few minutes of execution time on our workstation. For comparison, we note that methods based on computing the Schur factorization of the matrix $A$ used in our experiments require several hours of execution time on the same workstation.

We remark that Algorithm 3 performs similarly as Algorithm 2, and we therefore do not show computed examples with the former algorithm.

## 5. Conclusion

This paper describes three algorithms for the partial eigenvalue assignment problem for a control system (1.1) with a large matrix $A$ based on the implicitly restarted Arnoldi method. The algorithms do not require the matrix $A$ to be stored or factored; it is only necessary to evaluate matrix-vector products with the matrices $A$ or $A^T$.

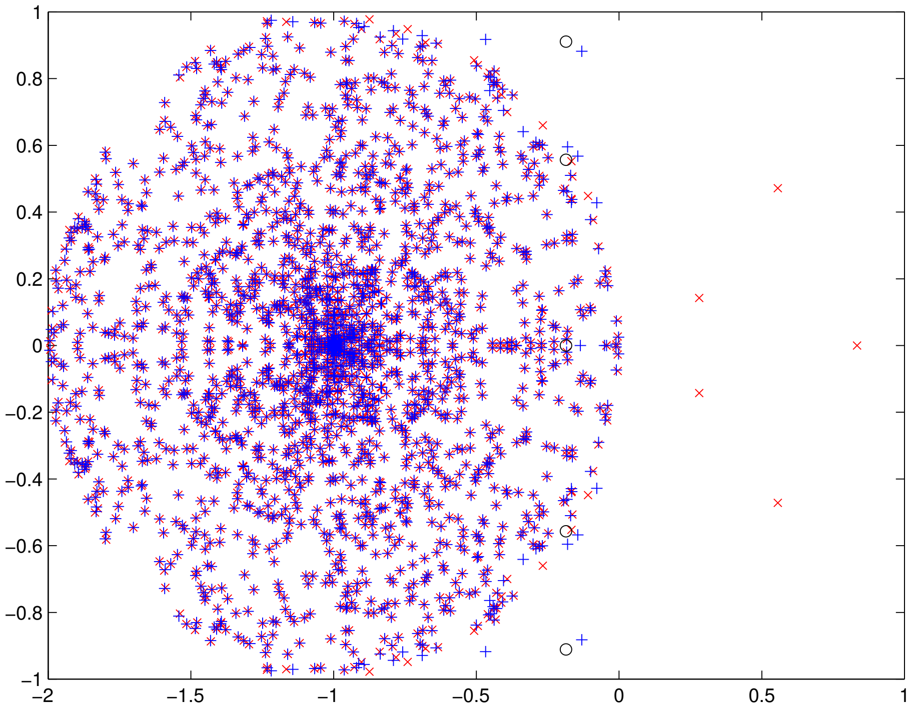


FIGURE 3. Solution determined by Algorithm 1 after four restarts in Step 1. The eigenvalues $\lambda_j$ of $A$ are marked by $\times$, the poles $\psi_j$ by $\circ$, and the eigenvalues of $A - bf^T$ by $+$.

| Quantity | Algorithm 2 | Algorithm 1 | | |
|---|---|---|---|---|
| Restarts | 2 | 2 | 3 | 4 |
| $\lVert AV_m - V_m H_m \rVert$ | $8.8 \cdot 10^{-2}$ | $8.8 \cdot 10^{-2}$ | $1.9 \cdot 10^{-2}$ | $1.3 \cdot 10^{-4}$ |
| $\mathrm{Re}(\lambda_{max})$ | $-2.3 \cdot 10^{-3}$ | $8.6 \cdot 10^{-2}$ | $1.1 \cdot 10^{-2}$ | $-1.7 \cdot 10^{-3}$ |
| $d(\lambda(A - bf^T), \mathbb{P} \cup \{\lambda_j\}_{j=m+1}^n)$ | $2.0 \cdot 10^{-2}$ | $3.9 \cdot 10^{-1}$ | $9.5 \cdot 10^{-2}$ | $7.4 \cdot 10^{-2}$ |

TABLE 1. Comparison of Algorithms 1 and 2. The quantity $\lambda_{max}$ denotes the eigenvalue of $A - bf^T$ with largest real part.

The computed examples of Section 4 show the spectrum of the matrix $A - bf^T$ produced by Algorithm 2 to be closer to the set $\mathbb{P} \cup \{\lambda_j\}_{j=m+1}^n$ than the spectrum of the analogous matrix determined by Algorithm 1 with more restarts in the IRA method. Thus, Algorithm 2 computed a better matrix $A - bf^T$ and required fewer matrix-vector product evaluations with the matrix $A^T$ than Algorithm 1. We found this to be typical for a large number of computed examples.

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