Generalized circulant Strang-type preconditioners

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Dedicated to Gilbert Strang on the Occasion of His 75th Birthday.

SUMMARY

Strang's proposal to use a circulant preconditioner for linear systems of equations with a Hermitian positive definite Toeplitz matrix has given rise to considerable research on circulant preconditioners. This paper presents an $\{e^{i\varphi}\}$-circulant Strang-type preconditioner.

KEY WORDS: Toeplitz matrix, generalized Hermitian matrix, circulant matrix, generalized circulant, preconditioning

1. Introduction

Linear systems of equations with a nonsingular Toeplitz matrix

$$Tx = b, \qquad T \in \mathbb{C}^{n\times n}, \qquad x, b \in \mathbb{C}^n, \tag{1}$$

arise in many applications in signal processing, Markov chains, and from the discretization of certain integral and differential equations; see, e.g., [1, 2].

Let $\theta \in (-\pi, \pi]$. A Toeplitz matrix

$$T = (n;\sigma,\delta,\tau) = \begin{bmatrix} \delta & \tau_1 & \tau_2 & & \cdots & & \tau_{n-1} \\ \sigma_1 & \delta & \tau_1 & \tau_2 & \cdots & & \tau_{n-2} \\ \sigma_2 & \sigma_1 & & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & & \\ & & & & & \tau_1 & \tau_2 \\ \sigma_{n-2} & & \cdots & & \sigma_1 & \delta & \tau_1 \\ \sigma_{n-1} & & \cdots & & \sigma_2 & \sigma_1 & \delta \end{bmatrix} \in \mathbb{C}^{n\times n} \tag{2}$$

is said to be $\{e^{i\theta}\}$-*Hermitian* if

$$\sigma_\ell = \bar{\tau}_\ell e^{i\theta}, \qquad 1 \le \ell < n, \tag{3}$$

where $\mathrm{i} = \sqrt{-1}$ and the bar denotes complex conjugation. The diagonal entries δ are arbitrary. Toeplitz matrices whose entries satisfy (3) for some angle θ are said to be *generalized Hermitian*. The angle $\theta = 0$ and $\delta \in \mathbb{R}$ yield Hermitian Toeplitz matrices; $\theta = \pi$ and $\delta = 0$ give skew-Hermitian ones.

The Toeplitz matrix (2) is referred as an $\{e^{\mathrm{i}\varphi}\}$-*circulant* if

$$\sigma_\ell = \tau_{n-\ell} e^{\mathrm{i}\varphi}, \qquad 1 \le \ell < n, \tag{4}$$

for some $\varphi \in (-\pi, \pi]$. Again, the diagonal entries are arbitrary. Toeplitz matrices whose entries satisfy (4) for some angle φ are said to be *generalized circulants*. The angles $\varphi = 0$ and $\varphi = \pi$ give (standard) circulants and skew-circulants, respectively.

Proposition 1.1. *Let $T = (n; \sigma, \delta, \tau)$ be an $\{e^{\mathrm{i}\theta}\}$-Hermitian Toeplitz matrix as well as an $\{e^{\mathrm{i}\varphi}\}$-circulant. Then T is determined by the two angles θ and φ, and by the first $\lfloor n/2 \rfloor$ entries of the first row, where $\lfloor n/2 \rfloor$ denotes the largest integer bounded above by $n/2$. If n is even, then only the magnitude of $\tau_{n/2}$ can be prescribed.*

Proof: Since T is a generalized Hermitian Toeplitz matrix, its entries satisfy $\sigma_\ell = \bar{\tau}_\ell e^{\mathrm{i}\theta}$, $1 \le \ell < n$. The fact that T is a generalized circulant yields the relation $\sigma_\ell = \tau_{n-\ell} e^{\mathrm{i}\varphi}$, $1 \le \ell < n$. It follows that $\tau_\ell = \bar{\tau}_{n-\ell} e^{\mathrm{i}(\theta-\varphi)}$, $1 \le \ell < n$. Thus, if n is odd, then T is determined by θ, φ, δ, and τ_k for $1 \le k \le (n-1)/2$. On the other hand, if n is even, then the argument of $\tau_{n/2}$ has to be equal to $(\theta - \varphi)/2$ (unless $\tau_{n/2} = 0$), and therefore only $|\tau_{n/2}|$ can be prescribed. ∎

We are concerned with the solution of linear systems of equations (1) with a generalized Hermitian Toeplitz matrix $T \in \mathbb{C}^{n \times n}$ by a Krylov subspace method. A preconditioner $C \in \mathbb{C}^{n \times n}$ often is used in order to increase the rate of convergence of the iterative method. When both T and C are Hermitian positive definite, we apply the conjugate gradient method to the solution of the preconditioned system

$$TC^{-1} y = b. \tag{5}$$

If T or C are not Hermitian positive definite, then the conjugate gradient method can be applied to the preconditioned normal equations

$$C^{-H} T^H T C^{-1} y = C^{-H} T^H b, \tag{6}$$

where the superscript ^{H} denotes transposition and complex conjugation. Alternatively, one may apply a Krylov subspace iterative method designed for the solution of linear systems of equations with a general nonsingular matrix to (5); see, e.g., [3, 4] for discussions on iterative methods and preconditioning.

Circulant matrices, which approximate T in some sense, have been used successfully as preconditioners for (1). Recent discussions and many references can be found in [2, 5]. The first circulant preconditioner is due to Strang [6], who proposed to precondition linear systems of equations (1) with a Hermitian positive definite Toeplitz matrix T by a circulant matrix with as many central diagonals equal to those of T as possible; see also [7]. The success of this preconditioner, known as the Strang preconditioner, sparked considerable research on the construction of circulant preconditioners and their properties. The present paper is concerned with generalizations to linear systems of equations (1) with an $\{e^{\mathrm{i}\theta}\}$-Hermitian Toeplitz matrix T and the application of $\{e^{\mathrm{i}\varphi}\}$-circulant Strang-type preconditioners. Properties of these preconditioners are investigated in Sections 2 and 4. Other generalized circulant preconditioners, including preconditioners based on skew-circulants, have been considered in [8, 9, 10, 12, 13]; see also [2, 5].

An attractive property of circulant and generalized circulant preconditioners is that the spectral factorization of such preconditioners of order n can be computed in $\mathcal{O}(n \log n)$ arithmetic floating point operations with the aid of the fast Fourier transform method. The spectral factorization allows the evaluation of the inverse of the preconditioner times a vector in only $\mathcal{O}(n \log n)$ arithmetic floating point operations. Many properties of circulants and generalized circulants can be found in [2, 5, 14]. We discuss properties relevant for the design of iterative solution methods for (5) and (6) in Section 3. Spectral properties of the preconditioned system are considered in Section 4. A few computed examples are presented in Section 5 and concluding remarks can be found in Section 6.

2. Generalized Strang-type preconditioners

The Strang preconditioner for a Hermitian Toeplitz matrix $T \in \mathbb{C}^{n\times n}$ is the Hermitian circulant matrix $S \in \mathbb{C}^{n\times n}$ obtained by first copying as many of the central diagonals of T into S as possible, and then completing S to a circulant; see, e.g., [2, 5, 6, 7] for details on the Strang preconditioner.

We describe $\{e^{i\varphi}\}$-*Strang-type preconditioners* for generalized Hermitian Toeplitz matrices. Let $T = (n; \sigma, \delta, \tau)$ be a generalized Hermitian Toeplitz matrix and let the $\{e^{i\varphi}\}$-circulant $S_\varphi = (n; \sigma^S, \delta^S, \tau^S)$ be obtained by copying as many central diagonals of T into S_φ as possible, and then completing S_φ to an $\{e^{i\varphi}\}$-circulant. We refer to the generalized circulant S_φ as an $\{e^{i\varphi}\}$-*Strang-type preconditioner*, or briefly as a *generalized Strang preconditioner*. The angle $-\pi < \varphi \le \pi$ is chosen to improve the spectral properties of the preconditioned matrix. The choice of φ depends on whether n is odd or even.

2.1. Generalized Strang preconditioners of odd order

We discuss properties of generalized Strang preconditioners for $\{e^{i\theta}\}$-Hermitian Toeplitz matrices $T = (n; \sigma, \delta, \tau)$ of odd order. Thus, n is assumed to be odd in this subsection. The entries of the generalized Strang preconditioner $S_\varphi = (n; \sigma^S, \delta^S, \tau^S)$ are given by

$$
\begin{aligned}
\delta^S &= \delta, \\
\sigma_k^S &= \begin{cases} \sigma_k, & 1 \le k \le \lfloor n/2 \rfloor, \\ \tau_{n-k} e^{i\varphi}, & \lfloor n/2 \rfloor < k < n, \end{cases} \\
\tau_k^S &= \begin{cases} \tau_k, & 1 \le k \le \lfloor n/2 \rfloor, \\ \sigma_{n-k} e^{-i\varphi}, & \lfloor n/2 \rfloor < k < n. \end{cases}
\end{aligned}
$$

Proposition 2.1. *Let T be an $\{e^{i\theta}\}$-Hermitian Toeplitz matrix. Then the $\{e^{i\varphi}\}$-Strang-type preconditioner $S_\varphi = \{n; \sigma^S, \delta^S, \tau^S\}$ is an $\{e^{i\theta}\}$-Hermitian Toeplitz matrix for all $-\pi < \varphi \le \pi$. Its entries are determined by θ, δ, and τ_k, $1 \le k \le \lfloor n/2 \rfloor$, and by the arbitrary angle*

$-\pi < \varphi \leq \pi$. *Specifically,*

$$\delta^S = \delta,$$
$$\sigma_k^S = \begin{cases} \bar{\tau}_k \mathrm{e}^{\mathrm{i}\theta}, & 1 \leq k \leq \lfloor n/2 \rfloor, \\ \tau_{n-k}\mathrm{e}^{\mathrm{i}\varphi}, & \lfloor n/2 \rfloor < k < n, \end{cases}$$
$$\tau_k^S = \begin{cases} \tau_k, & 1 \leq k \leq \lfloor n/2 \rfloor, \\ \bar{\tau}_{n-k}\mathrm{e}^{\mathrm{i}(\theta-\varphi)}, & \lfloor n/2 \rfloor < k < n. \end{cases}$$

In particular, when T is Hermitian, so is S_φ for any $-\pi < \varphi \leq \pi$.

Proof: Trivially, one has $\sigma_k^S = \bar{\tau}_k^S \mathrm{e}^{\mathrm{i}\theta}$ for $1 \leq k \leq \lfloor n/2 \rfloor$. We will show that this equality also holds for $\lfloor n/2 \rfloor < k < n$. Since $\sigma_k^S = \tau_{n-k}\mathrm{e}^{\mathrm{i}\varphi}$ and $\bar{\tau}_k^S \mathrm{e}^{\mathrm{i}\theta} = \bar{\sigma}_{n-k}\mathrm{e}^{\mathrm{i}(\varphi+\theta)}$, the proposition follows from the observation that $\tau = \bar{\sigma}\mathrm{e}^{\mathrm{i}\theta}$ implies that $\bar{\tau}_k^S \mathrm{e}^{\mathrm{i}\theta} = \tau_{n-k}\mathrm{e}^{\mathrm{i}\varphi} = \sigma_k^S$. In particular, $\theta = 0$ yields that $\bar{\tau}_k^S = \sigma_k^S$. ∎

The following result is a generalization of the optimality of the circulant Strang preconditioner [2, Theorem 4.1]. The norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ below denote the matrix norms induced by the ℓ_1 and uniform vector norms in $\mathbb{C}^n$, respectively.

Theorem 2.1. *Let $T = (n; \sigma, \delta, \tau)$, with n odd, be an $\{\mathrm{e}^{\mathrm{i}\theta}\}$-Hermitian Toeplitz matrix. The $\{\mathrm{e}^{\mathrm{i}\varphi}\}$-Strang-type preconditioner minimizes $\|C - T\|_1 = \|C - T\|_\infty$ over all $\{\mathrm{e}^{\mathrm{i}\theta}\}$-Hermitian generalized circulants for all angles $-\pi < \varphi \leq \pi$.*

Proof: The entries of the matrix T satisfy $\sigma_k = \bar{\tau}_k \mathrm{e}^{\mathrm{i}\theta}$ for $1 \leq k < n$. Since the minimizer $C^* = (n; \sigma^*, \delta^*, \tau^*)$ of $\|C - T\|_1$ is an $\{\mathrm{e}^{\mathrm{i}\theta}\}$-Hermitian matrix, one has $\sigma_k^* = \bar{\tau}_k^* \mathrm{e}^{\mathrm{i}\theta}$, $1 \leq k < n$. Moreover, in view of that C^* is a generalized circulant, there is an angle $-\pi < \varphi \leq \pi$, such that $\sigma_k^* = \tau_{n-k}^* \mathrm{e}^{\mathrm{i}\varphi}$, $1 \leq k < n$. It follows from Proposition 1.1 that C^* is determined by φ, θ, δ^*, and τ_k^*, $1 \leq k \leq \frac{n-1}{2}$. In particular, $\tau_k^* = \bar{\tau}_{n-k}^* \mathrm{e}^{\mathrm{i}(\theta-\varphi)}$, $1 \leq k < n$.

The hth absolute column sum c_h of $C^* - T$ is given by

$$c_h = \sum_{k=1}^{h-1} |\tau_k^* - \tau_k| + |\delta^* - \delta| + \sum_{k=1}^{n-h} |\sigma_k^* - \sigma_k|, \quad 1 \leq h \leq n.$$

Therefore, for $h < \frac{n+1}{2}$,

$$c_h = \sum_{k=1}^{h-1} |\tau_k^* - \tau_k| + |\delta^* - \delta| + \sum_{k=1}^{\frac{n-1}{2}} |\bar{\tau}_k^* - \bar{\tau}_k| + \sum_{k=\frac{n+1}{2}}^{n-h} |\tau_{n-k}^* \mathrm{e}^{\mathrm{i}(\varphi-\theta)} - \bar{\tau}_k|.$$

If $h = \frac{n+1}{2}$, then

$$c_h = \sum_{k=1}^{\frac{n-1}{2}} |\tau_k^* - \tau_k| + |\delta^* - \delta| + \sum_{k=1}^{\frac{n-1}{2}} |\bar{\tau}_k^* - \bar{\tau}_k|,$$

and, for $h > \frac{n+1}{2}$, we have

$$c_h = \sum_{k=1}^{\frac{n-1}{2}} |\tau_k^* - \tau_k| + \sum_{k=\frac{n+1}{2}}^{h-1} |\bar{\tau}_{n-k}^* \mathrm{e}^{\mathrm{i}(\theta-\varphi)} - \tau_k| + |\delta^* - \delta| + \sum_{k=1}^{n-h} |\bar{\tau}_k^* - \bar{\tau}_k|.$$

Since $c_h = c_{n-h}$ for $1 \le h \le n$, it suffices to consider c_h for $1 \le h \le \frac{n+1}{2}$. We would like C^* to minimize each absolute column sum, and first observe that this yields $\delta^* = \delta$. Moreover, the terms involving τ_k^* for $1 \le k \le \frac{n-1}{2}$ are either of the form $|\tau_k^* - \tau_k| + |\tau_k^* - \bar{\tau}_{n-k}\mathrm{e}^{\mathrm{i}(\theta-\varphi)}|$ (and the minimum is achieved at any point of the line segment between τ_k and $\bar{\tau}_{n-k}\mathrm{e}^{\mathrm{i}(\theta-\varphi)}$) or of the form $2|\tau_k^* - \tau_k|$. It follows that the absolute column sums attain their minima when $\tau_k^* = \tau_k$ for $1 \le k \le \frac{n-1}{2}$. ∎

The following result suggests a choice of the angle φ.

Theorem 2.2. *Assume that the entries of the Toeplitz matrix $T = (n; \sigma, \delta, \tau)$ of odd order $n = 2m+1$ satisfy*

$$\sum_{h=1}^{m} h(\sigma_h \overline{\tau}_{n-h} + \sigma_{n-h}\overline{\tau}_h) \neq 0. \tag{7}$$

Then $S_{\varphi^*} = (n; \sigma^S, \delta^S, \tau^S)$ with φ^* given by

$$\varphi^* = \arg\left(\sum_{h=1}^{m} h(\sigma_h \overline{\tau}_{n-h} + \sigma_{n-h}\overline{\tau}_h)\right) \tag{8}$$

minimizes the distance $d(\varphi)$ from T in the Frobenius norm over all generalized Strang preconditioners. We have

$$d(\varphi^*) = \sqrt{\sum_{h=1}^{m} h(|\sigma_h|^2 + |\tau_{n-h}|^2 + |\sigma_{n-h}|^2 + |\tau_h|^2) - 2\left|\sum_{h=1}^{m} h(\sigma_h \overline{\tau}_{n-h} + \sigma_{n-h}\overline{\tau}_h)\right|}.$$

Moreover, if (7) is violated and $(\sigma, \tau) \neq (0, 0)$, then there are infinitely many generalized Strang preconditioners S_φ, which depend on the arbitrary angle $-\pi < \varphi \le \pi$, at the same minimal distance from T.

Proof: The desired value of $-\pi < \varphi \le \pi$ is obtained by minimizing the squared distance

$$\begin{aligned} d^2(\varphi) &= \|S_\varphi - T\|_F^2 = \sum_{h=1}^{m} h(|\sigma_h \mathrm{e}^{-\mathrm{i}\varphi} - \tau_{n-h}|^2 + |\sigma_{n-h}\mathrm{e}^{-\mathrm{i}\varphi} - \tau_h|^2) \\ &= \sum_{h=1}^{m} h(|\sigma_h|^2 + |\tau_{n-h}|^2 + |\sigma_{n-h}|^2 + |\tau_h|^2) \\ &\quad -2\mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}\varphi}\sum_{h=1}^{m} h(\sigma_h \overline{\tau}_{n-h} + \sigma_{n-h}\overline{\tau}_h)\right). \end{aligned}$$

If (7) holds, then $d^2(\varphi)$ is minimal for the angle

$$\varphi^* = \arg(\sum_{h=1}^{m} h(\sigma_h \overline{\tau}_{n-h} + \sigma_{n-h}\overline{\tau}_h)),$$

and the proof is straightforward. On the other hand, if (7) is violated, then for all $-\pi < \varphi \le \pi$,

$$d(\varphi) = \sqrt{\sum_{h=1}^{m} h(|\sigma_h|^2 + |\tau_{n-h}|^2 + |\sigma_{n-h}|^2 + |\tau_h|^2)}.$$

∎

2.2. Generalized Strang preconditioners of even order

When n is even the entries $\sigma^S_{n/2}$ and $\tau^S_{n/2}$ of the Strang and generalized Strang preconditioners are not uniquely determined by the entries of $T = (n; \sigma, \delta, \tau)$. For instance, when T is Hermitian the choices

$$\begin{aligned} \sigma^S_{n/2} &= \tau^S_{n/2} = 0, && (9) \\ \sigma^S_{n/2} &= \tau^S_{n/2} = \frac{1}{2}(\sigma_{n/2} + \tau_{n/2}) && (10) \end{aligned}$$

for the Strang preconditioner have been considered; see, e.g., [5, p. 18]. We investigate properties of $\{e^{i\varphi}\}$-circulant Strang-type preconditioners for the analogous choices. The following results are similar to those of Proposition 2.1 and Theorem 2.2.

Proposition 2.2. *Let T be an $\{e^{i\theta}\}$-Hermitian Toeplitz matrix of order n. Then the $\{e^{i\varphi}\}$-Strang-type preconditioner S_φ, with the entries $\sigma^S_{n/2}$ and $\tau^S_{n/2}$ defined by (9), is an $\{e^{i\theta}\}$-Hermitian Toeplitz matrix for all $-\pi < \varphi \leq \pi$. In particular, S_φ is determined by θ, δ, and τ_k, $1 \leq k < \lfloor n/2 \rfloor$, and by the arbitrary angle $-\pi < \varphi \leq \pi$. Its entries are given by*

$$\begin{aligned} &\delta^S = \delta, \\ &\sigma^S_k = \begin{cases} \bar{\tau}_k e^{i\theta}, & 1 \leq k < n/2, \\ 0, & k = n/2, \\ \tau_{n-k} e^{i\varphi}, & n/2 < k < n, \end{cases} \\ &\tau^S_k = \begin{cases} \tau_k, & 1 \leq k < n/2, \\ 0, & k = n/2, \\ \bar{\tau}_{n-k} e^{i(\theta-\varphi)}, & n/2 < k < n. \end{cases} \end{aligned} \qquad (11)$$

When T is Hermitian, then so is S_φ for all $-\pi < \varphi \leq \pi$.

Proof: The result can be shown similarly as Proposition 2.1; just replace $\lfloor n/2 \rfloor$ by $\lfloor (n-1)/2 \rfloor$. ∎

Theorem 2.3. *Let $n = 2m + 2$ for some integer $m \geq 0$, and assume that the Toeplitz matrix $T = (n; \sigma, \delta, \tau)$ satisfies (7). Then S_{φ^*}, with φ^* determined by (8), minimizes the distance $d(\varphi)$ from T in the Frobenius norm over all generalized circulant Strang-type preconditioners. Moreover, if (7) is violated and $(\sigma, \tau) \neq (0, 0)$, then there are infinitely many matrices S_φ, which depend on the arbitrary angle $-\pi < \varphi \leq \pi$, at the same minimal distance from T. We have*

$$d^2(\varphi^*) = \sum_{h=1}^{m+1} h(|\sigma_h|^2 + |\tau_h|^2) + \sum_{h=1}^{m} h(|\tau_{n-h}|^2 + |\sigma_{n-h}|^2) - 2\left|\sum_{h=1}^{m} h(\sigma_h \overline{\tau}_{n-h} + \sigma_{n-h}\overline{\tau}_h)\right|.$$

Proof: The desired value of $-\pi < \varphi \le \pi$ is obtained by minimizing the squared distance

$$\begin{aligned} d^2(\varphi) &= \|S_\varphi - T\|_F^2 \\ &= \sum_{h=1}^{m} h(|\sigma_h e^{-i\varphi} - \tau_{n-h}|^2 + |\sigma_{n-h} e^{-i\varphi} - \tau_h|^2) + (m+1)(|\sigma_{m+1}|^2 + |\tau_{m+1}|^2) \\ &= \sum_{h=1}^{m+1} h(|\sigma_h|^2 + |\tau_h|^2) + \sum_{h=1}^{m} h(|\tau_{n-h}|^2 + |\sigma_{n-h}|^2) \\ &\quad -2\mathrm{Re}\left(e^{-i\varphi} \sum_{h=1}^{m} h(\sigma_h \overline{\tau}_{n-h} + \sigma_{n-h}\overline{\tau}_h)\right). \end{aligned}$$

If (7) holds, then $d^2(\varphi)$ is minimal for the angle

$$\varphi^* = \arg(\sum_{h=1}^{m} h(\sigma_h \overline{\tau}_{n-h} + \sigma_{n-h}\overline{\tau}_h))$$

and the desired result follows. Conversely, if (7) is violated, then for all $-\pi < \varphi \le \pi$,

$$d^2(\varphi) = \sum_{h=1}^{m+1} h(|\sigma_h|^2 + |\tau_h|^2) + \sum_{h=1}^{m} h(|\tau_{n-h}|^2 + |\sigma_{n-h}|^2).$$

■

Briefly consider the situation when $T = (n, \tau, \delta, \sigma)$ is real and symmetric and assume that $\tau_{n/2} \neq 0$. The real circulant Strang preconditioner with $\sigma_{n/2}^S$ and $\tau_{n/2}^S$ given by (10) matches two more diagonals of T than the circulant Strang preconditioner with entries $\sigma_{n/2}^S$ and $\tau_{n/2}^S$ defined by (9). However, this is not the case when T is Hermitian and $\tau_{n/2}$ has a nonvanishing imaginary part. In order to be able to match the diagonals with entries $\sigma_{n/2}$ and $\tau_{n/2}$ of a Hermitian Toeplitz matrix T, typically a generalized Strang preconditioner with a particular choice of angle φ has to be used. The remainder of this section discusses the choice of φ when T is a generalized Hermitian matrix.

Thus, let $T = (n; \sigma, \delta, \tau)$ be an $\{e^{i\theta}\}$-Hermitian Toeplitz matrix of even order. We will show below that the best $\{e^{i\varphi^*}\}$-Strang-type preconditioner $S_{\varphi^*} = (n; \sigma^S, \delta^S, \tau^S)$ is given by

$$\begin{aligned} &\delta^S = \delta, \\ &\sigma_k^S = \begin{cases} \sigma_k, & 1 \le k \le n/2, \\ \tau_{n-k} e^{i\varphi^*}, & n/2 < k < n, \end{cases} \\ &\tau_k^S = \begin{cases} \tau_k, & 1 \le k \le n/2, \\ \sigma_{n-k} e^{-i\varphi^*}, & n/2 < k < n, \end{cases} \\ &\varphi^* = \theta - 2\arg(\tau_{n/2}). \end{aligned} \tag{12}$$

We first note that Proposition 2.1 holds for the generalized Strang preconditioner $S_{\varphi^*} = (n; \sigma^S, \delta^S, \tau^S)$ defined by (12). However, the proposition generally does not hold when φ^* is replaced by an arbitrary angle $-\pi < \varphi \le \pi$. The following result is analogous to Theorem 2.1.

Theorem 2.4. *Let $T = (n;\sigma,\delta,\tau)$ with n even be an $\{\mathrm{e}^{\mathrm{i}\theta}\}$-Hermitian Toeplitz matrix. The $\{\mathrm{e}^{\mathrm{i}\varphi}\}$-Strang-type preconditioner defined by (12) minimizes $\|C-T\|_1 = \|C-T\|_\infty$ over all $\{\mathrm{e}^{\mathrm{i}\theta}\}$-Hermitian generalized circulants.*

Proof: The entries of the matrix T satisfy $\sigma_k = \bar{\tau}_k \mathrm{e}^{\mathrm{i}\theta}$ for $1 \le k < n$. Since the minimizer $C^* = (n;\sigma^*,\delta^*,\tau^*)$ of $\|C-T\|_1$ is an $\{\mathrm{e}^{\mathrm{i}\theta}\}$-Hermitian matrix, one has $\sigma_k^* = \bar{\tau}_k^* \mathrm{e}^{\mathrm{i}\theta}$, $1 \le k < n$. Moreover, in view of that C^* is a generalized circulant, there is an angle $-\pi < \varphi \le \pi$, such that $\sigma_k^* = \tau_{n-k}^* \mathrm{e}^{\mathrm{i}\varphi}$, $1 \le k < n$. It follows from Proposition 1.1 that C^* is determined by φ, θ, δ^*, and τ_k^*, $1 \le k \le \frac{n}{2}$. In particular, since $\tau_k^* = \bar{\tau}_{n-k}^* \mathrm{e}^{\mathrm{i}(\theta-\varphi)}$, $1 \le k < n$, the argument of $\tau_{n/2}^*$ has to be equal to $(\theta-\varphi)/2$ (unless $\tau_{n/2}^* = 0$).

It suffices to consider the hth absolute column sum c_h of $C^* - T$ for $1 \le h \le \frac{n}{2}$, since $c_h = c_{n-h}$ for $1 \le h \le n$. We have

$$c_h = \sum_{k=1}^{h-1} |\tau_k^* - \tau_k| + |\delta^* - \delta| + \sum_{k=1}^{\frac{n}{2}} |\bar{\tau}_k^* - \bar{\tau}_k| + \sum_{k=\frac{n+2}{2}}^{n-h} |\tau_{n-k}^* \mathrm{e}^{\mathrm{i}(\varphi-\theta)} - \bar{\tau}_k|, \quad 1 \le h \le \frac{n}{2}.$$

Minimizing each c_h first yields $\delta^* = \delta$. Moreover, the terms involving τ_k^* for $1 \le k \le \frac{n}{2}$ are either of the form $|\tau_k^* - \tau_k| + |\tau_k^* - \bar{\tau}_{n-k}\mathrm{e}^{\mathrm{i}(\theta-\varphi)}|$ or of the form $2|\tau_k^* - \tau_k|$. Taking into account the additional conditions either on $\tau_{n/2}^*$ or on the angle φ, it follows that the c_h attain their minima when $\tau_k^* = \tau_k$ for $1 \le k \le \frac{n}{2}$ and $\varphi^* = \theta - 2\arg(\tau_{n/2})$. ∎

The fact that the generalized Strang circulant with entries (12) generally matches more diagonal entries of the Toeplitz matrix $T = (n;\sigma,\delta,\tau)$ than the generalized Strang circulant defined by (11) translates into that the former, when used as a preconditioner, typically gives faster convergence. This is illustrated in Section 5.

3. Computation with generalized circulants

Let $C_\varphi = (n;\sigma,\delta,\tau)$ be an $\{e^{\mathrm{i}\varphi}\}$-circulant. It is well known that C_φ can be diagonalized by the product of the Fourier matrix and a unitary diagonal matrix D_φ. The following proposition provides some details; a proof can be found in [14].

Proposition 3.1. *The $\{\mathrm{e}^{\mathrm{i}\varphi}\}$-circulant C_φ has the spectral factorization*

$$C_\varphi = D_\varphi F \Lambda F^H D_\varphi^H, \tag{13}$$

where $F = [f_{j,k}]_{j,k=0}^{n-1}$ is the Fourier matrix with entries

$$f_{j,k} = \frac{1}{\sqrt{n}} \mathrm{e}^{-\mathrm{i}\frac{2\pi jk}{n}}, \qquad 0 \le j,k < n,$$

and

$$\begin{aligned} D_\varphi &= \mathrm{diag}[1, \mathrm{e}^{\mathrm{i}\varphi/n}, \mathrm{e}^{2\mathrm{i}\varphi/n}, \ldots, \mathrm{e}^{(n-1)\mathrm{i}\varphi/n}], \\ \Lambda &= \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]. \end{aligned}$$

The spectral factorization (13) can be used to evaluate matrix-vector products with C_φ in only $\mathcal{O}(n \log n)$ arithmetic floating point operations. Assuming that C_φ is invertible, matrix-vector products with C_φ^{-1} can be computed similarly.

It is well known that matrix-vector products with a Toeplitz matrix $T \in \mathbb{C}^{n\times n}$ can be evaluated in $\mathcal{O}(n \log n)$ arithmetic floating point operations either by embedding T in a circulant of order $2n$ and using the spectral factorization of the latter, or by expressing T as a sum of a circulant and a skew-circulant, and using the spectral factorizations of the latter matrices; see, e.g., [2, Section 3.4.3].

Application of the conjugate gradient method to the preconditioned linear systems of equations (5) or (6) with $C = C_\varphi$ requires the computation of matrix-vector products with the matrix TC_φ^{-1}. We evaluate these matrix-products without forming the matrix. Instead, we split T into two generalized circulants, one of which is an $\{\mathrm{e}^{\mathrm{i}\varphi}\}$-circulant, as described by the following result.

Proposition 3.2. *The Toeplitz matrix $T = (n; \sigma, \delta, \tau)$ can be expressed as*

$$T = \hat{C}_\varphi + \widetilde{C}_\psi, \tag{14}$$

where $\hat{C}_\varphi = (n; \hat{\sigma}, \hat{\delta}, \hat{\tau})$ is an $\{\mathrm{e}^{\mathrm{i}\varphi}\}$-circulant and $\widetilde{C}_\psi$ is an $\{\mathrm{e}^{\mathrm{i}\psi}\}$-circulant, for any $-\pi < \varphi \neq \psi \leq \pi$.

Proof: The off-diagonal entries of the circulant $\hat{C}_\varphi$ satisfy $\hat{\sigma}_\ell = \mathrm{e}^{\mathrm{i}\varphi}\hat{\tau}_{n-\ell}$ for $1 \leq \ell < n$; cf. (4). Therefore,

$$T - \hat{C}_\varphi = \begin{bmatrix} \delta - \hat{\delta} & \tau_1 - \hat{\tau}_1 & & \dots & \tau_{n-1} - \hat{\tau}_{n-1} \\ \sigma_1 - \mathrm{e}^{\mathrm{i}\varphi}\hat{\tau}_{n-1} & \delta - \hat{\delta} & \ddots & \dots & \tau_{n-2} - \hat{\tau}_{n-2} \\ \sigma_2 - \mathrm{e}^{\mathrm{i}\varphi}\hat{\tau}_{n-2} & \sigma_1 - \mathrm{e}^{\mathrm{i}\varphi}\hat{\tau}_{n-1} & \ddots & & \vdots \\ \vdots & & \ddots & \delta - \hat{\delta} & \tau_1 - \hat{\tau}_1 \\ \sigma_{n-1} - \mathrm{e}^{\mathrm{i}\varphi}\hat{\tau}_1 & \dots & & \sigma_1 - \mathrm{e}^{\mathrm{i}\varphi}\hat{\tau}_{n-1} & \delta - \hat{\delta} \end{bmatrix}.$$

This matrix is an $\{\mathrm{e}^{\mathrm{i}\psi}\}$-circulant provided that

$$\sigma_\ell - \mathrm{e}^{\mathrm{i}\varphi}\hat{\tau}_{n-\ell} = \mathrm{e}^{\mathrm{i}\psi}(\tau_{n-\ell} - \hat{\tau}_{n-\ell}), \qquad 1 \leq \ell < n,$$

or, equivalently,

$$(\mathrm{e}^{\mathrm{i}\psi} - \mathrm{e}^{\mathrm{i}\varphi})\hat{\tau}_{n-\ell} = \mathrm{e}^{\mathrm{i}\psi}\tau_{n-\ell} - \sigma_\ell, \qquad 1 \leq \ell < n.$$

These equations can be solved for $\hat{\tau}_{n-\ell}$, $1 \leq \ell < n$, when $-\pi < \varphi \neq \psi \leq \pi$. This shows the existence of the representation (14). ∎

In the computations, we let $\hat{\delta} = \delta$ and choose $-\pi < \psi \leq \pi$ so that $|\psi - \varphi| = \pi$. Then the $\{\mathrm{e}^{\mathrm{i}\varphi}\}$-circulant $\hat{C}_\varphi = (n; \hat{\sigma}, \delta, \hat{\tau})$ is determined by

$$\hat{\tau}_{n-\ell} = \frac{1}{2}(\tau_{n-\ell} + \mathrm{e}^{-\mathrm{i}\varphi}\sigma_\ell), \qquad 1 \leq \ell < n.$$

We now are in a position to discuss the evaluation of the matrix-vector products $TC_\varphi^{-1}x$, $x \in \mathbb{C}^n$. It follows from Proposition 3.1 that the inverse C_φ^{-1} is an $\{\mathrm{e}^{\mathrm{i}\varphi}\}$-circulant. The splitting (14), and the factorizations

$$\hat{C}_\varphi = D_\varphi F \hat{\Lambda} F^H D_\varphi^H, \qquad \widetilde{C}_\psi = D_\psi F \widetilde{\Lambda} F^H D_\psi^H,$$

and (13), yield

$$TC_\varphi^{-1} = (\hat{C}_\varphi + \widetilde{C}_\psi)C_\varphi^{-1} = D_\varphi F\hat{\Lambda}\Lambda^{-1}F^H D_\varphi^H + D_\psi F\widetilde{\Lambda}F^H D_\psi^H D_\varphi F\Lambda^{-1}F^H D_\varphi^H.$$

Assume for the moment that both T and C_φ are Hermitian and positive definite. Then the preconditioned linear system of equations (5) with $C = C_\varphi$ can be expressed in the equivalent form

$$\begin{aligned}(\hat{\Lambda}\Lambda^{-1} + \Lambda^{-1/2}F^H D_\varphi^H D_\psi F\widetilde{\Lambda}F^H D_\psi^H D_\varphi F\Lambda^{-1/2})z &= \Lambda^{-1/2}F^H D_\varphi^H b, \\ y &= D_\varphi F\Lambda^{1/2}z.\end{aligned} \tag{15}$$

The above system can be solved with the aid of the conjugate gradient method, because by the assumptions on T and C_φ, the matrix in (15) is Hermitian and positive definite. The evaluation of matrix-vector products with this matrix requires the computation of four fast Fourier transforms of n-vectors. The same number is required without preconditioning, i.e., when $C_\varphi = I$.

In case T is not Hermitian positive definite, the following modification of (15) often is more convenient to use,

$$(\hat{\Lambda}\Lambda^{-1} + F^H D_\varphi^H D_\psi F\widetilde{\Lambda}F^H D_\psi^H D_\varphi F\Lambda^{-1})z = F^H D_\varphi^H b, \qquad y = D_\varphi F z. \tag{16}$$

This linear system of equations can be solved by iterative methods designed for general linear systems of equations with a square nonsingular matrix, such as GMRES and BiCGStab, or by the conjugate gradient method applied to the normal equations associated with (16); see [4] for details on these methods. We discuss the performance of the conjugate gradient method in the following section.

4. Spectral properties

Consider the symbol

$$f(t) = \delta + \sum_{j=1}^{\infty} \left(\sigma_j e^{-ijt} + \tau_j e^{ijt}\right) \tag{17}$$

for the sequence of Toeplitz matrices $T^{(n)} = (n; \sigma, \delta, \tau)$, $n = 1, 2, \ldots$. A 2π-periodic real-valued function f is said to be in the Wiener class if its Fourier coefficients are absolutely summable. The Wiener class therefore is a proper subset of the space of 2π-periodic continuous real-valued functions. Toeplitz matrices with a symbol in the Wiener class are Hermitian.

Let $T = (n; \sigma, \delta, \tau)$ be a generalized Hermitian Toeplitz matrix with $\sigma_\ell = \bar{\tau}_\ell \mathrm{e}^{\mathrm{i}\theta}$, $1 \le \ell < n$, for some $-\pi < \theta \le \pi$. Then T can be written in the form

$$T = \delta I + \mathrm{e}^{\mathrm{i}\theta/2}H,$$

where $H = (n; \sigma, 0, \bar{\sigma})$ is a Hermitian Toeplitz matrix. In the special case when $\delta = 0$, the linear system of equations (1) can be expressed as

$$Hx = \mathrm{e}^{-\mathrm{i}\theta/2}b, \tag{18}$$

while when $\arg(\delta) = \theta/2$, the system (1) is equivalent to

$$(|\delta|I + H)x = \mathrm{e}^{-\mathrm{i}\theta/2}b. \tag{19}$$

Thus, linear systems of equations (1) with a generalized Hermitian Toeplitz matrix with special values of the diagonal entries δ can be replaced by systems with a Hermitian Toeplitz matrix. The generalized Strang preconditioner S_φ, when applied to a Hermitian Toeplitz matrix, is Hermitian for all $-\pi < \varphi \leq \pi$; see Propositions 2.1 and 2.2. Moreover, due to Proposition 3.1, the spectrum of the generalized Strang preconditioner S_φ is independent of φ and, in particular, the same as the spectrum of Strang's preconditioner. Therefore, properties of Strang's preconditioner ([2, Theorem 4.3]) carry over to the generalized Strang preconditioner. We obtain the following result.

Theorem 4.1. *Let f be a real-valued function in the Wiener class and let $\{T^{(n)}\}$ be a sequence of Toeplitz matrices generated by f. Let $\{S_\varphi^{(n)}\}$ be an associated sequence of generalized Strang preconditioners, where $S_\varphi^{(n)}$ is a preconditioner for $T^{(n)}$, $n = 1, 2, \ldots$. Then, for any φ, the spectra of the matrices $S_\varphi^{(n)} - T^{(n)}$, $n = 1, 2, \ldots$, are clustered at zero for large n.*

The above result can be shown under weaker assumptions on the symbol; see [2, Section 4.3]. Moreover, we may generalize [2, Theorem 4.2] and [2, Corollary 4.4] to generalized Strang preconditioners. This yields the following properties.

Theorem 4.2. *Let f be a positive function in the Wiener class and let $\{T^{(n)}\}$ be a sequence of Toeplitz matrices generated by f. Then*

i) the spectral norms of the matrices $S_\varphi^{(n)}$ and $[S_\varphi^{(n)}]^{-1}$ are uniformly bounded for large n;
ii) the spectra of the preconditioned Toeplitz matrices $T^{(n)}[S_\varphi^{(n)}]^{-1}$ are clustered at unity for large n;
iii) the smallest eigenvalues of the preconditioned Toeplitz matrices $T^{(n)}[S_\varphi^{(n)}]^{-1}$ are uniformly bounded away from zero as $n \to \infty$.

It follows from Theorem 4.2 that the preconditioned conjugate gradient method with a generalized Strang preconditioner converges superlinearly.

We turn to $\{\mathrm{e}^{\mathrm{i}\theta}\}$-Hermitian Toeplitz matrices. The following result discusses the spectrum of such matrices.

Theorem 4.3. *Let $\{T^{(n)}\}$ be a sequence of $\{\mathrm{e}^{\mathrm{i}\theta}\}$-Hermitian Toeplitz matrices and assume that the real-valued symbol g for the sequence $\mathrm{e}^{-\mathrm{i}\theta/2}(T^{(n)} - \delta I)$, $n = 1, 2, \ldots$, is in the Wiener class. Let*

$$m = \min_{t\in\mathbb{R}} g(t), \qquad M = \max_{t\in\mathbb{R}} g(t).$$

Then the spectrum of each one of the matrices $T^{(n)}$, $n = 1, 2, \ldots$, lives in the interval $\delta + \mathrm{e}^{\mathrm{i}\theta/2}[m, M]$.

Proof: The theorem follows from [1, Chapter 5]. ∎

If T is an $\{\mathrm{e}^{\mathrm{i}\theta}\}$-Hermitian Toeplitz matrix, then, due to Propositions 2.1 and 2.2, so is the generalized Strang preconditioner S_φ for any choice of φ. Moreover, the main diagonals of T and S_φ are the same. Therefore, the eigenvalues of T and S_φ are collinear.

Let $\{T^{(n)}\}$, with $T^{(n)} = \delta I + \mathrm{e}^{\mathrm{i}\theta/2}H^{(n)} \in \mathbb{C}^{n\times n}$, be a sequence of $\{\mathrm{e}^{\mathrm{i}\theta}\}$-Hermitian Toeplitz matrices, where $\{H^{(n)}\}$ is a sequence of Hermitian Toeplitz matrices associated with the real-valued symbol (17) in the Wiener class. Let $\hat{S}_\varphi^{(n)}$ be a (Hermitian) generalized Strang preconditioner for $H^{(n)}$. Then $S_\varphi^{(n)} = \delta I + e^{i\theta/2}\hat{S}_\varphi^{(n)}$ is an $\{e^{i\theta}\}$-Hermitian generalized Strang preconditioner for $T^{(n)}$. By Theorem 4.1, the eigenvalues of $\hat{S}_\varphi^{(n)} - H^{(n)}$ cluster at the origin for large n. Consequently, the eigenvalues of $S_\varphi^{(n)} - T^{(n)}$ also cluster at zero for large n.

5. Numerical examples

We illustrate the behavior of Strang-type preconditioners with a few computed examples. In all examples, the initial approximate solution is chosen to be $x_0 = 0$. Let x_k, $k = 1, 2, \ldots$, denote the sequence of computed approximate solutions. We terminate the iterations as soon as the relative residual error satisfies

$$\|Tx_k - b\|/\|b\| \leq 1 \cdot 10^{-7}. \tag{20}$$

This stopping criterion also is used in the computed examples reported in [5, 2]. Moreover, following [5, 2], we let $b = [1, 1, \ldots, 1]^T$ in the first two examples. All computations were carried out in MATLAB with about 16 significant decimal digits.

preconditioner	n	iter.	rel. res. norm	n	iter.	rel. res. norm
no preconditioner	32	15	$9.5 \cdot 10^{-8}$	64	18	$4.2 \cdot 10^{-8}$
generalized Strang	32	6	$4.8 \cdot 10^{-8}$	64	6	$9.3 \cdot 10^{-8}$
Strang with (9)	32	7	$7.3 \cdot 10^{-8}$	64	7	$6.4 \cdot 10^{-8}$
Strang with (10)	32	8	$9.1 \cdot 10^{-9}$	64	7	$8.4 \cdot 10^{-8}$
generalized T. Chan	32	6	$8.4 \cdot 10^{-8}$	64	7	$9.7 \cdot 10^{-9}$
T. Chan	32	6	$4.9 \cdot 10^{-8}$	64	7	$1.1 \cdot 10^{-8}$
no preconditioner	128	20	$4.5 \cdot 10^{-8}$	256	21	$7.4 \cdot 10^{-8}$
generalized Strang	128	7	$9.5 \cdot 10^{-10}$	256	7	$8.0 \cdot 10^{-9}$
Strang with (9)	128	7	$4.9 \cdot 10^{-8}$	256	7	$5.1 \cdot 10^{-8}$
Strang with (10)	128	7	$5.7 \cdot 10^{-8}$	256	7	$5.2 \cdot 10^{-8}$
generalized T. Chan	128	7	$5.6 \cdot 10^{-9}$	256	7	$8.8 \cdot 10^{-9}$
T. Chan	128	7	$9.7 \cdot 10^{-9}$	256	7	$8.9 \cdot 10^{-9}$
no preconditioner	512	22	$8.1 \cdot 10^{-8}$	1024	23	$7.4 \cdot 10^{-8}$
generalized Strang	512	7	$1.7 \cdot 10^{-8}$	1024	7	$3.3 \cdot 10^{-8}$
Strang with (9)	512	8	$1.8 \cdot 10^{-9}$	1024	8	$2.2 \cdot 10^{-9}$
Strang with (10)	512	8	$1.8 \cdot 10^{-9}$	1024	8	$2.3 \cdot 10^{-9}$
generalized T. Chan	512	7	$2.1 \cdot 10^{-8}$	1024	8	$2.6 \cdot 10^{-8}$
T. Chan	512	7	$4.7 \cdot 10^{-8}$	1024	7	$7.5 \cdot 10^{-10}$
no preconditioner	2048	23	$9.9 \cdot 10^{-8}$	4096	24	$6.8 \cdot 10^{-8}$
generalized Strang	2048	7	$7.2 \cdot 10^{-8}$	4096	8	$3.0 \cdot 10^{-9}$
Strang with (9)	2048	8	$3.5 \cdot 10^{-9}$	4096	8	$6.8 \cdot 10^{-9}$
Strang with (10)	2048	8	$3.5 \cdot 10^{-8}$	4096	8	$6.8 \cdot 10^{-9}$
generalized T. Chan	2048	8	$1.3 \cdot 10^{-9}$	4096	8	$2.8 \cdot 10^{-9}$
T. Chan	2048	8	$1.6 \cdot 10^{-9}$	4096	8	$4.1 \cdot 10^{-9}$

Table I. Example 5.1: Number of iterations and relative residual norm for the unpreconditioned and preconditioned conjugate gradient methods. The parameter n shows the order of the matrices. The entries $\sigma_{n/2}^S$ and $\tau_{n/2}^S$, as well as the angle $\varphi = \varphi^*$, of the generalized Strang preconditioners are defined by (12).

n	generalized Strang	rel. res. norm	Strang	rel. res. norm
31	6	$4.4 \cdot 10^{-8}$	8	$1.1 \cdot 10^{-8}$
63	6	$9.7 \cdot 10^{-8}$	7	$8.8 \cdot 10^{-8}$
127	7	$9.3 \cdot 10^{-10}$	7	$5.7 \cdot 10^{-8}$
255	7	$7.9 \cdot 10^{-9}$	7	$5.2 \cdot 10^{-8}$
511	7	$1.6 \cdot 10^{-8}$	8	$1.8 \cdot 10^{-9}$
1023	7	$3.3 \cdot 10^{-8}$	8	$2.3 \cdot 10^{-9}$
2047	7	$7.2 \cdot 10^{-8}$	8	$3.5 \cdot 10^{-9}$
4095	8	$3.0 \cdot 10^{-9}$	8	$6.8 \cdot 10^{-9}$

Table II. Example 5.1: Number of iterations and relative residual norm for the preconditioned conjugate gradient method with Strang and generalized Strang preconditioners. The parameter n shows the order of the matrices.

Example 5.1. Consider the Hermitian positive definite Toeplitz matrices $T = (n; \sigma, \delta, \tau) \in \mathbb{C}^{n\times n}$ generated by the symbol

$$f(t) = 2\sum_{k=0}^{\infty} \frac{\sin(kt) + \cos(kt)}{(1+k)^{1.1}}, \quad t \in [-\pi, \pi].$$

Then

$$\delta = 2, \quad \sigma_k = \frac{1+\mathrm{i}}{(1+k)^{1.1}}, \quad \tau_k = \bar{\sigma}_k, \quad k = 1 : n-1.$$

These matrices also are used in computations reported in [5, Section 2.5].

Table I shows the number of iterations, k, required to satisfy the stopping criterion (20) by the (unpreconditioned) conjugate gradient method applied (1) and by the (preconditioned) conjugate gradient method applied to (15) with several choices of preconditioners. The table reports results for matrices of even order n. The Strang preconditioner then is not uniquely defined; the common choices (9) or (10) determine the preconditioner uniquely. Table I reports the performance for both choices. The $n \times n$ generalized Strang preconditioners referred to in the table are the preconditioners described in Subsection 2.2, with the entries $\sigma^S_{n/2}$ and $\tau^S_{n/2}$, as well as the angle $\varphi = \varphi^*$, given by (12). We refer to these preconditioners as S_φ. They are $\{i\}$-circulants. The T. Chan preconditioner of Table I is the preconditioner proposed in [15], and the generalized T. Chan preconditioner is an analogue that allows the use of generalized circulants; see [10] for a description. Table I shows the preconditioned conjugate gradient method with the preconditioner S_φ often to require fewer iterations to satisfy (20) or to yield a smaller residual error with the same number of iterations than the other preconditioners in this comparison.

Let $S^{(0)}_\varphi$ denote the $n \times n$ generalized Strang preconditioner with $\sigma^S_{n/2} = \tau^S_{n/2} = 0$. We found that for this and other examples, the preconditioners $S^{(0)}_\varphi$ do not perform as well as the generalized Strang preconditioners S_φ determined by (12). For many systems of equations, the former preconditioners require one more iteration to satisfy the stopping criterion than the latter or give larger residual errors after the same number of iterations. In our experience, the preconditioner $S^{(0)}_\varphi$ never performed better than the generalized Strang preconditioners S_φ defined by (12). We therefore propose that the entries $\sigma^S_{n/2}$ and $\tau^S_{n/2}$ of generalized Strang preconditioners of even order be determined by (12). We remark that in Table I the Strang preconditioner performs the best with the entries $\sigma_{n/2}^S$ and $\tau_{n/2}^S$ given by (9).

Table II illustrates the performance of Strang and generalized Strang preconditioners for matrices of odd order n. The Strang preconditioner is uniquely determined for Toeplitz matrices of odd order; cf. Subsection 2.1. The generalized Strang preconditioners of Table II are $\{i\}$-circulants. □

n	generalized Strang	rel. res. norm	Strang	rel. res. norm	no preconditioner	rel. res. norm
31	13	$1.9 \cdot 10^{-8}$	18	$3.8 \cdot 10^{-8}$	26	$2.6 \cdot 10^{-8}$
63	14	$4.8 \cdot 10^{-8}$	19	$3.5 \cdot 10^{-8}$	44	$4.2 \cdot 10^{-8}$
127	14	$9.4 \cdot 10^{-8}$	19	$6.8 \cdot 10^{-8}$	72	$8.6 \cdot 10^{-8}$
255	15	$3.1 \cdot 10^{-8}$	21	$3.7 \cdot 10^{-8}$	131	$8.0 \cdot 10^{-8}$
511	16	$5.1 \cdot 10^{-8}$	21	$5.9 \cdot 10^{-8}$	232	$8.4 \cdot 10^{-8}$
1023	16	$8.5 \cdot 10^{-8}$	22	$8.8 \cdot 10^{-8}$	426	$9.5 \cdot 10^{-8}$
2047	17	$2.3 \cdot 10^{-8}$	23	$3.9 \cdot 10^{-8}$	798	$9.8 \cdot 10^{-8}$
4095	17	$5.0 \cdot 10^{-8}$	24	$5.7 \cdot 10^{-8}$	1554	$9.9 \cdot 10^{-8}$

Table III. Example 5.2: Number of iterations and relative residual norm for the conjugate gradient method applied to the normal equations associated with (16) with Strang and generalized Strang preconditioners, and for the conjugate gradient method applied to the (unpreconditioned) normal equations associated with (1). The parameter n shows the order of the matrices.

Example 5.2. We consider non-Hermitian Toeplitz matrices $T = (n; \sigma, \delta, \tau) \in \mathbb{C}^{n \times n}$ defined by

$$\delta = 1, \quad \sigma_k = -\frac{(n-k)^3}{n^3}, \quad \tau_k = \frac{n-k}{n}, \quad k = 1 : n-1.$$

Table III displays the number of iterations required to satisfy the stopping criterion (20) by the preconditioned conjugate gradient method applied to the normal equations associated with (16) for the Strang and generalized Strang preconditioners. The table also displays the number of iterations demanded by the (unpreconditioned) conjugate gradient method applied to the normal equations associated with (1). Preconditioning can be seen to reduce the number of iterations significantly, and the generalized Strang preconditioner requires fewer iterations than the Strang preconditioner. The generalized Strang preconditioners are $\{-1\}$-circulants.

When n is even, the generalized Strang preconditioner is a circulant and, thus, is identical with the Strang preconditioner. We therefore do not show results for this situation. □

Example 5.3. We consider the integral equation

$$\frac{1}{100} y(s) + \int_0^1 k(s-t) y(t) dt = f(s), \qquad 0 \le s \le 1,$$

with kernel

$$k(s) = (-1)^{\frac{1}{2}(1+\mathrm{sign}(s))} \cos(s),$$

and solution $y(t) \equiv 1$, which determines the right-hand side function $f(s)$. This equation is an example of a convolution, whose kernel has a jump discontinuity. Discretization by a Nyström method based on the n-node midpoint rule with nodes $s_i = t_i = (2i-1)/(2n)$, $i = 1, 2, \ldots n$,

n	preconditioner	iterations	rel. res. norm
32	no preconditioner	53	$6.8 \cdot 10^{-9}$
32	generalized Strang	8	$1.2 \cdot 10^{-8}$
32	Strang	41	$4.8 \cdot 10^{-8}$
64	no preconditioner	57	$7.8 \cdot 10^{-8}$
64	generalized Strang	8	$4.2 \cdot 10^{-9}$
64	Strang	46	$8.9 \cdot 10^{-8}$
128	no preconditioner	56	$7.9 \cdot 10^{-8}$
128	generalized Strang	8	$3.6 \cdot 10^{-9}$
128	Strang	48	$8.8 \cdot 10^{-8}$
256	no preconditioner	56	$9.4 \cdot 10^{-8}$
256	generalized Strang	7	$8.3 \cdot 10^{-8}$
256	Strang	50	$7.5 \cdot 10^{-8}$
512	no preconditioner	56	$9.5 \cdot 10^{-8}$
512	generalized Strang	7	$7.5 \cdot 10^{-8}$
512	Strang	50	$9.3 \cdot 10^{-8}$
1024	no preconditioner	56	$7.2 \cdot 10^{-8}$
1024	generalized Strang	7	$6.8 \cdot 10^{-8}$
1024	Strang	50	$9.2 \cdot 10^{-8}$
2048	no preconditioner	54	$9.7 \cdot 10^{-8}$
2048	generalized Strang	7	$5.6 \cdot 10^{-8}$
2048	Strang	50	$9.7 \cdot 10^{-8}$
4096	no preconditioner	53	$6.7 \cdot 10^{-8}$
4096	generalized Strang	7	$3.9 \cdot 10^{-8}$
4096	Strang	50	$9.1 \cdot 10^{-8}$

Table IV. Example 5.3: Number of iterations and relative residual norm for the unpreconditioned and preconditioned conjugate gradient methods applied to the normal equations associated with (16). Strang and generalized Strang preconditioners are compared. The parameter n shows the order of the matrices. The entries $\sigma^S_{n/2}$ and $\tau^S_{n/2}$, as well as the angle $\varphi = \varphi^*$, of the generalized Strang preconditioners are defined by (12). This choice gives the best results. For the Strang preconditioners, the choices (9) and (10) give the same results.

yields a $\{-1\}$-Hermitian Toeplitz matrix T of order n. The right-hand side vector b in (1) is chosen so that the system has the solution $x = [1, 1, \ldots, 1]^T$.

Table IV reports the number of iterations required by the conjugate gradient method applied to the normal equations associated with (1) (unpreconditioned iterations) and to the normal equations associated with (16) (preconditioned iterations). The generalized Strang preconditioners, which are $\{-1\}$-circulants, perform much better than the Strang preconditioners for this example. □

6. Conclusion

This paper describes a generalization of the Strang preconditioner that allows the use of generalized circulants. The reported numerical examples, as well as numerous other computed examples, show that the generalized Strang preconditioner may require fewer iterations than the Strang preconditioner. Moreover, it is competitive with other preconditioners, including the T. Chan preconditioner and the generalized T. Chan preconditioner.

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