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Eigenvector sensitivity under general and structured perturbations of tridiagonal Toeplitz-type matrices

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SUMMARY

The sensitivity of eigenvalues of structured matrices under general or structured perturbations of the matrix entries has been thoroughly studied in the literature. Error bounds are available and the pseudospectrum can be computed to gain insight. Few investigations have focused on analyzing the sensitivity of eigenvectors under general or structured perturbations. The present paper discusses this sensitivity for tridiagonal Toeplitz and Toeplitz-type matrices. Copyright © 0000 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The sensitivity of the eigenvalues of a structured matrix to general or structured perturbations of the matrix entries has received considerable attention in the literature. Both bounds and graphical tools such as the pseudospectrum or structured pseudospectrum have been developed; see, e.g., [5, 6, 16, 19, 25, 26]. While the pseudospectrum measures the sensitivity of the eigenvalues, it depends on the sensitivity of the eigenvectors of the matrix to perturbations of the matrix entries. However, we are not aware of investigations that focus on the sensitivity of the eigenvectors to general or structured perturbations of a structured matrix. It is the purpose of the present paper to carry out such an investigation for tridiagonal Toeplitz matrices and tridiagonal Toeplitz-type matrices that are obtained by modifying the first and last diagonal entries of a tridiagonal Toeplitz matrix. These kinds of matrices arise in numerous applications, including the solution of ordinary and partial differential equations [7, 8, 23, 28], time series analysis [12], and as regularization matrices in Tikhonov regularization for the solution of discrete ill-posed problems [10, 22]. It is therefore important to understand properties of these matrices relevant for computation. Our analysis is facilitated by the fact that the eigenvalues and eigenvectors of the matrices considered are known in closed form.

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Introduce the tridiagonal Toeplitz matrix

$$T = \begin{bmatrix} \delta & \tau & & & & O \\ \sigma & \delta & \tau & & & \\ & \sigma & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \tau \\ O & & & & \sigma & \delta \end{bmatrix} \in \mathbb{C}^{n\times n}. \tag{1.1}$$

We will denote this matrix by $T = (n; \sigma, \delta, \tau)$. It is well known that its eigenvalues are given by

$$\lambda_h = \delta + 2\sqrt{\sigma\tau}\,\cos\frac{h\pi}{n+1}, \quad h = 1:n; \tag{1.2}$$

see, e.g., [23]. Assume that $\sigma\tau \neq 0$. Then the matrix (1.1) has n simple eigenvalues, which lie on a line segment that is symmetric with respect to δ. The components of the right eigenvector $x_h = [x_{h,1}, x_{h,2}, \ldots, x_{h,n}]^T \in \mathbb{C}^n$, for $h = 1:n$, associated with the eigenvalue λ_h are given by

$$x_{h,k} = \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin\frac{hk\pi}{n+1}, \quad k = 1:n, \tag{1.3}$$

and the corresponding left eigenvector $y_h = [y_{h,1}, y_{h,2}, \ldots, y_{h,n}]^T \in \mathbb{C}^n$ has the components

$$y_{h,k} = \left(\sqrt{\frac{\bar\tau}{\bar\sigma}}\right)^k \sin\frac{hk\pi}{n+1}, \quad k = 1:n, \tag{1.4}$$

where the bar denotes complex conjugation. Throughout this paper the superscript $(\cdot)^T$ stands for transposition and the superscript $(\cdot)^H$ for transposition and complex conjugation.

If $\sigma = 0$ and $\tau \neq 0$ (or $\sigma \neq 0$ and $\tau = 0$), then the matrix (1.1) has the unique eigenvalue δ of geometric multiplicity one. The right and left eigenvectors are the first and last columns (or the last and first columns) of the identity matrix, respectively.

We also will consider tridiagonal Toeplitz-type matrices of the form

$$T_{\alpha,\beta} = \begin{bmatrix} \delta-\alpha & \tau & & & & O \\ \sigma & \delta & \tau & & & \\ & \sigma & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \tau \\ O & & & & \sigma & \delta-\beta \end{bmatrix} \in \mathbb{C}^{n\times n} \tag{1.5}$$

for certain parameters $\alpha, \beta \in \mathbb{C}$. These matrices arise in the solution of ordinary or partial differential equations on an interval. Thus, $T_{\alpha,\beta}$ is a tridiagonal Toeplitz matrix when $\alpha = \beta = 0$.

Formulas for eigenvalues and eigenvectors of the matrices (1.5) are explicitly known for several choices of the parameters α and β; they are derived in [27]. Table I reports expressions for the eigenvalues for several choices of α and β. When $\sigma\tau \neq 0$, the components of the right

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α	β	λ_h
0	$\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau}\,\cos\frac{2h\pi}{2n+1}$
$\sqrt{\sigma\tau}$	0	$\delta + 2\sqrt{\sigma\tau}\,\cos\frac{2h\pi}{2n+1}$
0	$-\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau}\,\cos\frac{(2h-1)\pi}{2n+1}$
$-\sqrt{\sigma\tau}$	0	$\delta + 2\sqrt{\sigma\tau}\,\cos\frac{(2h-1)\pi}{2n+1}$
$\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau}\,\cos\frac{(2h-1)\pi}{2n}$
$-\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau}\,\cos\frac{(2h-1)\pi}{2n}$
$\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau}\,\cos\frac{h\pi}{n}$
$-\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\delta + 2\sqrt{\sigma\tau}\,\cos\frac{(h-1)\pi}{n}$

Table I. Formulas for the eigenvalues λ_h of the matrix (1.5) for $h = 1 : n$ and several choices of α and β.

eigenvector $x_h = [x_{h,1}, x_{h,2}, \ldots, x_{h,n}]^T \in \mathbb{C}^n$ associated with the eigenvalue λ_h are given by

$$
\begin{array}{rcll}
x_{h,k} & = & \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin\frac{2hk\pi}{2n+1}, & \alpha = 0, \qquad \beta = \sqrt{\sigma\tau}; \\
x_{h,k} & = & \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin\frac{h(2k-1)\pi}{2n+1}, & \alpha = \sqrt{\sigma\tau}, \qquad \beta = 0; \\
x_{h,k} & = & \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin\frac{(2h-1)k\pi}{2n+1}, & \alpha = 0, \qquad \beta = -\sqrt{\sigma\tau}; \\
x_{h,k} & = & \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \cos\frac{(2h-1)(2k-1)\pi}{2(2n+1)}, & \alpha = -\sqrt{\sigma\tau}, \qquad \beta = 0; \\
x_{h,k} & = & \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin\frac{(2h-1)(2k-1)\pi}{4n}, & \alpha = \sqrt{\sigma\tau}, \qquad \beta = -\sqrt{\sigma\tau}; \\
x_{h,k} & = & \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \cos\frac{(2h-1)(2k-1)\pi}{4n}, & \alpha = -\sqrt{\sigma\tau}, \qquad \beta = \sqrt{\sigma\tau}; \\
x_{h,k} & = & \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \sin\frac{h(2k-1)\pi}{2n}, & \alpha = \sqrt{\sigma\tau}, \qquad \beta = \sqrt{\sigma\tau}; \\
x_{h,k} & = & \left(\sqrt{\frac{\sigma}{\tau}}\right)^k \cos\frac{(h-1)(2k-1)\pi}{2n}, & \alpha = -\sqrt{\sigma\tau}, \qquad \beta = -\sqrt{\sigma\tau},
\end{array}
$$

for $k = 1 : n$.

It is straightforward to show that the component $y_{h,k}$ of the left eigenvector $y_h = [y_{h,1}, \ldots, y_{h,n}]^T \in \mathbb{C}^n$ of (1.5) is obtained from the component $x_{h,k}$ of the corresponding right eigenvector by replacing the factor $(\sigma/\tau)^{k/2}$ by $(\bar{\tau}/\bar{\sigma})^{k/2}$.

This paper is organized as follows. Section 2 discusses the sensitivity of the eigenvalues of the matrices (1.1) and (1.5) to general (unstructured) perturbations. Eigenvalue condition numbers for the matrices (1.1) and (1.5) are given. Section 3 is concerned with the sensitivity of the eigenvectors of the matrices (1.1) and (1.5) to general perturbations. Eigenvector condition numbers are presented. Section 4 discusses eigenvalue and eigenvector sensitivity to structured perturbations. Condition numbers are defined. Section 5 describes two novel applications of tridiagonal Toeplitz matrices. The first part of the section shows how eigenvalues of a symmetric tridiagonal matrix can be estimated by using the explicitly known eigenvalues of the closest symmetric tridiagonal Toeplitz matrix. In the latter part of the section, we discuss how the eigenvectors of a severely nonsymmetric nearly tridiagonal Toeplitz matrix can be computed accurately by using the explicitly known spectral factorization of the closest tridiagonal Toeplitz matrix. Concluding remarks can be found in Section 6.

2. SENSITIVITY OF THE SPECTRUM

This section discusses the sensitivity of the eigenvalues of the matrices (1.1) and (1.5) to general (unstructured) perturbations of the matrix entries.

2.1. Eigenvalue distances

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Proposition 2.1
The eigenvalues (1.2) of the matrix T defined by (1.1) satisfy

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2(n+1)}\sin\frac{(2h-1)\pi}{2(n+1)}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2(n+1)}\sin\frac{(2h+1)\pi}{2(n+1)}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n. \end{cases} \tag{2.1}$$

In particular, the distance of the eigenvalue λ_h to the other eigenvalues of T only depends on h, n, and the product $|\sigma\tau|$. Moreover, the minimal distance between any two eigenvalues of T is

$$4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2(n+1)}\sin\frac{3\pi}{2(n+1)}.$$

This distance is achieved by $|\lambda_1 - \lambda_2|$ and $|\lambda_{n-1} - \lambda_n|$.

Proof
Let $1 \leq j, h \leq n$. The trigonometric identity

$$\cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$$

yields

$$\begin{aligned} \min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| &= \min\{|\lambda_h - \lambda_{h+1}|, |\lambda_h - \lambda_{h-1}|\} \\ &= 2\sqrt{|\sigma\tau|}\min\left\{\left|\cos\frac{h\pi}{n+1} - \cos\frac{(h+1)\pi}{n+1}\right|, \left|\cos\frac{h\pi}{n+1} - \cos\frac{(h-1)\pi}{n+1}\right|\right\} \\ &= 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2(n+1)}\min\left\{\left|\sin\frac{(2h+1)\pi}{2(n+1)}\right|, \left|\sin\frac{(2h-1)\pi}{2(n+1)}\right|\right\}. \end{aligned}$$

This shows (2.1). The remaining statements follow from this formula. □

Remark 2.2. Results on the spacing of the eigenvalues of Hermitian Toeplitz matrices with simple-loop symbols (e.g., of the eigenvalues of Hermitian tridiagonal Toeplitz matrices) are reported in [3,4]. Such results can be extended to non-Hermitian tridiagonal Toeplitz matrices by diagonal similarity transformation. To this end, we note that the matrix $T = (n; \sigma, \delta, \tau)$ is, via the diagonal matrix $D = \mathrm{diag}[1, v, \ldots, v^{n-1}]$, similar to $T' = (n; v\sigma, \delta, v^{-1}\tau)$. One can choose v so that $|v\sigma| = |v^{-1}\tau|$. The matrix T' then is normal; see [16, Theorem 3.1]. In particular, real tridiagonal Toeplitz matrices T can be transformed to symmetric matrices T' by letting v be such that $v\sigma = v^{-1}\tau$.

An analogue of Proposition 2.1 can be shown for the eigenvalues of the matrix (1.5) for the choices of α and β considered in Table I. The results follow from the expressions for the eigenvalues in this table and are formulated in the following proposition.

Proposition 2.3
The eigenvalues λ_h of the matrix $T_{\alpha,\beta}$ satisfy

(i) for $\alpha = 0$ and $\beta = \sqrt{\sigma\tau}$ or vice versa

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n+1}\sin\frac{(2h-1)\pi}{2n+1}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n+1}\sin\frac{(2h+1)\pi}{2n+1}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(ii) for $\alpha = 0$ and $\beta = -\sqrt{\sigma\tau}$ or vice versa

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n+1}\sin\frac{2(h-1)\pi}{2n+1}, & \text{for } 1 < h \leq \lceil\frac{n}{2}\rceil \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n+1}\sin\frac{2h\pi}{2n+1}, & \text{for } h = 1 \text{ or } \lceil\frac{n}{2}\rceil < h < n; \end{cases}$$

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α	β	Minimal distance
0	$\sqrt{\sigma\tau}$	$4\sqrt{\lvert\sigma\tau\rvert}\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}$
$\sqrt{\sigma\tau}$	0	$4\sqrt{\lvert\sigma\tau\rvert}\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}$
0	$-\sqrt{\sigma\tau}$	$4\sqrt{\lvert\sigma\tau\rvert}\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}$
$-\sqrt{\sigma\tau}$	0	$4\sqrt{\lvert\sigma\tau\rvert}\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}$
$\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$4\sqrt{\lvert\sigma\tau\rvert}\sin^2\frac{\pi}{n}$
$-\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$4\sqrt{\lvert\sigma\tau\rvert}\sin^2\frac{\pi}{n}$
$\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$4\sqrt{\lvert\sigma\tau\rvert}\sin^2\frac{\pi}{2n}$
$-\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$4\sqrt{\lvert\sigma\tau\rvert}\sin^2\frac{\pi}{2n}$

Table II. Minimal distance between eigenvalues of the matrix (1.5) for several choices of α and β.

(iii) for $\alpha = \sqrt{\sigma\tau}$ and $\beta = -\sqrt{\sigma\tau}$ or vice versa

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{n}\sin\frac{(h-1)\pi}{n}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{n}\sin\frac{h\pi}{n}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(iv) for $\alpha = \sqrt{\sigma\tau}$ and $\beta = \sqrt{\sigma\tau}$

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n}\sin\frac{(2h-1)\pi}{2n}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n}\sin\frac{(2h+1)\pi}{2n}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(v) for $\alpha = -\sqrt{\sigma\tau}$ and $\beta = -\sqrt{\sigma\tau}$

$$\min_{\lambda_j \neq \lambda_h} |\lambda_h - \lambda_j| = \begin{cases} 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n}\sin\frac{(2h-3)\pi}{2n}, & \text{for } 1 < h \leq \lceil\frac{n}{2}\rceil \text{ or } h = n, \\ 4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n}\sin\frac{(2h-1)\pi}{2n}, & \text{for } h = 1 \text{ or } \lceil\frac{n}{2}\rceil < h < n. \end{cases}$$

Table II shows the minimal distance between any two eigenvalue of $T_{\alpha,\beta}$ for the choices of α and β of Table I.

2.2. Eigenvalue condition numbers

We first consider the eigenvalues of the Toeplitz matrix (1.1). Condition numbers for these eigenvalues also have been discussed in [16]. When $\sigma\tau \neq 0$, eigenvalue condition numbers can be obtained from (1.3) and (1.4). Standard computations and the trigonometric identity

$$\sum_{k=1}^{n} \sin^2 \frac{hk\pi}{n+1} = \frac{n+1}{2}, \quad h = 1:n, \tag{2.2}$$

show that, for $h = 1:n$,

$$\begin{aligned} \|x_h\|_2^2 &= \sum_{k=1}^{n} \left|\frac{\sigma}{\tau}\right|^k \sin^2 \frac{hk\pi}{n+1}, \\ \|y_h\|_2^2 &= \sum_{k=1}^{n} \left|\frac{\tau}{\sigma}\right|^k \sin^2 \frac{hk\pi}{n+1}, \\ |y_h^H x_h| &= \sum_{k=1}^{n} \sin^2 \frac{hk\pi}{n+1} = \frac{n+1}{2}. \end{aligned}$$

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Consequently, the condition numbers for the eigenvalues λ_h, $h = 1 : n$, of the matrix (1.1) are given by

$$\begin{aligned} \kappa(\lambda_h) &= \frac{\|x_h\|_2 \|y_h\|_2}{\left|y_h^H\, x_h\right|} \\ &= \frac{2}{n+1}\sqrt{\sum_{k=1}^{n}\left|\frac{\sigma}{\tau}\right|^k \sin^2\frac{hk\pi}{n+1}\cdot\sum_{k=1}^{n}\left|\frac{\tau}{\sigma}\right|^k \sin^2\frac{hk\pi}{n+1}}. \end{aligned} \tag{2.3}$$

Note that the eigenvalue condition numbers $\kappa(\lambda_h)$ only depend on h, n, and the ratio $\left|\frac{\sigma}{\tau}\right|$. When $|\sigma| = |\tau|$, we have

$$\|x_h\|_2^2 = \|y_h\|_2^2 = \sum_{k=1}^{n}\sin^2\frac{hk\,\pi}{n+1} = \frac{n+1}{2}\,, \quad h = 1 : n,$$

and it follows that

$$\kappa(\lambda_h) = \frac{\|x_h\|_2\,\|y_h\|_2}{\left|y_h^H\, x_h\right|} = 1.$$

Thus, the eigenvalues are perfectly conditioned. This is in agreement with the observation that the matrix T is normal when $|\sigma| = |\tau|$; see [16, Theorem 3.1].

We turn to the condition numbers of the eigenvalues of the Toeplitz-like matrix $T_{\alpha,\beta}$ defined by (1.5) for parameters α and β considered in Table I. The condition numbers, which are reported in Tables III and IV, can be derived by using the trigonometric identities

$$\begin{aligned} \textstyle\sum_{k=1}^{n}\sin^2\frac{2hk\,\pi}{2n+1} &= \tfrac{2n+1}{4}, & \textstyle\sum_{k=1}^{n}\sin^2\frac{h(2k-1)\,\pi}{2n+1} &= \tfrac{2n+1}{4}, \\ \textstyle\sum_{k=1}^{n}\sin^2\frac{(2h-1)k\,\pi}{2n+1} &= \tfrac{2n+1}{4}, & \textstyle\sum_{k=1}^{n}\cos^2\frac{(2h-1)(2k-1)\,\pi}{2(2n+1)} &= \tfrac{2n+1}{4}, \\ \textstyle\sum_{k=1}^{n}\sin^2\frac{(2h-1)(2k-1)\,\pi}{4n} &= \tfrac{n}{2}, & \textstyle\sum_{k=1}^{n}\cos^2\frac{(2h-1)(2k-1)\,\pi}{4n} &= \tfrac{n}{2}, \end{aligned}$$

for $h = 1 : n$, and

$$\sum_{k=1}^{n}\sin^2\frac{h(2k-1)\,\pi}{2n} = \frac{n}{2}\,, \text{ if } h \neq n; \quad \sum_{k=1}^{n}\cos^2\frac{(h-1)(2k-1)\,\pi}{2n} = \frac{n}{2}, \text{ if } h \neq 1.$$

Notice that $\kappa(\lambda_h) = 1$ when $\alpha = \beta = \sqrt{\sigma\tau}$ and $h = n$, and when $\alpha = \beta = -\sqrt{\sigma\tau}$ and $h = 1$.

Proposition 2.4
Let $|\sigma| = |\tau| > 0$, and let α and β be defined as in Table I. Then the tridiagonal Toeplitz-like matrix given by (1.5) is normal. Consequently, all eigenvalues have condition number one.

Proof
To show normality, we may apply [1, Theorem 1] and note, using the notation of this reference, that we have $\delta \pm \sqrt{\sigma\tau} = (\hat{r} + id)e^{i\phi}$, with $\hat{r} = r \pm |\sigma|$, if $\delta = (r + id)e^{i\phi}$ and $\tau = \sigma e^{2i\phi}$ for some $r, d, \phi \in \mathbb{R}$, and if $|\sigma| = |\tau| \neq 0$. Here and below i denotes the imaginary unit. □

3. SENSITIVITY OF THE EIGENVECTORS

The beginning of this section reviews results by Stewart [24]. These results are subsequently applied to the matrices (1.1) and (1.5).

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α	β	$\kappa(\lambda_h)$
0	$\sqrt{\sigma\tau}$	$\frac{4}{2n+1}\sqrt{\sum_{k=1}^{n}\left\lvert\frac{\sigma}{\tau}\right\rvert^k \sin^2\frac{2hk\pi}{2n+1}\cdot\sum_{k=1}^{n}\left\lvert\frac{\tau}{\sigma}\right\rvert^k \sin^2\frac{2hk\pi}{2n+1}}$
$\sqrt{\sigma\tau}$	0	$\frac{4}{2n+1}\sqrt{\sum_{k=1}^{n}\left\lvert\frac{\sigma}{\tau}\right\rvert^k \sin^2\frac{h(2k-1)\pi}{2n+1}\cdot\sum_{k=1}^{n}\left\lvert\frac{\tau}{\sigma}\right\rvert^k \sin^2\frac{h(2k-1)\pi}{2n+1}}$
0	$-\sqrt{\sigma\tau}$	$\frac{4}{2n+1}\sqrt{\sum_{k=1}^{n}\left\lvert\frac{\sigma}{\tau}\right\rvert^k \sin^2\frac{(2h-1)k\pi}{2n+1}\cdot\sum_{k=1}^{n}\left\lvert\frac{\tau}{\sigma}\right\rvert^k \sin^2\frac{(2h-1)k\pi}{2n+1}}$
$-\sqrt{\sigma\tau}$	0	$\frac{4}{2n+1}\sqrt{\sum_{k=1}^{n}\left\lvert\frac{\sigma}{\tau}\right\rvert^k \cos^2\frac{(2h-1)(2k-1)\pi}{2(2n+1)}\cdot\sum_{k=1}^{n}\left\lvert\frac{\tau}{\sigma}\right\rvert^k \cos^2\frac{(2h-1)(2k-1)\pi}{2(2n+1)}}$
$\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\frac{2}{n}\sqrt{\sum_{k=1}^{n}\left\lvert\frac{\sigma}{\tau}\right\rvert^k \sin^2\frac{(2h-1)(2k-1)\pi}{4n}\cdot\sum_{k=1}^{n}\left\lvert\frac{\tau}{\sigma}\right\rvert^k \sin^2\frac{(2h-1)(2k-1)\pi}{4n}}$
$-\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\frac{2}{n}\sqrt{\sum_{k=1}^{n}\left\lvert\frac{\sigma}{\tau}\right\rvert^k \cos^2\frac{(2h-1)(2k-1)\pi}{4n}\cdot\sum_{k=1}^{n}\left\lvert\frac{\tau}{\sigma}\right\rvert^k \cos^2\frac{(2h-1)(2k-1)\pi}{4n}}$

Table III. Condition numbers of the eigenvalues λ_h, $h = 1 : n$, of the matrix (1.5) for several choices of $\alpha \neq \beta$.

α	β	$\kappa(\lambda_h)$
$\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\frac{2}{n}\sqrt{\sum_{k=1}^{n}\left\lvert\frac{\sigma}{\tau}\right\rvert^k \sin^2\frac{h(2k-1)\pi}{2n}\cdot\sum_{k=1}^{n}\left\lvert\frac{\tau}{\sigma}\right\rvert^k \sin^2\frac{h(2k-1)\pi}{2n}}$
$-\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\frac{2}{n}\sqrt{\sum_{k=1}^{n}\left\lvert\frac{\sigma}{\tau}\right\rvert^k \cos^2\frac{(h-1)(2k-1)\pi}{2n}\cdot\sum_{k=1}^{n}\left\lvert\frac{\tau}{\sigma}\right\rvert^k \cos^2\frac{(h-1)(2k-1)\pi}{2n}}$

Table IV. Condition numbers of the eigenvalues λ_h, $h = 1 : n-1$, of the matrix (1.5) for $\alpha = \beta = \sqrt{\sigma\tau}$, and of the eigenvalues λ_h, $h = 2 : n$, for $\alpha = \beta = -\sqrt{\sigma\tau}$.

3.1. Eigenvector condition numbers

Definition 3.1. ([24]) Let $A \in \mathbb{C}^{n\times n}$ and let x be an eigenvector of unit norm associated with the simple eigenvalue μ. Let $U \in \mathbb{C}^{n\times(n-1)}$ be a matrix whose columns form an orthonormal basis for Range$(A - \mu I)$. The condition number of x (i.e., the condition number of the one-dimensional invariant subspace spanned by x) is defined by

$$\kappa(x) = \|(\mu I - U^H AU)^{-1}\|_2^{-1}.$$

Let $A^\varepsilon = A + \varepsilon E$, where $\varepsilon \in \mathbb{R}$ is of small magnitude and $E \in \mathbb{C}^{n\times n}$ is a matrix with $\|E\|_F = 1$. Here and below $\|\cdot\|_F$ denotes the Frobenius norm. Let x^ε be the unit eigenvector of A^ε corresponding to x, i.e., there is a continuous mapping $t \to x^t$ for $0 \le t \le \varepsilon$ such that $x^t = x$ for $t = 0$ and $x^t = x^\varepsilon$ for $t = \varepsilon$. Then for the induced perturbation in the direction between x and the pseudoeigenvector x^ε one has

$$\sin\theta_{x,x^\varepsilon} \le \kappa(x)\varepsilon, \tag{3.1}$$

where $\sin\theta_{x,x^\varepsilon} := \sqrt{1-\cos^2\theta_{x,x^\varepsilon}}$ and $\cos\theta_{x,x^\varepsilon} := |x^H x^\varepsilon|$; see Stewart [24, pp. 48–50] for more details.

3.2. Eigenvector condition numbers in the normal case

Let the matrix $A \in \mathbb{C}^{n\times n}$ be normal and denote its spectrum by $\Lambda(A)$. Consider the expression $\|(\mu I - U^H AU)^{-1}\|_2$ of Definition 3.1. It is straightforward to show that the upper bound

$$\|(\mu I - U^H AU)^{-1}\|_2 \le \min_{\substack{\lambda\neq\mu\\ \lambda\in\Lambda(A)}} |\mu - \lambda|$$

is attained because A is unitarily diagonalizable. Thus, if the matrix $A \in \mathbb{C}^{n\times n}$ is normal, the condition number of a unit eigenvector x only depends on how well the associated eigenvalue μ is separated from the other eigenvalues of the matrix. This result leads to the following proposition, which is shown in [24].

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Proposition 3.2
Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix and let x be a unit eigenvector associated with the simple eigenvalue μ. The condition number of x (i.e., the condition number of the one-dimensional invariant subspace spanned by x) is given by

$$\kappa(x) = \left(\min_{\substack{\lambda\neq\mu \\ \lambda\in\Lambda(A)}} |\mu - \lambda| \right)^{-1}.$$

Consider the tridiagonal Toeplitz matrix $T = (n; \sigma, \delta, \tau)$ and introduce the right and left unit eigenvectors,

$$\widetilde{x}_h = \frac{x_h}{\|x_h\|}, \quad \widetilde{y}_h = \frac{y_h}{\|y_h\|}, \quad h = 1:n,$$

where the vectors x_h and y_h are defined by (1.3) and (1.4), respectively.

Proposition 3.3
Let the Toeplitz matrix $T = (n; \sigma, \delta, \tau)$ be normal. Then the condition number of $\widetilde{x}_h$ is given by

$$\kappa(\widetilde{x}_h) = \begin{cases} \left(4|\sigma| \sin \frac{\pi}{2(n+1)} \sin \frac{(2h-1)\pi}{2(n+1)}\right)^{-1}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ \left(4|\sigma| \sin \frac{\pi}{2(n+1)} \sin \frac{(2h+1)\pi}{2(n+1)}\right)^{-1}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n. \end{cases} \tag{3.2}$$

In particular, $\kappa(\widetilde{x}_h)$ depends only on h, n, and $|\sigma|$. Moreover,

$$\max_{h=1:n} \kappa(\widetilde{x}_h) = \left(4|\sigma| \sin \frac{\pi}{2(n+1)} \sin \frac{3\pi}{2(n+1)}\right)^{-1}.$$

The maximum is attained by the eigenvectors $\widetilde{x}_h$ associated with the four extremal eigenvalues with indices $h = 1, 2, n-1, n$.

Proof
The proof follows from Propositions 2.1 and 3.2, by using the characterization in [16, Theorem 3.1]. □

Figure 1 shows the condition numbers $\kappa(\widetilde{x}_h)$ of normalized eigenvectors of a 100×100 normal tridiagonal Toeplitz matrix with $|\sigma| = |\tau| = 1$.

Let $T^\varepsilon = T + \varepsilon E$, where $\varepsilon \in \mathbb{R}$ is a constant of small magnitude, and $E \in \mathbb{C}^{n\times n}$ satisfies $\|E\|_F = 1$. Introduce the unit pseudoeigenvector $\widetilde{x}_h^\varepsilon$ of T^ε corresponding to the unit eigenvector $\widetilde{x}_h$ of T. Thus, there is a continuous mapping $t \to \widetilde{x}_h^t$ for $0 \leq t \leq \varepsilon$ such that $\widetilde{x}_h^t = \widetilde{x}_h$ for $t = 0$ and $\widetilde{x}_h^t = \widetilde{x}_h^\varepsilon$ for $t = \varepsilon$. We obtain from (3.2) that

$$0 \leq \sin\theta_{\widetilde{x}_h, \widetilde{x}_h^\varepsilon} \leq \begin{cases} \left(4|\sigma| \sin \frac{\pi}{2(n+1)} \sin \frac{(2h-1)\pi}{2(n+1)}\right)^{-1} \varepsilon, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ \left(4|\sigma| \sin \frac{\pi}{2(n+1)} \sin \frac{(2h+1)\pi}{2(n+1)}\right)^{-1} \varepsilon, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n. \end{cases} \tag{3.3}$$

Proposition 3.4
Let the matrix $T = (n; \sigma, \delta, \tau)$ be Hermitian. Given the unit pseudoeigenvector $\widetilde{x}_h^\varepsilon$, define the associated Rayleigh quotient,

$$\widetilde{\lambda}_h^\varepsilon = (\widetilde{x}_h^\varepsilon)^H T \widetilde{x}_h^\varepsilon,$$

and introduce the associated residual norm

$$r_h^\varepsilon = \|T\widetilde{x}_h^\varepsilon - \widetilde{\lambda}_h^\varepsilon \widetilde{x}_h^\varepsilon\|_2.$$

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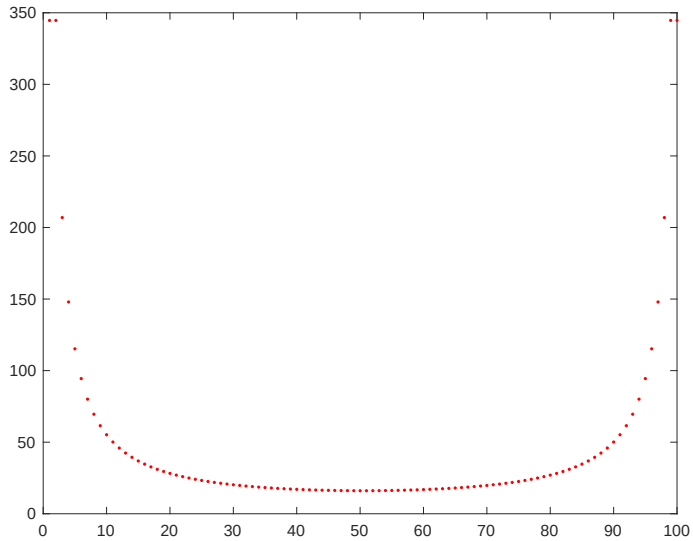

Figure 1. Eigenvector condition numbers for the matrix $T = (100; \exp i\theta_1, \delta, \exp i\theta_2)$, where $\delta \in \mathbb{C}$ and $\theta_1, \theta_2 \in \mathbb{R}$ are arbitrarily chosen parameters, and $i = \sqrt{-1}$. The horizontal axis shows the index of the eigenvalues λ_h, $h = 1 : 100$, and the vertical axis shows the condition numbers $\kappa(\widetilde{x}_h)$. The condition numbers are independent of δ, θ_1, and θ_2.

Then

$$\frac{r_h^\varepsilon}{2|\sigma| \cos \frac{\pi}{n+1}} \leq \sin \theta_{\widetilde{x}_h, \widetilde{x}_h^\varepsilon} \leq \frac{r_h^\varepsilon}{\min_{k \neq h} |\lambda_k - \widetilde{\lambda}_h^\varepsilon|} \,. \tag{3.4}$$

Proof
The proof follows from [20, Theorem 11.7.1] by observing that $\mathrm{spread}(T) := \lambda_1 - \lambda_n = 2|\sigma| \cos \frac{\pi}{n+1}$. □

We turn to the condition number of the eigenvectors of the matrix (1.5) for α- and β-values of Table I. Using Proposition 2.3, we obtain the following expressions.

Proposition 3.5
Let the matrix $T_{\alpha,\beta} \in \mathbb{C}^{n \times n}$ be normal and let $\widetilde{x}_h$, for $h = 1 : n$, be unit eigenvectors. Then

(i) for $\alpha = 0$ and $\beta = \sqrt{\sigma\tau}$ or vice versa

$$\kappa(\widetilde{x}_h) = \begin{cases} \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{(2h-1)\pi}{2n+1}\right)^{-1}, & \text{for } 1 < h \leq \frac{n}{2} \text{ or } h = n, \\ \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{(2h+1)\pi}{2n+1}\right)^{-1}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(ii) for $\alpha = 0$ and $\beta = -\sqrt{\sigma\tau}$ or vice versa

$$\kappa(\widetilde{x}_h) = \begin{cases} \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{2(h-1)\pi}{2n+1}\right)^{-1}, & \text{for } 1 < h \leq \lceil \frac{n}{2} \rceil \text{ or } h = n, \\ \left(4\sqrt{|\sigma\tau|} \sin \frac{\pi}{2n+1} \sin \frac{2h\pi}{2n+1}\right)^{-1}, & \text{for } h = 1 \text{ or } \lceil \frac{n}{2} \rceil < h < n; \end{cases}$$

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α	β	$\max_{h=1:n} \kappa(\widetilde{x}_h)$
0	$\sqrt{\sigma\tau}$	$\left(4\lvert\sigma\rvert \sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\right)^{-1}$
$\sqrt{\sigma\tau}$	0	$\left(4\lvert\sigma\rvert \sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\right)^{-1}$
0	$-\sqrt{\sigma\tau}$	$\left(4\lvert\sigma\rvert \sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\right)^{-1}$
$-\sqrt{\sigma\tau}$	0	$\left(4\lvert\sigma\rvert \sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\right)^{-1}$
$\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\left(4\lvert\sigma\rvert \sin^2\frac{\pi}{n}\right)^{-1}$
$-\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\left(4\lvert\sigma\rvert \sin^2\frac{\pi}{n}\right)^{-1}$
$\sqrt{\sigma\tau}$	$\sqrt{\sigma\tau}$	$\left(4\lvert\sigma\rvert \sin^2\frac{\pi}{2n}\right)^{-1}$
$-\sqrt{\sigma\tau}$	$-\sqrt{\sigma\tau}$	$\left(4\lvert\sigma\rvert \sin^2\frac{\pi}{2n}\right)^{-1}$

Table V. Maximal eigenvector condition numbers for the eigenvectors of the matrix (1.5) for several choices of α and β.

(iii) for $\alpha = \sqrt{\sigma\tau}$ and $\beta = -\sqrt{\sigma\tau}$ or vice versa

$$\kappa(\widetilde{x}_h) = \begin{cases} \left(4\sqrt{|\sigma\tau|}\sin\frac{\pi}{n}\sin\frac{(h-1)\pi}{n}\right)^{-1}, & \text{for } 1 < h \le \frac{n}{2} \text{ or } h = n, \\ \left(4\sqrt{|\sigma\tau|}\sin\frac{\pi}{n}\sin\frac{h\pi}{n}\right)^{-1}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(iv) for $\alpha = \sqrt{\sigma\tau}$ and $\beta = \sqrt{\sigma\tau}$

$$\kappa(\widetilde{x}_h) = \begin{cases} \left(4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n}\sin\frac{(2h-1)\pi}{2n}\right)^{-1}, & \text{for } 1 < h \le \frac{n}{2} \text{ or } h = n, \\ \left(4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n}\sin\frac{(2h+1)\pi}{2n}\right)^{-1}, & \text{for } h = 1 \text{ or } \frac{n}{2} < h < n; \end{cases}$$

(v) for $\alpha = -\sqrt{\sigma\tau}$ and $\beta = -\sqrt{\sigma\tau}$

$$\kappa(\widetilde{x}_h) = \begin{cases} \left(4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n}\sin\frac{(2h-3)\pi}{2n}\right)^{-1}, & \text{for } 1 < h \le \lceil\frac{n}{2}\rceil \text{ or } h = n, \\ \left(4\sqrt{|\sigma\tau|}\sin\frac{\pi}{2n}\sin\frac{(2h-1)\pi}{2n}\right)^{-1}, & \text{for } h = 1 \text{ or } \lceil\frac{n}{2}\rceil < h < n; \end{cases}$$

The maximal eigenvector condition numbers are reported in Table V.

4. SENSITIVITY TO STRUCTURED PERTURBATIONS

Observe that the smaller $0 < |\sigma/\tau| < 1$ is, the larger is the first component of the unit right eigenvector $\widetilde{x}_h$ and the last component of the unit left eigenvector $\widetilde{y}_h$. Similarly, the larger $1 < |\sigma/\tau| < \infty$ is, the larger is the last component of $\widetilde{x}_h$ and the first component of $\widetilde{y}_h$.

Consider the Wilkinson perturbation

$$W_h = \widetilde{y}_h \widetilde{x}_h^H$$

of the matrix T defined by (1.1) associated with the eigenvalue λ_h. This is a unit-norm perturbation of T that yields the largest perturbation in λ_h; see, e.g., [26]. The entries of

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largest magnitude of W_h are in the bottom-left corner of W_h when $|\sigma/\tau| < 1$ and in the top-right corner when $|\sigma/\tau| > 1$. The entries of W_h close to the diagonal are of small magnitude. In particular, the entries of largest magnitude of W_h are not present in $W_h|_{\mathcal{T}}$, the orthogonal projection of W_h in the subspace $\mathcal{T}$ of tridiagonal Toeplitz matrices. This projection is used in the following proposition which summarizes results from [14] and yields useful formulations for both the $\mathcal{T}$-structured eigenvalue condition number (see, e.g., [11]) and the worst-case $\mathcal{T}$-structured perturbations [13, 14].

Proposition 4.1
Let λ_h be a simple eigenvalue of a Toeplitz matrix $T \in \mathcal{T} \subset \mathbb{C}^{n\times n}$ with associated unit right and left eigenvectors $\widetilde{x}_h$ and $\widetilde{y}_h$, respectively. Given any matrix $E \in \mathcal{T}$ with $\|E\|_F = 1$, let $\lambda_h(t)$ be an eigenvalue of $T + tE$ converging to λ_h as $t \to 0$. Then

$$|\dot{\lambda}_h(0)| \leq \max\left\{ \left| \frac{\widetilde{y}_h^H E \widetilde{x}_h}{\widetilde{y}_h^H \widetilde{x}_h} \right|,\ \|E\|_F = 1,\ E \in \mathcal{T} \right\} = \frac{\|W_h|_{\mathcal{T}}\|_F}{|\widetilde{y}_h^H \widetilde{x}_h|}$$

and

$$\dot{\lambda}_h(0) = \frac{\|W_h|_{\mathcal{T}}\|_F}{|\widetilde{y}_h^H \widetilde{x}_h|} \qquad \text{if} \qquad E = \eta \frac{W_h|_{\mathcal{T}}}{\|W_h|_{\mathcal{T}}\|_F},$$

for any unimodular $\eta \in \mathbb{C}$. Here $\dot{\lambda}_h(t)$ denotes the derivative of $\lambda_h(t)$ with respect to the parameter t.

It follows from Proposition 4.1 that the $\mathcal{T}$-structured condition number of the eigenvalue λ_h of the tridiagonal Toeplitz matrix T is given by

$$\kappa_{\mathcal{T}}(\lambda_h) = \kappa(\lambda_h)\|W_h|_{\mathcal{T}}\|_F.$$

This expression shows that the $\mathcal{T}$-structured condition number $\kappa_{\mathcal{T}}(\lambda_h)$ may be small even when the traditional condition number $\kappa(\lambda_h)$ is large. Thus, an eigenvalue λ_h may be much more sensitive to a general perturbation of T than to a structured perturbation. The worst-case structured perturbation [14] is given by the structured analogue of the Wilkinson perturbation

$$W_h|_{\widehat{\mathcal{T}}} := \frac{W_h|_{\mathcal{T}}}{\|W_h|_{\mathcal{T}}\|_F}.$$

We have the following result.

Proposition 4.2
The $\mathcal{T}$-structured condition number of a simple eigenvalue λ_h of a tridiagonal Toeplitz matrix $T = (n; \sigma, \delta, \tau)$ is given by

$$\kappa_{\mathcal{T}}(\lambda_h) = \sqrt{\frac{1}{n} + \frac{1}{n-1}\left(\left|\frac{\sigma}{\tau}\right| + \left|\frac{\tau}{\sigma}\right|\right)\cos^2\frac{h\pi}{n+1}}. \tag{4.1}$$

In particular, $\kappa_{\mathcal{T}}(\lambda_h)$ only depends on h, n, and the ratio $\left|\frac{\sigma}{\tau}\right|$.

Proof
Let σ_h, δ_h, and τ_h denote the subdiagonal, diagonal, and superdiagonal entries of $W_h|_{\mathcal{T}}$,

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respectively. Then

$$\begin{aligned}
\sigma_h &= \frac{\sqrt{\frac{\overline{\tau}}{\overline{\sigma}}}\sum_{k=1}^{n-1}\sin\frac{hk\pi}{n+1}\sin\frac{h(k+1)\pi}{n+1}}{(n-1)\sqrt{\sum_{k=1}^{n}\left|\frac{\sigma}{\tau}\right|^k\sin^2\frac{hk\pi}{n+1}\cdot\sum_{k=1}^{n}\left|\frac{\tau}{\sigma}\right|^k\sin^2\frac{hk\pi}{n+1}}} \\
&= \frac{(n+1)\sqrt{\frac{\overline{\tau}}{\overline{\sigma}}}\cos\frac{h\pi}{n+1}}{2(n-1)\sqrt{\sum_{k=1}^{n}\left|\frac{\sigma}{\tau}\right|^k\sin^2\frac{hk\pi}{n+1}\cdot\sum_{k=1}^{n}\left|\frac{\tau}{\sigma}\right|^k\sin^2\frac{hk\pi}{n+1}}}, \\
\delta_h &= \frac{n+1}{2n\sqrt{\sum_{k=1}^{n}\left|\frac{\sigma}{\tau}\right|^k\sin^2\frac{hk\pi}{n+1}\cdot\sum_{k=1}^{n}\left|\frac{\tau}{\sigma}\right|^k\sin^2\frac{hk\pi}{n+1}}}, \\
\tau_h &= \frac{\sqrt{\frac{\overline{\sigma}}{\overline{\tau}}}\sum_{k=1}^{n-1}\sin\frac{hk\pi}{n+1}\sin\frac{h(k+1)\pi}{n+1}}{(n-1)\sqrt{\sum_{k=1}^{n}\left|\frac{\sigma}{\tau}\right|^k\sin^2\frac{hk\pi}{n+1}\cdot\sum_{k=1}^{n}\left|\frac{\tau}{\sigma}\right|^k\sin^2\frac{hk\pi}{n+1}}} \\
&= \frac{(n+1)\sqrt{\frac{\overline{\sigma}}{\overline{\tau}}}\cos\frac{h\pi}{n+1}}{2(n-1)\sqrt{\sum_{k=1}^{n}\left|\frac{\sigma}{\tau}\right|^k\sin^2\frac{hk\pi}{n+1}\cdot\sum_{k=1}^{n}\left|\frac{\tau}{\sigma}\right|^k\sin^2\frac{hk\pi}{n+1}}}.
\end{aligned}$$

The above expressions were obtained by exploiting the trigonometric identities (2.2) and

$$\sum_{k=1}^{n-1}\sin\frac{hk\pi}{n+1}\sin\frac{h(k+1)\pi}{n+1} = \frac{n+1}{2}\cos\frac{h\pi}{n+1}, \qquad h = 1:n; \tag{4.2}$$

see, e.g., [6, Appendix A]. Hence,

$$\begin{aligned}
\|W_h|_{\mathcal{T}}\|_F &= \sqrt{n|\delta_h|^2 + (n-1)|\sigma_h|^2 + (n-1)|\tau_h|^2} \\
&= \frac{\frac{n+1}{2}\sqrt{\frac{1}{n} + \frac{1}{n-1}\left(\left|\frac{\sigma}{\tau}\right| + \left|\frac{\tau}{\sigma}\right|\right)\cos^2\frac{h\pi}{n+1}}}{\sqrt{\sum_{k=1}^{n}\left|\frac{\sigma}{\tau}\right|^k\sin^2\frac{hk\pi}{n+1}\cdot\sum_{k=1}^{n}\left|\frac{\tau}{\sigma}\right|^k\sin^2\frac{hk\pi}{n+1}}}.
\end{aligned}$$

Finally $\kappa_{\mathcal{T}}(\lambda_h)$ is the product of $\kappa(\lambda_h)$ and $\|W_h|_{\mathcal{T}}\|_F$. The proof now follows by using (2.4). □

4.1. Eigenvector structured sensitivity in the normal case

When E is a (tridiagonal Toeplitz) structured perturbation of T, the perturbed matrix $T^\varepsilon = T + \varepsilon E$ is a tridiagonal Toeplitz matrix. Assume that T is normal. Unfortunately, $T^\varepsilon = (n; \sigma^\varepsilon, \delta^\varepsilon, \tau^\varepsilon)$ might not be normal because $|\sigma^\varepsilon|$ may differ from $|\tau^\varepsilon|$. For the components of the eigenvector $x_h^\varepsilon = [x_{h,1}^\varepsilon, x_{h,2}^\varepsilon, \ldots, x_{h,n}^\varepsilon]^T$ associated with the hth eigenvalue of T^ε, we have

$$x_{h,k}^\varepsilon = \left(\sqrt{\frac{\sigma^\varepsilon}{\tau^\varepsilon}}\right)^k \sin\frac{hk\pi}{n+1}, \quad k = 1:n, \quad h = 1:n,$$

so that

$$\cos\theta_{\widetilde{x}_h,\widetilde{x}_h^\varepsilon} = \frac{\left|\sum_{k=1}^{n}\left(\sqrt{\frac{\overline{\sigma}}{\overline{\tau}}}\right)^k\left(\sqrt{\frac{\sigma^\varepsilon}{\tau^\varepsilon}}\right)^k\sin^2\frac{hk\pi}{n+1}\right|}{\sqrt{\frac{n+1}{2}\sum_{k=1}^{n}\left|\frac{\sigma^\varepsilon}{\tau^\varepsilon}\right|^k\sin^2\frac{hk\pi}{n+1}}}, \quad h = 1:n, \tag{4.3}$$

where $\widetilde{x}_h$ and $\widetilde{y}_h$ are normalized vectors. Notice that the perturbations induced in the eigenvectors do not depend on δ^ε. In fact, the induced perturbations only depend on the ratio $\frac{\sigma^\varepsilon}{\tau^\varepsilon}$.

Proposition 4.3
The right and left eigenvectors of normal tridiagonal Toeplitz matrices $T = (n; \sigma, \delta, \tau)$ only depend on the dimension n and on the angle $\theta = \arg(\sigma) - \arg(\tau)$.

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Proof
From (1.3) and (1.4), it is clear that, given the dimension of the matrix, the ratio σ/τ uniquely determines the right and left eigenvectors of T up to a scaling factor. Since $|\sigma| = |\tau|$, one has

$$x_{h,k} = y_{h,k} = e^{i\frac{k}{2}\theta} \sin \frac{hk\pi}{n+1}, \quad k = 1:n, \quad h = 1:n. \tag{4.4}$$

□

Remark 4.4. When T is Hermitian, we have $\theta = 2\arg(\sigma)$, whereas in the skew-Hermitian case, one has $\theta = 2\arg(\sigma) - \pi$.

Proposition 4.5
If the perturbation εE of the Hermitian matrix $T = (n; \sigma, \delta, \bar{\sigma})$ has the same structure as T, then the right eigenvector x_h^ε [the left eigenvector y_h^ε] associated to the hth eigenvalue of

$$T^\varepsilon := T + \varepsilon E = (n; \sigma^\varepsilon, \delta^\varepsilon, \bar{\sigma}^\varepsilon)$$

has the components

$$x_{h,k}^\varepsilon = y_{h,k}^\varepsilon = e^{ik\arg(\sigma^\varepsilon)} \sin \frac{hk\pi}{n+1}, \quad k = 1:n,$$

for $h = 1:n$. Moreover, the associated Rayleigh quotient is given by

$$\widetilde{\lambda}_h^\varepsilon := \frac{x_h^{\varepsilon H} T x_h^\varepsilon}{x_h^{\varepsilon H} x_h^\varepsilon} = \delta + 2|\sigma| \cos(\arg(\sigma) - \arg(\sigma^\varepsilon)) \cos \frac{h\pi}{n+1}, \quad h = 1:n, \tag{4.5}$$

and the following inequalities hold

$$\frac{\|Tx_h^\varepsilon - \widetilde{\lambda}_h^\varepsilon x_h^\varepsilon\|_2}{\sqrt{2(n+1)}\cos\frac{\pi}{n+1}} \le \sin\theta_{\widetilde{x}_h, \widetilde{x}_h^\varepsilon} \le \frac{\|Tx_h^\varepsilon - \widetilde{\lambda}_h^\varepsilon x_h^\varepsilon\|_2}{\sqrt{2(n+1)}|\sigma|\,|1 - \cos(\arg(\sigma) - \arg(\sigma^\varepsilon))| \cos\frac{h\pi}{n+1}}.$$

Proof
If T is Hermitian, then T^ε is Hermitian as well (i.e., $\tau^\varepsilon = \bar{\sigma^\varepsilon}$). The angle θ in (4.4) is equal to $2\arg(\sigma^\varepsilon)$; see Remark 4.4. Further, one has

$$\widetilde{\lambda}_h^\varepsilon = \frac{\delta \sum_{k=1}^{n-1} \sin^2 \frac{hk\pi}{n+1} + (\sigma e^{-i\arg(\sigma^\varepsilon)} + \bar{\sigma} e^{i\arg(\sigma^\varepsilon)}) \sum_{k=1}^{n-1} \sin\frac{hk\pi}{n+1} \sin\frac{h(k+1)\pi}{n+1}}{\frac{n+1}{2}}.$$

Exploiting the identities (2.2) and (4.2), we obtain

$$\widetilde{\lambda}_h^\varepsilon = \delta + (\sigma e^{-i\arg(\sigma^\varepsilon)} + \bar{\sigma} e^{i\arg(\sigma^\varepsilon)}) \cos \frac{h\pi}{n+1}.$$

Moreover (4.5) follows from $\sigma e^{-i\arg(\sigma^\varepsilon)} + \bar{\sigma} e^{i\arg(\sigma^\varepsilon)} = |\sigma| \Re(e^{i(\arg(\sigma) - \arg(\sigma^\varepsilon))})$, where $\Re(\cdot)$ denotes the real part of the argument. The proof is concluded by using Proposition 3.4, observing that $\Re(e^{it}) = \cos(t)$, $\|x_h^\varepsilon\|_2^2 = \frac{n+1}{2}$, and

$$\min_{k \ne h} |\lambda_k - \widetilde{\lambda}_h^\varepsilon| = |\lambda_h - \widetilde{\lambda}_h^\varepsilon| = 2|\sigma|\,|1 - \cos(\arg(\sigma) - \arg(\sigma^\varepsilon))| \cos \frac{h\pi}{n+1}.$$

□

When T is skew-Hermitian and the perturbation E has the same structure, we have that T^ε is skew-Hermitian as well (i.e., $\tau^\varepsilon = -\bar{\sigma}^\varepsilon$). Thus, in both the Hermitian and skew-Hermitian cases, the structured ε-pseudospectrum lies in a closed line segment, i.e., on the real axis or on the imaginary axis, respectively. In other situations when $|\sigma| = |\tau|$ and $|\sigma^\varepsilon| \ne |\tau^\varepsilon|$, the structured ε-pseudospectrum is bounded by the ellipse $\{\tau z + \delta + \sigma z^{-1} : z \in \mathbb{C}, |z| = 1\}$, which is the boundary of the spectrum of the Toeplitz operator $T_\infty = (\infty; \sigma, \delta, \tau)$; see, e.g., [6, 16, 18, 21].

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4.1.1. The real case The following results are concerned with with normal real tridiagonal Toeplitz matrices.

Proposition 4.6
All real symmetric tridiagonal Toeplitz matrices of a given dimension have the same right and left eigenvectors.

Proof
If T is real and symmetric (i.e., $\sigma = \tau$), then

$$x_{h,k} = y_{h,k} = \sin \frac{hk\pi}{n+1}, \quad k = 1:n, \quad h = 1:n.$$

□

Corollary 4.7
The eigenvectors of a real symmetric tridiagonal Toeplitz matrix are perfectly conditioned with respect to any structured perturbation that respects symmetry.

Proof
If T is symmetric, then T^ε is symmetric as well (i.e., $\sigma^\varepsilon = \tau^\varepsilon$). It follows from Proposition 4.6 that $\widetilde{x}_h = \widetilde{x}_h^\varepsilon$ for $h = 1:n$. □

Corollary 4.8
The eigenvectors of a real shifted skew-symmetric tridiagonal Toeplitz matrix are perfectly conditioned with respect to structured perturbations that respect both the skew-symmetry and the signs of the (sub- and) super-diagonals.

Proof
If T is shifted skew-symmetric, then $\sigma = -\tau$, and one has

$$x_{h,k} = y_{h,k} = (\mathrm{sgn}(\tau)i)^k \sin \frac{hk\pi}{n+1}, \quad k = 1:n, \quad h = 1:n.$$

By assumption T^ε is a real shifted skew-symmetric tridiagonal Toeplitz matrix and $\mathrm{sgn}(\tau) = \mathrm{sgn}(\tau^\varepsilon)$. Thus, from (4.3), we have

$$\cos\theta_{\widetilde{x}_h,\widetilde{x}_h^\varepsilon} = \frac{\left|\sum_{k=1}^n (\mathrm{sgn}(\tau)i)^k(-\mathrm{sgn}(\tau^\varepsilon)i)^k \sin^2 \frac{hk\pi}{n+1}\right|}{\sqrt{\frac{n+1}{2}\sum_{k=1}^n \sin^2 \frac{hk\pi}{n+1}}} = \frac{\sum_{k=1}^n \sin^2 \frac{hk\pi}{n+1}}{\sqrt{\frac{n+1}{2}\sum_{k=1}^n \sin^2 \frac{hk\pi}{n+1}}} = 1\,.$$

□

Proposition 4.9
The eigenvectors of a real normal tridiagonal Toeplitz matrix are perfectly conditioned with respect to any structured perturbation that respects the symmetry [skew-symmetry and signature].

Proof
A real tridiagonal matrix T is normal if and only if it is symmetric or shifted skew-symmetric; see, e.g., [15, Theorem 7.1] or [17, Corollary 2.2]. The proof now follows from Corollaries 4.7 and 4.8. □

Let $\mathcal{S}_\mathcal{T}$ denote the subspace of real symmetric tridiagonal Toeplitz matrices and let $\mathcal{A}_\mathcal{T}$ be the subspace of real shifted skew-symmetric tridiagonal Toeplitz matrices. The above results show that the unstructured measure (3.2) of the sensitivity to perturbations of the eigenvectors of a tridiagonal Toeplitz matrix in $\mathcal{S}_\mathcal{T}$ or $\mathcal{A}_\mathcal{T}$ is not accurate in case of structured perturbations E of the matrix T, i.e., when $E \in \mathcal{S}_\mathcal{T}$ or $E \in \mathcal{A}_\mathcal{T}$ with E small enough.

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4.2. Eigenvalue structured sensitivity in the normal case

For normal matrices, the right and left unit eigenvectors can be chosen to be the same. Then the Wilkinson perturbation W_h is symmetric for $h = 1 : n$.

Corollary 4.10
The $\mathcal{T}$-structured condition number of the eigenvalue λ_h of a normal tridiagonal Toeplitz matrix T is given by

$$\kappa_{\mathcal{T}}(\lambda_h) = \sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2\frac{h\pi}{n+1}}, \quad h = 1 : n. \tag{4.6}$$

Proof
The proof trivially follows from (4.1), since $|\sigma| = |\tau|$. □

4.2.1. The real case We recall that a real tridiagonal matrix T is normal if and only if it is symmetric or shifted skew-symmetric. Notice that Proposition 4.1 can be generalized to several other structures and that, in particular, it holds true if one everywhere replaces $\mathcal{T}$ by either $\mathcal{S}_{\mathcal{T}}$ or $\mathcal{A}_{\mathcal{T}}$, or other subspaces of matrices with a given symmetry-pattern; see [14]. It follows that, for $h = 1 : n$, the $\mathcal{S}_{\mathcal{T}}$-structured [$\mathcal{A}_{\mathcal{T}}$-structured] condition number of the eigenvalue λ_h of a real symmetric [shifted skew-symmetric] tridiagonal Toeplitz matrix T is given by

$$\kappa_{\mathcal{S}_{\mathcal{T}}}(\lambda_h) = \|W_h|_{\mathcal{S}_{\mathcal{T}}}\|_F \qquad [\kappa_{\mathcal{A}_{\mathcal{T}}}(\lambda_h) = \|W_h|_{\mathcal{A}_{\mathcal{T}}}\|_F],$$

$\kappa(\lambda_h)$ being equal to 1, and that the worst-case structured perturbation [14] is given by the structured analogue of the Wilkinson perturbation:

$$W_h|_{\widehat{\mathcal{S}_{\mathcal{T}}}} := \frac{W_h|_{\mathcal{S}_{\mathcal{T}}}}{\|W_h|_{\mathcal{S}_{\mathcal{T}}}\|_F} \qquad [W_h|_{\widehat{\mathcal{A}_{\mathcal{T}}}} := \frac{W_h|_{\mathcal{A}_{\mathcal{T}}}}{\|W_h|_{\mathcal{A}_{\mathcal{T}}}\|_F}].$$

The following result is concerned with symmetric tridiagonal Toeplitz matrices and eigenvalue sensitivity to $\mathcal{S}_{\mathcal{T}}$-structured perturbations, i.e., to real symmetric tridiagonal Toeplitz matrix perturbations.

Proposition 4.11
The eigenvalues λ_h of any symmetric tridiagonal Toeplitz matrix $T \in \mathbb{R}^{n\times n}$ have condition numbers

$$\kappa_{\mathcal{S}_{\mathcal{T}}}(\lambda_h) = \sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2\frac{h\pi}{n+1}}, \quad h = 1 : n,$$

with respect to any structured perturbation that respects the symmetry.

Proof
It is straightforward that $\kappa_{\mathcal{S}_{\mathcal{T}}}(\lambda_h) \le \kappa_{\mathcal{T}}(\lambda_h)$. In addition, in the real symmetric case, i.e., when $\sigma = \tau$, the Wilkinson perturbation associated with λ_h, $W_h = \widetilde{y}_h\widetilde{x}_h^H$, is real and symmetric. Thus, the orthogonal projection of W_h in the subspace of real symmetric tridiagonal Toeplitz matrices coincides with $W_h|_{\mathcal{T}}$. This concludes the proof, since $\kappa_{\mathcal{S}_{\mathcal{T}}}(\lambda_h)$ coincides with the condition number $\kappa_{\mathcal{T}}(\lambda_h)$ in (4.6), i.e.,

$$\kappa_{\mathcal{S}_{\mathcal{T}}}(\lambda_h) = \|W_h|_{\mathcal{S}_{\mathcal{T}}}\|_F = \|W_h|_{\mathcal{T}}\|_F = \kappa_{\mathcal{T}}(\lambda_h).$$

□

Figure 2 shows the structured eigenvalue condition numbers $\kappa_{\mathcal{S}_{\mathcal{T}}}(\lambda_h)$ for a 100×100 symmetric tridiagonal Toeplitz matrix.

Remark 4.12. Let σ_h, δ_h, and τ_h denote the subdiagonal, diagonal, and superdiagonal entries, respectively, of the orthogonal projection of the Wilkinson perturbation W_h associated with the eigenvalue λ_h of a real symmetric tridiagonal Toeplitz matrix $T = (n; \sigma, \delta, \sigma)$ (i.e.,

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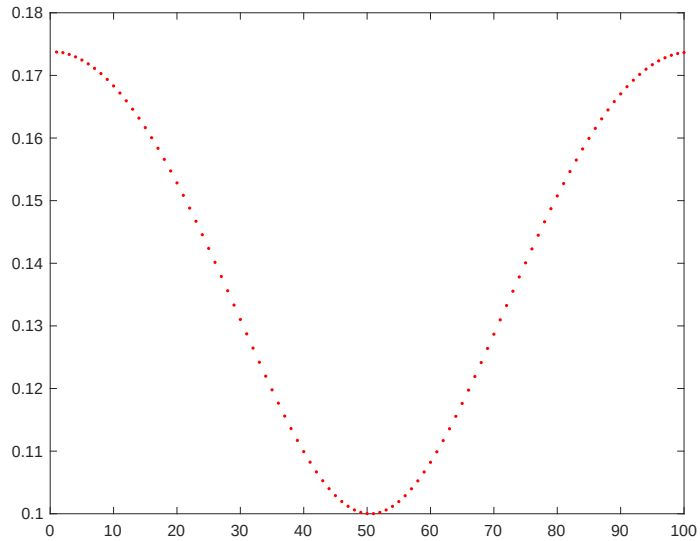

Figure 2. Structured eigenvalue condition numbers for the matrix $T = (100; \sigma, \delta, \sigma)$, where σ and τ are arbitrarily chosen real parameters. The horizontal axis shows the index of the eigenvalue λ_h, $h = 1 : 100$, and the vertical axis the structured condition numbers $\kappa_{\mathcal{S}_\mathcal{T}}(\lambda_h)$. The condition numbers are independent of σ.

$W_h|_{\mathcal{S}_\mathcal{T}} \equiv W_h|_{\mathcal{T}}$; cf. the proof of Proposition 4.11). It is easy to show that

$$\sigma_h = \tau_h = \frac{1}{n-1} \cos \frac{h\pi}{n+1}; \qquad \delta_h = \frac{1}{n}.$$

Moreover, one has

$$\widehat{\sigma}_h = \widehat{\tau}_h = \frac{\cos \frac{h\pi}{n+1}}{(n-1)\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{h\pi}{n+1}}}; \quad \widehat{\delta}_h = \frac{1}{n\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{h\pi}{n+1}}},$$

where $\widehat{\sigma}_h$, $\widehat{\tau}_h$, and $\widehat{\delta}_h$ denote the subdiagonal, diagonal, and superdiagonal entries, respectively, of the unit-norm $\mathcal{S}_\mathcal{T}$-structured analogue of the Wilkinson perturbation, $W_h|_{\widehat{\mathcal{S}_\mathcal{T}}}$. Thus, if we perturb T by the real symmetric tridiagonal Toeplitz matrix $\varepsilon W_j|_{\widehat{\mathcal{S}_\mathcal{T}}}$ $[-\varepsilon W_j|_{\widehat{\mathcal{S}_\mathcal{T}}}]$, for a given $j \in \{1, \ldots, n\}$, the spectrum of the perturbed matrix T_j^ε $[T_j^{-\varepsilon}]$ contains the eigenvalue

$$\lambda_j^\varepsilon = \delta + \frac{\varepsilon}{n\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{j\pi}{n+1}}} + 2\left(\sigma + \frac{\varepsilon \cos \frac{j\pi}{n+1}}{(n-1)\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{j\pi}{n+1}}}\right) \cos \frac{j\pi}{n+1}$$

$$[\lambda_j^{-\varepsilon} = \delta - \frac{\varepsilon}{n\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{j\pi}{n+1}}} + 2\left(\sigma - \frac{\varepsilon \cos \frac{j\pi}{n+1}}{(n-1)\sqrt{\frac{1}{n} + \frac{2}{n-1}\cos^2 \frac{j\pi}{n+1}}}\right) \cos \frac{j\pi}{n+1}].$$

Straightforwardly, the $\mathcal{S}_\mathcal{T}$-structured ε-pseudospectrum, for ε small enough, is given by the union of the real intervals $[\lambda_h^{-\varepsilon}, \lambda_h^{+\varepsilon}]$ of width $2\,\kappa_{\mathcal{S}_\mathcal{T}}(\lambda_h)\varepsilon$, for $h = 1 : n$.

Let us turn to the shifted skew-symmetric case.

Proposition 4.13
All the eigenvalues of a shifted skew-symmetric tridiagonal Toeplitz matrix $T \in \mathbb{R}^{n\times n}$ have

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the same condition number

$$\kappa_{\mathcal{A}_\mathcal{T}}(\lambda_h) = \frac{1}{\sqrt{n}}$$

with respect to any structured perturbation that respects the shifted skew-symmetry.

Proof
Odd eigenvector components of real shifted skew-symmetric tridiagonal Toeplitz matrices are purely imaginary numbers. Hence, the Wilkinson perturbation associated with λ_h is symmetric. By using the same notation as in Remark 4.12, we obtain

$$\sigma_h = \bar{\tau}_h = \frac{\mathrm{sgn}(\tau)i}{n-1}\cos\frac{h\pi}{n+1}; \qquad \delta_h = \frac{1}{n}.$$

Thus, the orthogonal projection of W_h in the subspace of real shifted skew-symmetric tridiagonal Toeplitz matrices is the matrix $\frac{1}{n}I$. Its Frobenius norm $\frac{1}{\sqrt{n}}$ gives the structured condition number $\kappa_{\mathcal{A}_\mathcal{T}}(\lambda_h) = \|W_h|_{\mathcal{A}_\mathcal{T}}\|_F$. □

Remark 4.14. Perturbing the real shifted skew-symmetric tridiagonal matrix $T = (n; \sigma, \delta, -\sigma)$ by $\pm\varepsilon W_h|_{\widehat{\mathcal{A}_\mathcal{T}}}$, where $W_h|_{\widehat{\mathcal{A}_\mathcal{T}}}$ is the $\mathcal{A}_\mathcal{T}$-structured unit-norm analogue of the Wilkinson perturbation, gives the pseudoeigenvalues $\lambda_h^{\pm\varepsilon} = \delta \pm \frac{\varepsilon}{\sqrt{n}} + 2i|\sigma|\cos\frac{h\pi}{n+1}$ for $h = 1:n$.

We conclude this section by noticing that Proposition 4.6 can be extended to real symmetric tridiagonal Toeplitz-type matrices. We have the following result.

Proposition 4.15
Any real symmetric tridiagonal Toeplitz-type matrix of a fixed order n of the types considered in Table I has the same right and left eigenvectors.

Proof
One has

$$\begin{array}{rcll}
x_{h,k} & = & \sin\frac{2hk\pi}{2n+1}, & \alpha = 0, \quad \beta = \sigma; \\
x_{h,k} & = & \sin\frac{h(2k-1)\pi}{2n+1}, & \alpha = \sigma, \quad \beta = 0; \\
x_{h,k} & = & \sin\frac{(2h-1)k\pi}{2n+1}, & \alpha = 0, \quad \beta = -\sigma; \\
x_{h,k} & = & \cos\frac{(2h-1)(2k-1)\pi}{2(2n+1)}, & \alpha = -\sigma, \quad \beta = 0; \\
x_{h,k} & = & \sin\frac{(2h-1)(2k-1)\pi}{4n}, & \alpha = \sigma, \quad \beta = -\sigma; \\
x_{h,k} & = & \cos\frac{(2h-1)(2k-1)\pi}{4n}, & \alpha = -\sigma, \quad \beta = \sigma; \\
x_{h,k} & = & \sin\frac{h(2k-1)\pi}{2n}, & \alpha = \sigma, \quad \beta = \sigma; \\
x_{h,k} & = & \cos\frac{(h-1)(2k-1)\pi}{2n}, & \alpha = -\sigma, \quad \beta = -\sigma
\end{array}$$

for $k = 1:n$. □

5. APPLICATIONS

This section discusses how the theory developed in the previous sections can be applied to approximate the eigenvalues or accurately evaluate the spectral factorization of certain matrices.

5.1. Approximation of the spectrum of a real symmetric tridiagonal matrix

Let $A_n \in \mathbb{R}^{n\times n}$ be a symmetric tridiagonal matrix. Denote the jth subdiagonal entry of A_n by σ_j, $j = 1:n-1$, and let δ_j be the jth diagonal entry, $j = 1:n$. The matrix A_n may, for instance, have been determined by carrying out n steps of the symmetric Lanczos algorithm applied to a large symmetric matrix A; see, e.g., [9] for a discussion on this algorithm.

Let $T := A_n|_\mathcal{T}$ be the orthogonal projection of A_n in the subspace $\mathcal{T}$ of tridiagonal Toeplitz matrices. We are interested in the matrix T because its eigenvalues are known in closed form and can be used to estimate the eigenvalues of A_n.

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Proposition 5.1
T is a real symmetric tridiagonal Toeplitz matrix.

Proof
The proof is straightforward, because both the subdiagonal and superdiagonal entries of T are equal to $\frac{\sum_{j=1}^{n-1}\sigma_j}{n-1}$. □

Proposition 5.2
If the trace of A_n vanishes, then the spectrum of T is real and symmetric with respect to the origin. Moreover, if n is odd, then T is singular. For n even,

$$\kappa_2(T) = \frac{\cos\frac{\pi}{n+1}}{\cos\frac{n\pi}{2(n+1)}}.$$

Proof
The diagonal entries of T, given by $\delta = \frac{\sum_{j=1}^{n}\delta_j}{n}$, vanish. Therefore the spectrum $\{\lambda_j\}_{j=1}^{n}$ of T is symmetric with respect to the origin. If n is odd, zero is an eigenvalue; otherwise, if n is even, one has $\kappa_2(T) = \lambda_1/\lambda_{\frac{n}{2}}$, where the eigenvalues are defined by (1.2) with $\tau = \sigma$. This concludes the proof. □

We have that T coincides with A_n if and only if A_n is a Toeplitz matrix. Thus, trivially, if A_n is a scalar, then T coincides with A_n. Moreover, the following inequality holds.

Proposition 5.3
Let $\lambda_1(A_n) \geq \ldots \geq \lambda_n(A_n)$ denote the eigenvalues of A_n in decreasing order and let λ_i be the eigenvalues of T given by (1.2) with $\tau = \sigma$. Then the average of the squared distances between the eigenvalues of A_n and T satisfies

$$\frac{1}{n}\sum_{i=1}^{n}(\lambda_i(A_n) - \lambda_i)^2 \leq \frac{1}{n}\|A_n - T\|_F^2.$$

Proof
Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices. Denote by $\lambda_\downarrow(M)$ $[\lambda_\uparrow(M)]$ the vector whose entries are the eigenvalues of a symmetric matrix M sorted in decreasing [increasing] order. Then

$$\|\lambda_\downarrow(A) - \lambda_\downarrow(B)\| \leq \|A - B\|_F \leq \|\lambda_\downarrow(A) - \lambda_\uparrow(B)\|;$$

see, e.g., [2]. This shows the proposition. □

Remark 5.4. Notice that A_n being symmetric positive definite does not guarantee that T is positive definite. Indeed, T is positive definite if and only if

$$\frac{\sum_{j=1}^{n}\delta_j}{n} > 2\frac{\sum_{j=1}^{n-1}\sigma_j}{n-1}\cos\frac{\pi}{n+1}.$$

Let $T = (n; \sigma, \delta, \tau)$ be symmetric and define the Toeplitz-type matrix $A_n := T_{\alpha,\beta}$, where $\alpha = \pm\sqrt{\sigma\tau}$ and $\beta = \mp\sqrt{\sigma\tau}$. The eigenvalues of the matrix A_n are symmetric with respect to δ; expressions for the eigenvalues are provided in the fifth and sixth rows of Table I. It is easy to show that T is the closest tridiagonal Toeplitz matrix to A_n in the Frobenius norm. Moreover, if $\delta = 0$, then the eigenvalues of A_n are symmetric with respect to the origin and A_n has null trace so that, due to Proposition 5.2, the spectrum of T is symmetric with respect to the origin.

We illustrate Proposition 5.3 with an example. Let the matrix $A_n = [a_{i,j}] \in \mathbb{R}^{n\times n}$ differ from the symmetric Toeplitz matrix $T = (n, \sigma, \delta, \sigma)$ only in the entry $a_{2,2}$. Then the proposition shows that

$$\frac{1}{n}\sum_{i=1}^{n}(\lambda_i(A_n) - \lambda_i)^2 \leq \frac{1}{n}|a_{2,2} - \delta|^2.$$

In particular, the sum in the left-hand side converges to zero as n increases. Hence, the spectrum of T furnishes an accurate approximation of the spectrum of A_n when n is large.

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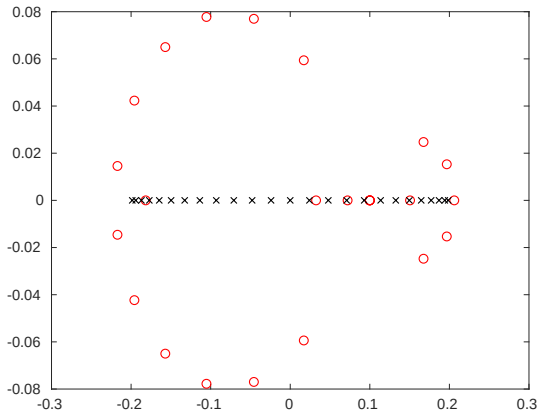

Figure 3. Exact eigenvalues (5.2) of $T = (25; 1, 0, 0.01)$ (marked with black x) and the approximate eigenvalues computed by the function eig (marked with red o).

5.2. *Accurate computation of the spectrum of nonsymmetric nearly tridiagonal Toeplitz matrices*

Let the tridiagonal Toeplitz matrix $T = (n; \delta, \sigma, \tau)$ be nonsymmetric. It has the spectral factorization

$$T = X \Lambda X^{-1}, \tag{5.1}$$

where $X \in \mathbb{C}^{n \times n}$ is the eigenvector matrix whose columns are given by (1.3) and the entries of the matrix $\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ are the eigenvalues given by (1.2).

When $T \in \mathbb{R}^{n \times n}$ is far from symmetric, then the MATLAB function eig is only able to compute the spectral factorization (5.1) with reduced accuracy. For instance, consider the matrix $T = (25; 1, 0, 0.01)$. The eigenvalues of T are given by

$$\lambda_h = 0.2 \cos \frac{h\pi}{26}, \quad h = 1 : 25, \tag{5.2}$$

while many of the eigenvalues determined by the function eig have a significant imaginary part; see Figure 3.

Also the spectrum of other nonsymmetric matrices can be difficult to compute accurately by the function eig. When the matrix of interest, $A_n \in \mathbb{R}^{n \times n}$, is close to a Toeplitz matrix $T = (n; \sigma, \delta, \tau)$, the spectral factorization (5.1) may be used to determine a more accurate spectral factorization of A_n than can be computed with eig in the following manner:

1. Determine the tridiagonal Toeplitz matrix T closest to A_n in the Frobenius norm.
2. Determine the spectral factorization (5.1) of T by using (1.2) and (1.3).
3. Evaluate the matrix $B = X^{-1} A_n X = \Lambda + X^{-1}(A_n - T)X$. If T is close to A_n, then this matrix is closer to a symmetric matrix than A_n.
4. Compute the spectral factorization $B = Y D Y^{-1}$ by using the MATLAB function eig. Thus, Y is the eigenvector matrix of B, and D is a diagonal matrix, whose nontrivial entries are the eigenvalues. Typically, the matrix Y is fairly well conditioned and can be computed by the function eig with quite high accuracy. The matrix $Z = XY$ is (an approximation of) the eigenvector matrix of A_n.

We illustrate the computations outlined with an example. Let $T = (25; 1, 0, 0.01)$, and let $A_n = T_{\alpha,\beta} \in \mathbb{R}^{n \times n}$ be a tridiagonal Toeplitz-type matrix (1.5) obtained from T with $\alpha = 0.1$ and $\beta = -\alpha$. The eigenvalues of A_n are real and symmetric with respect to the origin; their formulas are shown in the fifth and sixth rows of Table I. The eigenvectors of A_n are described

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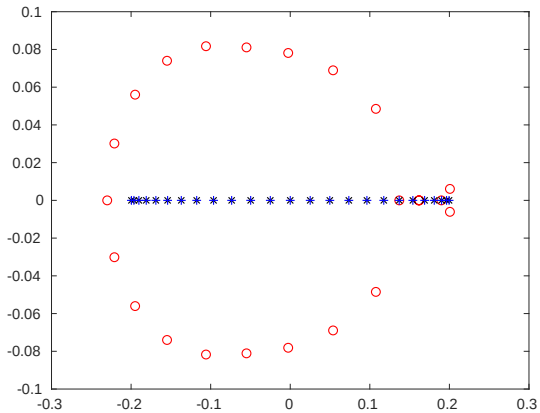

Figure 4. Exact eigenvalues of A_n (marked with black x), the approximate eigenvalues computed by the function eig (marked with red o), and eigenvalues computed by the algorithm outlined above (marked with blue +).

in Section 1. Hence, it is straightforward to assess the accuracy of the computational method described. It is easy to see that T is the closest tridiagonal Toeplitz matrix to A_n. Its eigenvalues and eigenvectors are given by (1.2) and (1.3).

Figure 4 displays the spectrum of A_n computed by using the relevant formulas of Table I (marked with black +), and approximations of the spectrum computed by the MATLAB function eig (marked with red o) and the procedure described above (marked with blue x). The eigenvalues determined in the latter manner cannot be distinguished from the exact ones in Figure 4, while some of the approximate eigenvalues computed by eig applied to A_n can be seen to have large imaginary components. The maximum pairwise difference of the exact eigenvalues and the eigenvalues computed by the MATLAB function eig, ordered in the same manner, is $4.3 \cdot 10^{-1}$, while the maximum pairwise difference of the exact eigenvalues and the eigenvalues computed by our approach described above only is $3.3 \cdot 10^{-8}$. Thus, the approximation of a tridiagonal matrix by the closest Toeplitz matrix and using the spectral factorization of the latter may yield a more accurate spectral factorization than the one determined by the MATLAB function eig.

6. CONCLUSIONS

The paper discusses the sensitivity of eigenvectors of tridiagonal Toeplitz matrices under general and structured perturbations. The eigenvectors are found to be quite sensitive to perturbations when the Toeplitz matrix is far from normal, but the eigenvectors are insensitive to structured perturbation when the Toeplitz matrix has additional structure, such as being real symmetric. Our analysis suggests a novel method for computing the spectral factorization of a general nonsymmetric tridiagonal matrix.

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