Weighted averaged Gaussian quadrature rules for modified Chebyshev measures

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Abstract

This paper is concerned with the approximation of integrals of a real-valued integrand over the interval $[-1, 1]$ by Gauss quadrature. The averaged and optimal averaged quadrature rules ([13, 21]) provide a convenient method for approximating the error in the Gauss quadrature. However, they are applicable to all integrands that are continuous on the interval $[-1, 1]$ only if their nodes are internal, i.e. if they belong to this interval.

We discuss two approaches to determine averaged quadrature rules with nodes in $[-1, 1]$: (i) truncating the Jacobi matrix associated with the optimal averaged rule, and (ii) weighting the optimal averaged quadrature rule. We consider Chebyshev measures of the first, second, and third kinds that are modified by a linear over linear rational factor, and discuss the internality of averaged, optimal averaged, and truncated optimal averaged quadrature

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rules. Moreover, we show that the weighting yields internal averaged rules if a weighting parameter is properly chosen, and we provide bounds for this parameter that guarantee internality. Finally, we illustrate that the weighted averaged rules give more accurate estimates of the quadrature error than the truncated optimal averaged rules.

Keywords: Gauss quadrature rule, averaged Gauss rules, generalized averaged Gauss rule, internality of quadrature rule, modified Chebyshev measure

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1. Introduction

Let $\mathrm{d}\lambda$ be a nonnegative measure with infinitely many points of support on the interval $[a, b]$ with $-\infty \leq a < b \leq \infty$, and assume that the measure is such that all moments are well defined. We let $\{P_k\}_{k=0}^{\infty}$ denote the sequence of monic orthogonal polynomials with respect to this measure. In particular, $\deg(P_k) = k$. The polynomials P_k satisfy a three-term recurrence relation of the form

$$P_{k+1}(x) = (x - \alpha_k)P_k(x) - \beta_k P_{k-1}(x), \quad k = 1, 2, \dots\ , \tag{1}$$

where $P_{-1}(x) \equiv 0$, $P_0(x) \equiv 1$, β_0 is an arbitrary constant, and $\beta_k > 0$ for all $k \geqslant 1$.

The n-node Gauss quadrature rule

$$\mathcal{G}_n(f) = \sum_{i=1}^{n} w_i^{(n)} f(x_i^{(n)}) \tag{2}$$

is the interpolatory quadrature rule with maximum degree of exactness, $2n - 1$, among all rules with n nodes for approximating the integral

$$I(f) = \int_a^b f(x)\,\mathrm{d}\lambda(x). \tag{3}$$

Thus, $I(f) = \mathcal{G}_n(f)$ for all polynomials f of degree at most $2n - 1$. Gauss rules are by far the most common quadrature rules used to approximate integrals (3) with a general integrand f.

The nodes $x_i^{(n)}$ $(i = 1, 2, \dots, n)$ of the rule (2) are the zeros of the monic orthogonal polynomial P_n and lie in the convex hull $\mathbb{H}$ of the support of the measure $\mathrm{d}\lambda$; the weights $w_i^{(n)}$ $(i = 1, 2, \dots, n)$ are positive; see, e.g., [9].

It is important to be able to estimate the quadrature error when applying the quadrature rule (2) to ensure that the rule determines an approximation of the integral with desired accuracy, and to avoid the evaluation of the integrand at needlessly many nodes. A popular approach to estimate the magnitude of the quadrature error,

$$E_n(f) = |(I - \mathcal{G}_n)(f)|, \tag{4}$$

is to use another quadrature rule, $\mathcal{A}_\ell$, with $\ell > n$ nodes and of degree of exactness larger than $2n-1$. One then can use

$$|(\mathcal{A}_\ell - \mathcal{G}_n)(f)| \tag{5}$$

as an estimate of (4).

A classical choice of the rule $\mathcal{A}_\ell$ is the Gauss-Kronrod quadrature rule associated with the Gauss rule (2). It has $2n+1$ nodes, n of which are the nodes of $\mathcal{G}_n$, and its degree of exactness is at least $3n+1$. However, the $n+1$ non-Gauss nodes are not necessarily real; see Notaris [15] for a nice fairly recent discussion on Gauss-Kronrod rules, as well as Peherstorfer and Petras [16].

Alternative choices for the rule $\mathcal{A}_\ell$, that recently have gained some attention, are the so-called averaged rules; see [8, 17, 18, 19, 20, 21, 22]. The first such rule was introduce by Laurie [13],

$$\mathcal{Q}^L_{2n+1} = \frac{1}{2}\left(\mathcal{G}_n + \widehat{\mathcal{G}}_{n+1}\right),$$

where $\widehat{\mathcal{G}}_{n+1}$ denotes the anti-Gauss rule associated with $\mathcal{G}_n$. It is determined by the requirement

$$\left(\widehat{\mathcal{G}}_{n+1} - I\right)(x^k) = -\left(\mathcal{G}_n - I\right)(x^k) \qquad (k = 0, 1, \ldots, 2n+1).$$

The nodes of the anti-Gauss rule are the zeros of the polynomial

$$\pi_{n+1} = P_{n+1} - \beta_n P_{n-1},$$

where β_n is a recursion coefficient for the sequence of monic orthogonal polynomials P_k; see (1). The averaged rule $\mathcal{Q}^L_{2n+1}$ has degree of exactness at least $2n+1$. We remark that anti-Gauss and associated averaged rules also can be used to estimate the error in the computed solution of Fredholm integral equations of the second kind defined on an interval; see Díaz de Alba et al. [2] for details.

Erich [8] defined for $0 < \theta < 1$ the weighted averaged formula

$$\mathcal{Q}^\theta_{2n+1} = \frac{1}{1+\theta}\left(\theta\mathcal{G}_n + \mathcal{G}^\theta_{n+1}\right), \tag{6}$$

where the quadrature rule $\mathcal{G}_{n+1}^{\theta}$ is determined by the conditions

$$\left(\mathcal{G}_{n+1}^{\theta} - I\right)(x^k) = -\theta\,(\mathcal{G}_n - I)\,(x^k) \qquad (k = 0, 1, \ldots, 2n+1).$$

The nodes of the rule $\mathcal{G}_{n+1}^{\theta}$ are the zeros of the polynomial

$$\pi_{n+1}^{\theta} = P_{n+1} - \theta\beta_n P_{n-1}. \tag{7}$$

Obviously, the anti-Gauss formula $\widehat{\mathcal{G}}_{n+1}$ corresponds to $\theta = 1$. Weighted averaged formulas also were introduced in [1].

The degree of exactness of the rule (6) is at least $2n+1$ for any fixed $0 < \theta < 1$. Erich [8] found for Gauss-Laguerre and Gauss-Hermite measures values of θ that makes the quadrature formula (6) have degree of exactness at least $2n+2$. This is the maximum degree of exactness for any $0 < \theta < 1$. We refer to the rule (6) with this value of θ as an optimal averaged rule.

Spalević [20] extended the result by Ehrich [8] to more general measures. He observed that it follows from results in [8] that the choice

$$\theta = \frac{\beta_{n+1}}{\beta_n} \tag{8}$$

yields a weighted averaged rule (6) with degree of exactness at least $2n+2$ for quite general measures. For this value of θ, the nodes of the quadrature rule $\mathcal{G}_{n+1}^{\theta}$ are the zeros of the polynomial

$$F_{n+1} = P_{n+1} - \beta_{n+1}P_{n-1}.$$

From now on we will use the notation $\mathcal{Q}_{2n+1}^{\beta}$ for the weighted averaged rule (6), where $\beta = \theta\beta_n$. In the special case when θ is defined by (8), we will refer to the quadrature formula $\mathcal{Q}_{2n+1}^{\beta}$ as $\mathcal{Q}_{2n+1}^{S}$.

The weighted averaged quadrature rules $\mathcal{Q}_{2n+1}^{\beta}$ (as well as their special cases $\mathcal{Q}_{2n+1}^{L}$ and $\mathcal{Q}_{2n+1}^{S}$) are associated with Jacobi matrices of order $2n+1$. These matrices are determined by recursion coefficients of the monic orthogonal polynomials (1); see [5, 18] for details. The eigenvalues of these matrices are the nodes of the weighted averaged rule, and the square of the first components of suitably normalized eigenvectors yields the weights. This property is used by the Golub-Welsch [10] algorithm for computing the nodes and weights of Gauss-type quadrature rules in an efficient manner.

The quadrature formulas $\mathcal{Q}_{2n+1}^{L}$ and $\mathcal{Q}_{2n+1}^{S}$ have real nodes, positive weights, and they are easy to compute. However, they are not guaranteed to be internal, i.e., they may have the outermost nodes outside of the set $\mathbb{H}$.

This means that they may yield poor accuracy, or may not be applicable, when the integrand has a singularity close to or at a boundary point of $\mathbb{H}$.

It is the purpose of this paper to discuss modifications of the rule $\mathcal{Q}^S_{2n+1}$ to make it internal, i.e., to have all nodes in $\mathbb{H}$, when $\mathcal{Q}^S_{2n+1}$ does not. One approach to achieve this is to truncate the Jacobi matrix associated with the quadrature rule $\mathcal{Q}^S_{2n+1}$. The truncated optimal averaged rules were introduced in [5]. The simplest one of them is obtained by removing the last $n-1$ rows and columns from the Jacobi matrices of order $2n+1$ for the quadrature rule $\mathcal{Q}^S_{2n+1}$. This determines the truncated optimal averaged rule $\mathcal{Q}^T_{n+2}$ with $n+2$ nodes. This rule has the same degree of exactness as the optimal averaged rule $\mathcal{Q}^S_{2n+1}$. The nodes of $\mathcal{Q}^T_{n+2}$ are the zeros of the polynomial

$$t_{n+2}(x) = (x - \alpha_{n-1})P_{n+1}(x) - \beta_{n+1}P_n(x), \tag{9}$$

where α_{n-1} and β_{n+1} are recursion coefficients for the sequence of monic orthogonal polynomials (1). The fact that the quadrature rule $\mathcal{Q}^T_{n+2}$ may be internal when $\mathcal{Q}^S_{2n+1}$ is not, follows from the Cauchy interlacing theorem for eigenvalues when applied to the Jacobi matrices associated with the rules $\mathcal{Q}^S_{2n+1}$ and $\mathcal{Q}^T_{n+2}$; see, e.g., [11]. Here one exploits that the eigenvalues of a Jacobi matrix are the nodes of the corresponding quadrature rule. A more detailed analysis of how the eigenvalues of these Jacobi matrices relate is provided by Hill and Parlett [12, Theorem 1]. For simplicity, we will henceforth refer to the rule $\mathcal{Q}^T_{n+2}$ as a truncated averaged rule.

However, there are some drawbacks of the quadrature rule $\mathcal{Q}^T_{n+2}$. The main one is that the quadrature error achieved with the rule $\mathcal{Q}^S_{2n+1}$ may be significantly smaller than the quadrature error of the rule $\mathcal{Q}^T_{n+2}$ when both rules can be applied. Therefore, the error estimate (5) is more accurate when $\mathcal{A}_\ell$ in (5) is chosen to be the rule $\mathcal{Q}^S_{2n+1}$ than the rule $\mathcal{Q}^T_{n+2}$ when both rules can be used. This depends on that for many measures and integrands, the quadrature error achieved with $\mathcal{Q}^S_{2n+1}$ is smaller than what might be anticipated from the degree of exactness of these rules; see [18] for discussions and numerous computed examples. A minor drawback of the rule $\mathcal{Q}^T_{n+2}$ is that the evaluation of the pair of quadrature rules $\{\mathcal{G}_n(f), \mathcal{Q}^T_{n+2}(f)\}$ typically requires one more evaluation of the integrand f than the evaluation of the pair of rules $\{\mathcal{G}_n(f), \mathcal{Q}^S_{2n+1}(f)\}$.

Another method for determining internal averaged rules is to choose a suitable parameter θ in the weighted averaged rule (6) that ensures that all quadrature nodes are in $\mathbb{H}$. This approach was first described in [18]. Weighted averaged rules determined in this manner typically yield a smaller quadrature error than truncated averaged rules. We will discuss this approach in detail and, in particular, derive bounds for the parameter $\beta = \theta\beta_n$ that ensure internality.

Recently, internality of averaged rules, optimal averaged rules, and the truncated version of the latter rules for Chebyshev measures modified by a linear divisor or a certain linear rational function was considered in [3, 4, 7]. The averaged and optimal averaged rules were found not to be internal in some situations. The internality of these quadrature rules for the Bernstein-Szegő weight function was analyzed in [6]. We also should mention the recent paper by Milovanović [14], where the author finds explicit expressions for the coefficients in the three-term recurrence relation for monic orthogonal polynomials with respect to this weight function.

This paper considers modifications of the Chebyshev measure by a general linear rational function, and we demonstrate how to obtain internal averaged rules by simple weighting, i.e., by choosing a suitable value of β. In the situation when neither the averaged nor the optimal averaged rules are internal, weighted averaged rules $\mathcal{Q}^{\beta}_{2n+1}$ generally achieve higher accuracy than the truncated averaged rules $\mathcal{Q}^{T}_{n+2}$. Our interest in the modification of Chebyshev measures stems from the attention that modification methods and their applications, e.g., to computing the Hilbert transform, have received in the literature; see Gautschi [9, Section 2.4] for a thorough discussion of modification algorithms and some applications.

This paper is organized as follows. Section 2 introduces the modification of Chebyshev measures to be considered and reviews available results. The discussions in Sections 3, 4, and 5 are concerned with modifications in the cases when the original measures are Chebyshev measures of the first, second, and third kinds, respectively. Section 6 presents applications to error estimation of the quadrature rules and illustrates the accuracy of the error estimates obtained. Concluding remarks can be found in Section 7.

2. Modification of the measure by a linear over linear rational factor

Consider the measure

$$\mathrm{d}\widehat{\lambda}(x) = (x+\gamma)\,\mathrm{d}\widetilde{\lambda}(x) = \frac{x+\gamma}{x+\delta}\,\mathrm{d}\lambda(x) \tag{10}$$

for $-1 < x < 1$ and $\gamma, \delta \in \mathbb{R} \setminus [-1, 1]$, where $\mathrm{d}\lambda$ is one of the four Chebyshev measures,

$$\mathrm{d}\lambda(x) = (1+x)^{\pm 1/2}(1-x)^{\pm 1/2}\,\mathrm{d}x.$$

We will represent the parameters γ and δ in (10) as

$$\gamma = \frac{1}{2}\left(v + v^{-1}\right) \quad \text{and} \quad \delta = \frac{1}{2}\left(u + u^{-1}\right) \tag{11}$$

for some $u, v \in (-1, 1)$. Clearly, $\gamma \neq \delta$ implies $u \neq v$. Given monic orthogonal polynomials P_k and their recurrence coefficients α_k and β_k for the measure $\mathrm{d}\lambda$, Gautschi [9, Section 2.4] describes algorithms for computing monic orthogonal polynomials $\widetilde{P}_k$ and their recurrence coefficients $\widetilde{\alpha}_k$ and $\widetilde{\beta}_k$ for the measure $\mathrm{d}\widetilde{\lambda}$, as well as monic orthogonal polynomials $\widehat{P}_k$ and their recurrence coefficients $\widehat{\alpha}_k$ and $\widehat{\beta}_k$ for the measure $\mathrm{d}\widehat{\lambda}$.

For the measure $\mathrm{d}\widetilde{\lambda}$, the averaged rules $\mathcal{Q}^L_{2n+1}$ and optimal averaged rules $\mathcal{Q}^S_{2n+1}$ coincide with the Gauss-Kronrod rule for $n \geqslant 3$ and are internal; see [3, 4, 7]. From now on, we therefore will consider optimal averaged rules associated with the measure $\mathrm{d}\widehat{\lambda}$.

3. Modification of the Chebyshev measure of the first kind

Let us consider the measure

$$\mathrm{d}\widehat{\lambda}(x) = \frac{x+\gamma}{x+\delta} \cdot \frac{\mathrm{d}x}{\sqrt{1-x^2}} \quad \text{for} \quad -1 < x < 1,$$

where $\gamma, \delta \in \mathbb{R} \setminus [-1, 1]$. This is the measure (10) when $\mathrm{d}\lambda(x) = \frac{\mathrm{d}x}{\sqrt{1-x^2}}$ is the Chebyshev measure of the first kind. The monic orthogonal polynomials with respect to $\mathrm{d}\lambda$ are the monic Chebyshev polynomials of the first kind, $\frac{1}{2^{n-1}} T_n(x)$, and the polynomial T_n of degree n is characterized by $T_n(\cos(\xi)) = \cos(n\xi)$; consequently, $T_n(\pm 1) = (\pm 1)^n$.

Since switching the signs of γ, δ and x yields the same measure, we may assume that $\delta > 1$. Then u and v given by (11) satisfy $0 < u < 1$ and $-1 < v < 1$.

For the measure $\mathrm{d}\widetilde{\lambda}(x) = \frac{\mathrm{d}\lambda(x)}{x+\delta}$ we obtain

$$\widetilde{P}_k(x) = \frac{1}{2^{k-1}}\left(T_k(x) + u\, T_{k-1}(x)\right) \quad \text{for} \quad k \geqslant 2,$$

with $\widetilde{P}_0(x) \equiv 1$ and $\widetilde{P}_1(x) = x + u$; see, e.g., [3, 9]. The corresponding recursion coefficients are

$$\begin{aligned}
&\widetilde{\alpha}_0 = -u, && \widetilde{\alpha}_1 = \frac{1}{2}u, && \widetilde{\alpha}_k = 0 \quad \text{for} \quad k \geqslant 2,\\
&\widetilde{\beta}_0 = \frac{2\pi u}{1-u^2}, && \widetilde{\beta}_1 = \frac{1}{2}(1-u^2), && \widetilde{\beta}_k = \frac{1}{4} \quad \text{for} \quad k \geqslant 2.
\end{aligned}$$

Further, for the measure $\mathrm{d}\widehat{\lambda}$, the orthogonal polynomials are

$$\widehat{P}_k(x) = \frac{\widetilde{P}_{k+1}(x) - r_k \widetilde{P}_k(x)}{x+\gamma}, \quad \text{where} \quad r_k = \frac{\widetilde{P}_{k+1}(-\gamma)}{\widetilde{P}_k(-\gamma)},$$

under the assumption that $\widetilde{P}_k(-\gamma) \neq 0$ for all k. The quotients r_k can be computed by the relations

$$r_0 = -\gamma - \widetilde{\alpha}_0, \quad r_1 = -\gamma - \widetilde{\alpha}_1 - \frac{\widetilde{\beta}_1}{r_0}, \quad r_k = -\gamma - \frac{1}{4r_{k-1}} \quad (k \geqslant 2),$$

and the recursion coefficients are

$$\begin{aligned} \widehat{\alpha}_k &= \widetilde{\alpha}_{k+1} + r_{k+1} - r_k \quad \text{for} \quad k \geqslant 0, \\ \widehat{\beta}_0 = -r_0\widetilde{\beta}_0, \quad \widehat{\beta}_k &= \widetilde{\beta}_k r_k / r_{k-1} \quad \text{for} \quad k \geqslant 1. \end{aligned} \tag{12}$$

The above relations yield

$$r_0 = u - \frac{1}{2}\left(v + v^{-1}\right), \qquad r_k = -\frac{1}{2v} \cdot \frac{v^{2k+1} - A}{v^{2k-1} - A} \quad (k \geqslant 1), \tag{13}$$

where

$$A = \frac{1-uv}{u-v}. \tag{14}$$

Note that $|A| > 1$ and $\operatorname{sgn}(A) = \operatorname{sgn}(u-v)$. Moreover, $v > u$ is equivalent to $\delta > \gamma > 1$, and $v(v-u) > 0$ is equivalent to $\delta > \gamma$.

3.1. Internality of averaged rules

The weighted averaged quadrature rule $\mathcal{Q}^{\beta}_{2n+1}$ is internal if and only if

$$\frac{\widehat{P}_{n+1}(x)}{\widehat{P}_{n-1}(x)} \geqslant \beta \quad \text{for} \quad x = \pm 1. \tag{15}$$

For $x = 1$ and $x = -1$ these inequalities reduce to

$$\beta \leqslant \beta_+ := \frac{1}{4} \cdot \frac{1 - 2r_{n+1}}{1 - 2r_{n-1}} \quad \text{and} \quad \beta \leqslant \beta_- := \frac{1}{4} \cdot \frac{1 + 2r_{n+1}}{1 + 2r_{n-1}},$$

respectively. Using (13), we obtain

$$\beta_+ = \frac{1}{4} - \frac{1}{4}\frac{Av^{2n-3}(1-v)(1-v^4)}{(A-v^{2n-2})(A-v^{2n+1})}, \quad \beta_- = \frac{1}{4} - \frac{1}{4}\frac{Av^{2n-3}(1+v)(1-v^4)}{(A+v^{2n-2})(A-v^{2n+1})}. \tag{16}$$

Since $|A| > 1$, we have $\operatorname{sgn}(A \pm v^l) = \operatorname{sgn}(A) = \operatorname{sgn}(u-v)$. This leads to the following result.

Theorem 3.1. *To ensure that the quadrature rule* $\mathcal{Q}_{2n+1}^{\beta}$ *is internal, it is sufficient for* β *to satisfy* $\beta \leqslant \beta_+$ *when* $\delta > \gamma > 1$*, and* $\beta \leqslant \beta_-$ *in all other cases, where* β_+, β_- *are given by* (16).

Note that for $v(v-u) > 0$, i.e. for $\gamma < \delta$, one can take $\beta = 1/4$ to secure internality.

If $\beta = \widehat{\beta}_n$, then the rule $\mathcal{Q}_{2n+1}^{\beta}$ becomes the averaged rule $\mathcal{Q}_{2n+1}^{L}$. Using results from [3], the relations (15) reduce to

$$Av^{2n-2}\frac{(1+v)(1-v)^3(A+v^{2n})(A-v^{2n-3})}{(A-v^{2n-1})^2(A-v^{2n-2})(A-v^{2n+1})} \geqslant 0 \tag{17}$$

for $x = 1$, and to

$$Av^{2n-2}\frac{(1-v)(1+v)^3(A-v^{2n})(A-v^{2n-3})}{(A-v^{2n-1})^2(A+v^{2n-2})(A-v^{2n+1})} \leqslant 0 \tag{18}$$

for $x = -1$.

Now we have that condition (17) is satisfied if and only if $A > 0$, i.e., if and only if $u > v$, and condition (18) holds if and only if $A < 0$, i.e., if and only if $u < v$. We have shown the following result.

Theorem 3.2. *The quadrature rule* $\mathcal{Q}_{2n+1}^{L}$ *associated with the measure* $d\widehat{\lambda}$ *has an external node: for* $\delta > \gamma > 1$ *the largest node is external, and in all other cases the smallest node is external.*

We turn to the situation when $\beta = \widehat{\beta}_{n+1}$. Then one obtains the optimal averaged rule $\mathcal{Q}_{2n+1}^{S}$ introduced in [21], and the inequalities (15) reduce to

$$Av^{2n-3}\frac{(1-v)(1-v^2)(1-v^3)(A+v^{2n})}{(A-v^{2n-2})(A-v^{2n+1})^2} \leqslant 0$$

for $x = 1$, and to

$$Av^{2n-3}\frac{(1+v)(1-v^2)(1+v^3)(A-v^{2n})}{(A+v^{2n-2})(A-v^{2n+1})^2} \leqslant 0$$

for $x = -1$. Consequently, we have the following theorem.

Theorem 3.3. *The quadrature rule* $\mathcal{Q}_{2n+1}^{S}$ *associated with the measure* $d\widehat{\lambda}$ *is internal if and only if* $\delta > \gamma$.

All computation in the paper are carried out using Matlab and high-precision arithmetic. In this and the following numerical examples, we use the notation $\mathcal{Q}_{2n+1}^{W}$ for the weighted averaged rule with the maximum allowable β, which is $\beta = \min\{\beta_+, \beta_-\}$. Thus, β is determined by the parameters γ, δ and n.

Example 3.4. *The values of the outermost nodes of the quadrature rules $\mathcal{Q}_{2n+1}^{W}$ for some values of the parameters γ, δ, and n, and of the rule $\mathcal{Q}_{2n+1}^{1/4}$ for some values of n are shown in Table 1. Analogous results for the rules $\mathcal{Q}_{2n+1}^{L}$ and $\mathcal{Q}_{2n+1}^{S}$ are displayed in Table 2.*

(γ,δ)	n	$\beta-\frac{1}{4}$	$1+x_1^W$	$1-x_{2n+1}^W$	$1+x_1^{1/4}$	$1-x_{2n+1}^{1/4}$
$(1.01, 1.25)$	5	2.5006×10^{-3}	1.5971×10^{-2}	0	1.6712×10^{-2}	4.8080×10^{-4}
	10	7.7125×10^{-4}	1.8131×10^{-3}	0	1.9320×10^{-3}	7.5643×10^{-5}
	15	1.9979×10^{-4}	2.6563×10^{-4}	0	2.8494×10^{-4}	1.3174×10^{-5}
	20	4.9389×10^{-5}	4.4562×10^{-5}	0	4.7902×10^{-5}	2.4516×10^{-6}
	30	2.9399×10^{-6}	1.5854×10^{-6}	0	1.7056×10^{-6}	9.7562×10^{-8}
$(-1.25, 1.01)$	5	8.6982×10^{-4}	0	4.8897×10^{-4}	7.2470×10^{-5}	7.3028×10^{-4}
	10	8.5299×10^{-7}	0	1.9987×10^{-7}	5.0406×10^{-8}	2.9981×10^{-7}
	15	8.3300×10^{-10}	0	1.2309×10^{-10}	3.7998×10^{-11}	1.8463×10^{-10}
	20	8.1348×10^{-13}	0	8.7776×10^{-14}	3.0216×10^{-14}	1.3166×10^{-13}
	30	7.7579×10^{-19}	0	5.4374×10^{-20}	2.1012×10^{-20}	8.1562×10^{-20}
$(5, 1.0001)$	5	-2.9043×10^{-8}	0	1.1598×10^{-9}	-3.8672×10^{-10}	-5.1603×10^{-9}
	10	-3.2146×10^{-18}	0	6.1478×10^{-20}	-4.0133×10^{-20}	-2.7354×10^{-19}
	15	-3.5582×10^{-28}	0	4.4736×10^{-30}	-4.1812×10^{-30}	-1.9905×10^{-29}
	20	-3.9384×10^{-38}	0	3.6882×10^{-40}	-4.3712×10^{-40}	-1.6411×10^{-39}
	30	-4.8252×10^{-58}	0	2.9918×10^{-60}	-4.8204×10^{-60}	-1.3312×10^{-59}

Table 1: The value of β and the outermost nodes of the weighted averaged rules $\mathcal{Q}_{2n+1}^{W}$ and $\mathcal{Q}_{2n+1}^{1/4}$ for the measure $d\widehat{\lambda}(x)=\frac{x+\gamma}{x+\delta}\cdot\frac{dx}{\sqrt{1-x^2}}$ for some values of γ, δ, and n.

(γ,δ)	n	$1+x_1^L$	$1-x_{2n+1}^L$	$1+x_1^S$	$1-x_{2n+1}^S$
$(1.01, 1.25)$	5	1.5935×10^{-2}	-2.3547×10^{-5}	1.6080×10^{-2}	7.0320×10^{-5}
	10	1.8058×10^{-3}	-4.6239×10^{-6}	1.8349×10^{-3}	1.3906×10^{-5}
	15	2.6439×10^{-4}	-8.4663×10^{-7}	2.6937×10^{-4}	2.5536×10^{-6}
	20	4.4345×10^{-5}	-1.5938×10^{-7}	4.5218×10^{-5}	4.8114×10^{-7}
	30	1.5776×10^{-6}	-6.3633×10^{-9}	1.6091×10^{-6}	1.9217×10^{-8}
$(-1.25, 1.01)$	5	-1.4600×10^{-5}	4.4053×10^{-4}	5.0716×10^{-5}	6.5768×10^{-4}
	10	-1.0081×10^{-8}	1.7989×10^{-7}	3.5285×10^{-8}	2.6983×10^{-7}
	15	-7.5997×10^{-12}	1.1078×10^{-10}	2.6599×10^{-11}	1.6617×10^{-10}
	20	-6.0432×10^{-15}	7.8999×10^{-14}	2.1151×10^{-14}	1.1850×10^{-13}
	30	-4.2023×10^{-21}	4.8937×10^{-20}	1.4708×10^{-20}	7.3405×10^{-20}
$(5, 1.0001)$	5	-4.2578×10^{-11}	4.6390×10^{-10}	-3.8321×10^{-10}	-5.1029×10^{-9}
	10	-4.4187×10^{-21}	2.4591×10^{-20}	-3.9768×10^{-20}	-2.7050×10^{-19}
	15	-4.6035×10^{-31}	1.7894×10^{-30}	-4.1432×10^{-30}	-1.9684×10^{-29}
	20	-4.8127×10^{-41}	1.4753×10^{-40}	-4.3315×10^{-40}	-1.6228×10^{-39}
	30	-5.3073×10^{-61}	1.1967×10^{-60}	-4.7766×10^{-60}	-1.3164×10^{-59}

Table 2: The outermost nodes of the averaged rule $\mathcal{Q}_{2n+1}^{L}$ and the optimal averaged rule $\mathcal{Q}_{2n+1}^{S}$ for the measure $d\widehat{\lambda}(x)=\frac{x+\gamma}{x+\delta}\cdot\frac{dx}{\sqrt{1-x^2}}$ for some values of γ, δ, and n.

4. Modification of the Chebyshev measure of the second kind

This section considers the measure

$$\mathrm{d}\widehat{\lambda}(x)=\frac{x+\gamma}{x+\delta}\sqrt{1-x^2}\,\mathrm{d}x \quad \text{for} \quad -1<x<1,$$

where $\gamma,\delta\in\mathbb{R}\setminus[-1,1]$. This is the measure (10) when $\mathrm{d}\lambda(x)=\sqrt{1-x^2}\,\mathrm{d}x$ is the Chebyshev measure of the second kind. The monic orthogonal polynomials with respect to $\mathrm{d}\lambda$ are the monic Chebyshev polynomials of the second kind, $\frac{1}{2^{n-1}}U_n(x)$, and the polynomial U_n of degree n is characterized by $U_n(\cos\xi)=\frac{\sin((n+1)\xi)}{\sin(\xi)}$; consequently, $U_n(\pm1)=(\pm1)^n(n+1)$.

Similarly as in Section 3, γ and δ are determined by the parameters u and v and, due to symmetry, we may assume that $0<u<1$ and $-1<v<1$.

For the measure $\mathrm{d}\widetilde{\lambda}(x)=\frac{\mathrm{d}\lambda(x)}{x+\delta}$ the orthogonal polynomials are

$$\widetilde{P}_k(x)=\frac{1}{2^k}\left(U_k(x)+u\,U_{k-1}(x)\right)\quad\text{for}\quad k\geqslant 1,$$

with $\widetilde{P}_0(x)\equiv 1$; see [7, 9]. The recursion coefficients of the polynomials $\widetilde{P}_k$ are given by

$$\widetilde{\alpha}_0=-\frac{u}{2},\quad \widetilde{\alpha}_k=0 \text{ for } k\geqslant 1,$$
$$\widetilde{\beta}_0=u\pi,\quad \widetilde{\beta}_k=\frac{1}{4} \text{ for } k\geqslant 1;$$

see [7, 9]. Moreover, for the measure $\mathrm{d}\widehat{\lambda}$, the orthogonal polynomials are

$$\widehat{P}_k(x)=\frac{\widetilde{P}_{k+1}(x)-r_k\widetilde{P}_k(x)}{x+\gamma},\quad\text{where}\quad r_k=\frac{\widetilde{P}_{k+1}(-\gamma)}{\widetilde{P}_k(-\gamma)},$$

under the assumption that $\widetilde{P}_k(-\gamma)\neq 0$ for all k. The quotients r_k satisfy the relations

$$r_0=-\gamma-\widetilde{\alpha}_0,\quad r_k=-\gamma-\frac{1}{4r_{k-1}}\quad(k\geqslant 1),$$

and the recursion coefficients are given by (12). We obtain

$$r_0=\frac{1}{2}\left(u-v-v^{-1}\right),\qquad r_k=-\frac{1}{2v}\cdot\frac{v^{2k+3}+A}{v^{2k+1}+A}\quad(k\geqslant 1),\tag{19}$$

where A is defined by (14).

4.1. Internality of the averaged rules

The weighted averaged quadrature formula $\mathcal{Q}_{2n+1}^{\beta}$ is internal if and only if conditions (15) hold. It follows from (19) that for $x=1$ and $x=-1$, these conditions reduce to

$$\begin{aligned}\beta\leqslant\beta_+ &:=\frac{1}{4}\cdot\frac{1+u+(1+(1+u)(n+1))\left(1+\frac{1}{v}\cdot\frac{v^{2n+5}+A}{v^{2n+3}+A}\right)}{1+u+(1+(1+u)(n-1))\left(1+\frac{1}{v}\cdot\frac{v^{2n+1}+A}{v^{2n-1}+A}\right)}\\ &=\frac{1}{4}+\frac{(1+u)(1+v)(2v^{2n+4}+A)}{v(v^{2n+3}+A)M_1}+\frac{n+(n-1)u}{4M_1}\frac{Av^{2n-2}(v^2-1)^2(v^2+1)}{(v^{2n-1}+A)(v^{2n+3}+A)}\end{aligned}\tag{20}$$

and

$$\beta \leqslant \beta_- := \frac{1}{4} \cdot \frac{1-u+(1+(1-u)(n+1))\left(1-\frac{1}{v}\cdot\frac{v^{2n+5}+A}{v^{2n+3}+A}\right)}{1-u+(1+(1-u)(n-1))\left(1-\frac{1}{v}\cdot\frac{v^{2n+1}+A}{v^{2n-1}+A}\right)}$$
$$= \frac{1}{4} + \frac{(1-u)(1-v)(2v^{2n+4}-A)}{v(v^{2n+3}+A)M_2} - \frac{n-(n-1)u}{4M_2}\frac{Av^{2n-2}(v^2-1)^2(v^2+1)}{(v^{2n-1}+A)(v^{2n+3}+A)}, \tag{21}$$

respectively, where

$$M_1 = 1+u+\big(n+(n-1)u\big)\left(1+\frac{1}{v}\cdot\frac{v^{2n+1}+A}{v^{2n-1}+A}\right),$$
$$M_2 = 1-u+\big(n-(n-1)u\big)\left(1-\frac{1}{v}\cdot\frac{v^{2n+1}+A}{v^{2n-1}+A}\right).$$

Note that, as $n\to\infty$,

$$M_1 = \frac{v-u+n(1+u)(1+v)}{v}+o(1), \quad -M_2 = \frac{u-v+n(1-u)(1-v)}{v}+o(1).$$

Thus,

$$\beta_+ = \frac{1}{4}\left(1+\frac{2}{n+\frac{v-u}{(1+u)(1+v)}+o(1)}\right), \quad \beta_- = \frac{1}{4}\left(1+\frac{2}{n+\frac{u-v}{(1-u)(1-v)}+o(1)}\right) \tag{22}$$

as $n\to\infty$.

Theorem 4.1. *To ensure that the quadrature rule $\mathcal{Q}^{\beta}_{2n+1}$ is internal when n is large enough, it is sufficient for β to satisfy* (20) *when $\delta>\gamma>1$, and* (21) *in all other cases.*

Proof. Indeed, for n large enough, we have $\beta_+ > \frac{1}{4}(1+\frac{2}{n}) > \beta_-$ if $u>v$, and $\beta_+ < \frac{1}{4}(1+\frac{2}{n}) < \beta_-$ if $u<v$. $\square$

In particular, letting $\beta=\widehat{\beta}_n$ and $\beta=\widehat{\beta}_{n+1}$ gives the averaged rules $\mathcal{Q}^{L}_{2n+1}$ and $\mathcal{Q}^{S}_{2n+1}$, respectively. Since

$$\widehat{\beta}_n = \frac{1}{4}\left(1+\frac{Av^{2n-1}(v^2-1)^2}{(v^{2n+1}+A)^2}\right) = \frac{1}{4}\left(1+o\Big(\frac{1}{n}\Big)\right),$$

we obtain the following result.

Theorem 4.2. *The quadrature rules $\mathcal{Q}^{L}_{2n+1}$ and $\mathcal{Q}^{S}_{2n+1}$ associated with the measure $d\widehat{\lambda}$ are internal if n is large enough.*

Typically, small values of n already produce internal quadrature rules. This is illustrated by the following examples. Again, $\beta=\min\{\beta_+,\beta_-\}$.

Example 4.3. *The values of β and the outermost nodes of the rules $\mathcal{Q}^W_{2n+1}$ and $\mathcal{Q}^{1/4}_{2n+1}$ for some values of γ, δ and n are shown in Table 3.*

(γ,δ)	n	$\beta-\frac{1}{4}$	$1+x^W_1$	$1-x^W_{2n+1}$	$1+x^{1/4}_1$	$1-x^{1/4}_{2n+1}$
$(1.01, 1.25)$	5	9.321×10^{-2}	2.6482×10^{-2}	0	6.1602×10^{-2}	3.1636×10^{-2}
	10	4.859×10^{-2}	7.8666×10^{-3}	0	1.9316×10^{-2}	9.8233×10^{-3}
	15	3.287×10^{-2}	3.0383×10^{-3}	0	8.5966×10^{-3}	4.7182×10^{-3}
	20	2.480×10^{-2}	1.3750×10^{-3}	0	4.6048×10^{-3}	2.7580×10^{-3}
	30	1.659×10^{-2}	3.9876×10^{-4}	0	1.8503×10^{-3}	1.2725×10^{-3}
$(-1.25, 1.01)$	5	4.169×10^{-2}	0	4.0034×10^{-2}	9.3261×10^{-3}	5.6598×10^{-2}
	10	2.955×10^{-2}	0	7.0309×10^{-3}	4.2053×10^{-3}	1.3464×10^{-2}
	15	2.281×10^{-2}	0	2.4179×10^{-3}	2.4545×10^{-3}	5.8244×10^{-3}
	20	1.857×10^{-2}	0	1.1132×10^{-3}	1.6224×10^{-3}	3.2289×10^{-3}
	30	1.354×10^{-2}	0	3.6412×10^{-4}	8.6663×10^{-4}	1.4137×10^{-3}

Table 3: The value of β and the outermost nodes of the weighted averaged rules $\mathcal{Q}^W_{2n+1}$ and $\mathcal{Q}^{1/4}_{2n+1}$ for the measure $d\widehat{\lambda}(x)=\frac{x+\gamma}{x+\delta}\sqrt{1-x^2}\,dx$ for some values of γ, δ and n.

Example 4.4. *The values of the outermost nodes of the rules $\mathcal{Q}^L_{2n+1}$ and $\mathcal{Q}^S_{2n+1}$ for some values of γ, δ and n are shown in Table 4.*

(γ,δ)	n	$1+x^L_1$	$1-x^L_{2n+1}$	$1+x^S_1$	$1-x^S_{2n+1}$
$(1.01, 1.25)$	5	6.2949×10^{-2}	3.2825×10^{-2}	6.2543×10^{-2}	3.2467×10^{-2}
	10	1.9471×10^{-2}	9.9519×10^{-3}	1.9431×10^{-2}	9.9186×10^{-3}
	15	8.6216×10^{-3}	4.7387×10^{-3}	8.6154×10^{-3}	4.7336×10^{-3}
	20	4.6094×10^{-3}	2.7618×10^{-3}	4.6083×10^{-3}	2.7609×10^{-3}
	30	1.8505×10^{-3}	1.2727×10^{-3}	1.8504×10^{-3}	1.2727×10^{-3}
$(-1.25, 1.01)$	5	9.3809×10^{-3}	5.6701×10^{-2}	9.3398×10^{-3}	5.6623×10^{-2}
	10	4.2054×10^{-3}	1.3464×10^{-2}	4.2053×10^{-3}	1.3464×10^{-2}
	15	2.4545×10^{-3}	5.8244×10^{-3}	2.4545×10^{-3}	5.8244×10^{-3}
	20	1.6224×10^{-3}	3.2289×10^{-3}	1.6224×10^{-3}	3.2289×10^{-3}
	30	8.6663×10^{-4}	1.4137×10^{-3}	8.6663×10^{-4}	1.4137×10^{-3}

Table 4: The outermost nodes of the averaged rule $\mathcal{Q}^L_{2n+1}$ and the optimal averaged rule $\mathcal{Q}^S_{2n+1}$ for the measure $d\widehat{\lambda}(x)=\frac{x+\gamma}{x+\delta}\sqrt{1-x^2}\,dx$ for some values of γ, δ and n.

5. Modification of the Chebyshev measure of the third kind

We finally consider the measure

$$\mathrm{d}\widehat{\lambda}(x)=\frac{x+\gamma}{x+\delta}\sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x \quad \text{for} \quad -1<x<1,$$

where $\gamma,\delta\in\mathbb{R}\setminus[-1,1]$. This is the measure (10) when $\mathrm{d}\lambda(x)=\sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x$ is the Chebyshev measure of the third kind. The monic orthogonal polynomials with respect to $\mathrm{d}\lambda$ are the monic Chebyshev polynomials of the third kind, $\frac{1}{2^{n-1}}V_n(x)$, and the polynomial V_n of degree n is characterized by $V_n(\cos\xi)=\frac{\cos((n+\frac{1}{2})\xi)}{\cos(\frac{\xi}{2})}$; consequently, $V_n(1)=1$ and $V_n(-1)=(-1)^n(2n+1)$.

This time, we allow u, v in (11) from the entire range: $-1 < u, v < 1$. Switching the signs of γ, δ and x yields a modification of the Chebyshev measure of the fourth kind.

For the measure $\mathrm{d}\widetilde{\lambda}(x) = \frac{\mathrm{d}\lambda(x)}{x+\delta}$ we obtain

$$\widetilde{P}_k(x) = \frac{1}{2^k}\left(V_k(x) + u\, V_{k-1}(x)\right) \quad \text{for} \quad k \geqslant 0;$$

see [4, 9]. The recursion coefficients of the polynomials $\widetilde{P}_k$ are given by

$$\begin{aligned} &\widetilde{\alpha}_0 = \frac{1-u}{2}, && \widetilde{\alpha}_k = 0 \quad \text{for} \quad k \geqslant 1, \\ &\widetilde{\beta}_0 = \frac{2\pi u}{1+u}, \quad \widetilde{\beta}_1 = \frac{1+u}{4}, && \widetilde{\beta}_k = \frac{1}{4} \quad \text{for} \quad k \geqslant 2. \end{aligned}$$

Moreover, for the measure $\mathrm{d}\widehat{\lambda}$, the orthogonal polynomials are

$$\widehat{P}_k(x) = \frac{\widetilde{P}_{k+1}(x) - r_k \widetilde{P}_k(x)}{x+\gamma}, \quad \text{where} \quad r_k = \frac{\widetilde{P}_{k+1}(-\gamma)}{\widetilde{P}_k(-\gamma)},$$

under the assumption that $\widetilde{P}_k(-\gamma) \neq 0$ for all k. The quotients r_k satisfy the relations

$$r_0 = -\gamma - \widetilde{\alpha}_0, \quad r_1 = -\gamma - \widetilde{\alpha}_1 - \frac{\widetilde{\beta}_1}{r_0}, \quad r_k = -\gamma - \frac{1}{4r_{k-1}} \quad (k \geqslant 2),$$

and the recursion coefficients are given by (12); see [4, 9]. We obtain

$$r_0 = -\frac{1}{2}\left(1 - u + v + v^{-1}\right), \qquad r_k = -\frac{1}{2v} \cdot \frac{v^{2k+2} + A}{v^{2k} + A} \quad (k \geqslant 1), \tag{23}$$

where A is defined by (14).

5.1. *Internality of the averaged rules*

The weighted averaged quadrature formula $\mathcal{Q}^{\beta}_{2n+1}$ is internal if and only if conditions (15) are satisfied. For $x = 1$ and $x = -1$ these inequalities reduce to

$$\beta \leqslant \beta_+ := \frac{1}{4} \cdot \frac{1 - 2r_{n+1}}{1 - 2r_{n-1}}$$

and

$$\beta \leqslant \beta_- := \frac{1}{4} \cdot \frac{2(1-u) + (2 + (1-u)(2n+1))(1 + 2r_{n+1})}{2(1-u) + (2 + (1-u)(2n-3))(1 + 2r_{n-1})},$$

respectively. Using formula (23), we obtain

$$\beta_+ = \frac{1}{4} + \frac{1}{4}\frac{Av^{2n-2}(1-v)(1-v^4)}{(A + v^{2n-1})(A + v^{2n+2})} \tag{24}$$

and

$$\beta_- = \frac{1}{4} \cdot \frac{2(1-u) + (2 + (1-u)(2n+1))\left(1 - \frac{1}{v} \cdot \frac{v^{2n+4}+A}{v^{2n+2}+A}\right)}{2(1-u) + (2 + (1-u)(2n-3))\left(1 - \frac{1}{v} \cdot \frac{v^{2n}+A}{v^{2n-2}+A}\right)}. \tag{25}$$

Defining

$$M = 2(1-u) + (2 + (2n-3)(1-u))\left(1 - \frac{1}{v} \cdot \frac{v^{2n}+A}{v^{2n-2}+A}\right),$$

we obtain

$$\begin{aligned}\beta_- &= \frac{1}{4} + \frac{(1-u)(1-v)(v^{2n+3}-A)}{vM(v^{2n+2}+A)} \\ &\quad + \frac{(2 + (2n-3)(1-u))(v-1)}{vM} \frac{Av^{2n-2}(1-v^4)(1+v)}{4(v^{2n+2}+A)(v^{2n-2}+A)}.\end{aligned}$$

Moreover,

$$M = \frac{2n(1-u)(v-1)}{v} + o(n) \quad \text{as} \quad n \to \infty.$$

Therefore, $\beta_+ \leqslant \beta_-$, which leads us to the following result.

Theorem 5.1. *To ensure that the quadrature rule $\mathcal{Q}_{2n+1}^{\beta}$ is internal when n is large, it is sufficient to have $\beta \leqslant \beta_+$, with β_+ given by* (24)*.*

When $\beta = \widehat{\beta}_n$, the rule $\mathcal{Q}_{2n+1}^{\beta}$ becomes the averaged rule $\mathcal{Q}_{2n+1}^{L}$; see [13]. Using results from [4], the inequalities (15) reduce to

$$Av^{2n-1}\frac{(1-v)^3(1+v)(A+v^{2n-2})(A-v^{2n+1})}{(A+v^{2n})^2(A+v^{2n-1})(A+v^{2n+2})} \leqslant 0 \tag{26}$$

for $x = 1$, and to

$$\frac{1}{4} + \frac{Av^{2n-2}(1-v^2)^2}{4(A+v^{2n})^2} \leqslant \beta_- \tag{27}$$

for $x = -1$.

When n is large enough, inequality (26) holds if and only if $Av < 0$, which is equivalent to $\delta(\delta - \gamma) > 0$, and inequality (27) always holds. Thus, we have shown the following

Theorem 5.2. *The quadrature rule $\mathcal{Q}_{2n+1}^{L}$ associated with the measure $d\widehat{\lambda}$ is internal for n sufficiently large if and only if $\delta(\delta - \gamma) > 0$.*

If instead $\beta = \widehat{\beta}_{n+1}$, then one obtains the optimal averaged rule $\mathcal{Q}_{2n+1}^{S}$. The inequalities (15) then reduce to

$$Av^{2n-2}\frac{(1-v)(1-v^2)(1-v^3)(A-v^{2n+1})}{(A+v^{2n-1})(A+v^{2n+2})^2} \geqslant 0$$

for $x = 1$, and to

$$\frac{1}{4} + \frac{Av^{2n}(1-v^2)^2}{4(A+v^{2n+2})^2} \leqslant \beta_-$$

for $x = -1$. This leads to the following result.

Theorem 5.3. *The quadrature rule $\mathcal{Q}^S_{2n+1}$ associated with the measure $d\widehat{\lambda}$ is internal when n is sufficiently large if and only if $u > v$, i.e. if and only if $\gamma\delta(\gamma-\delta) > 0$.*

Example 5.4. *Table 5 displays the value of $\beta = \beta_+$ and the outermost nodes of the quadrature rules $\mathcal{Q}^W_{2n+1}$ and $\mathcal{Q}^{1/4}_{2n+1}$ for some values of the parameters γ, δ and n. Table 6 displays the outermost nodes of the quadrature rules $\mathcal{Q}^L_{2n+1}$ and $\mathcal{Q}^S_{2n+1}$.*

(γ,δ)	n	$\beta - \frac{1}{4}$	$1 + x_1^W$	$1 - x_{2n+1}^W$	$1 + x_1^{1/4}$	$1 - x_{2n+1}^{1/4}$
$(-5,-1.0001)$	5	-2.9339×10^{-9}	4.7164×10^{-2}	0	4.7164×10^{-2}	-3.8808×10^{-11}
	10	-3.2474×10^{-19}	1.2081×10^{-2}	0	1.2081×10^{-2}	-4.0291×10^{-21}
	15	-3.5945×10^{-29}	5.4093×10^{-3}	0	5.4093×10^{-3}	-4.1992×10^{-31}
	20	-3.9786×10^{-39}	3.0535×10^{-3}	0	3.0535×10^{-3}	-4.3914×10^{-41}
	30	-4.8744×10^{-59}	1.3618×10^{-3}	0	1.3618×10^{-3}	-4.8453×10^{-61}
$(5,1.0001)$	5	2.3955×10^{-9}	1.4133×10^{-3}	0	1.4133×10^{-3}	4.7014×10^{-10}
	10	2.6515×10^{-19}	6.8285×10^{-4}	0	6.8285×10^{-4}	2.6265×10^{-20}
	15	2.9349×10^{-29}	4.4335×10^{-4}	0	4.4335×10^{-4}	1.9442×10^{-30}
	20	3.2485×10^{-39}	3.2455×10^{-4}	0	3.2455×10^{-4}	1.6166×10^{-40}
	30	3.9800×10^{-59}	2.0670×10^{-4}	0	2.0670×10^{-4}	1.3224×10^{-60}

Table 5: The value of β and the outermost nodes of averaged rules $\mathcal{Q}^W_{2n+1}$ and $\mathcal{Q}^{1/4}_{2n+1}$ for the measure $d\widehat{\lambda}(x) = \frac{x+\gamma}{x+\delta}\sqrt{\frac{1+x}{1-x}}\,dx$ for some values of γ, δ, and n.

(γ,δ)	n	$1 + x_1^L$	$1 - x_{2n+1}^L$	$1 + x_1^S$	$1 - x_{2n+1}^S$
$(-5,-1.0001)$	5	4.7164×10^{-2}	-4.2729×10^{-12}	4.7164×10^{-2}	-3.8456×10^{-11}
	10	1.2081×10^{-2}	-4.4361×10^{-22}	1.2081×10^{-2}	-3.9925×10^{-21}
	15	5.4093×10^{-3}	-4.6234×10^{-32}	5.4093×10^{-3}	-4.1610×10^{-31}
	20	3.0535×10^{-3}	-4.8350×10^{-42}	3.0535×10^{-3}	-4.3515×10^{-41}
	30	1.3618×10^{-3}	-5.3348×10^{-62}	1.3618×10^{-3}	-4.8013×10^{-41}
$(5,1.0001)$	5	1.4133×10^{-3}	-4.2265×10^{-11}	1.4133×10^{-3}	4.6491×10^{-10}
	10	6.8285×10^{-4}	-2.3612×10^{-21}	6.8285×10^{-4}	2.5973×10^{-20}
	15	4.4335×10^{-4}	-1.7478×10^{-31}	4.4335×10^{-4}	1.9226×10^{-30}
	20	3.2455×10^{-4}	-1.4533×10^{-41}	3.2455×10^{-4}	1.5986×10^{-40}
	30	2.0670×10^{-4}	-1.1889×10^{-41}	2.0670×10^{-4}	1.3077×10^{-60}

Table 6: The outermost nodes of averaged rules $\mathcal{Q}^L_{2n+1}$ and optimal averaged rules $\mathcal{Q}^S_{2n+1}$ for the measure $d\widehat{\lambda}(x) = \frac{x+\gamma}{x+\delta}\sqrt{\frac{1+x}{1-x}}\,dx$ for some values of γ, δ, and n.

5.2. Internality of truncated optimal averaged rule

This subsection considers truncated averaged rules $\mathcal{Q}^T_{n+2}$, whose nodes are the zeros of the polynomials (9). These quadrature rules are internal if and only if (see, e.g., [3])

$$\frac{(x-\widehat{\alpha}_{n-1})\widehat{P}_{n+1}(x)}{\widehat{\beta}_{n+1}\widehat{P}_n(x)} \geqslant 1 \quad \text{for} \quad x = \pm 1,$$

which is equivalent to

$$2(1+r_n-r_{n-1})\frac{r_n}{r_{n+1}} \cdot \frac{1+2r_{n+1}}{1+2r_n} \geqslant 1 \ \text{ for } \ x=1,$$
$$2(1-r_n+r_{n-1})\frac{r_n}{r_{n+1}} \cdot \frac{1-2r_{n+1}}{1-2r_n} \geqslant 1 \ \text{ for } \ x=-1.$$

From (13), (19), and (23) one can notice that $r_n \to -\frac{1}{2v}$ when $n \to \infty$. Hence, for all Chebyshev weights, the previous inequalities are satisfied for n sufficiently large. We obtain the following theorem.

Theorem 5.5. *The truncated averaged rule $\mathcal{Q}^T_{n+2}$ is internal for the measure (10), where $d\lambda$ is one of four Chebyshev measures, when n is large enough.*

The following example illustrates that n does not have to be very large in order for the rule $\mathcal{Q}^T_{n+2}$ to be internal.

Example 5.6. *Consider the measure*

$$d\widehat{\lambda}(x) = \frac{x+\gamma}{x+\delta} \cdot \frac{dx}{\sqrt{1-x^2}}.$$

Table 7 shows the outermost nodes of truncated averaged rules $\mathcal{Q}^T_{n+2}$ for some values of the parameters γ, δ, and n.

(γ, δ)	n	x_1^T	x_{n+2}^T
$(1.2, 1.01)$	5	-0.989617650796475	0.974131930155257
	10	-0.995527156361191	0.991277828763488
	15	-0.997431051900731	0.995675625680682
	20	-0.998317300421766	0.997425153658672
	30	-0.999110214098954	0.998786714682353
$(1.0001, 1.25)$	5	-0.930746165060798	0.976077006397617
	10	-0.973037919917524	0.991676686601152
	15	-0.985768154010461	0.995816156554298
	20	-0.991257699163812	0.997490025315858
	30	-0.995791471964076	0.998807789973846

Table 7: The outermost nodes of the truncated rule $\mathcal{Q}^T_{n+2}$ for the measure $d\widehat{\lambda}(x) = \frac{x+\gamma}{x+\delta} \cdot \frac{dx}{\sqrt{1-x^2}}$ for some values of γ, δ and n.

6. Numerical examples of error estimation

The examples of this section illustrate the application of the quadrature rules $\mathcal{Q}^L_{2n+1}$, $\mathcal{Q}^S_{2n+1}$, $\mathcal{Q}^W_{2n+1}$, $\mathcal{Q}^{1/4}_{2n+1}$, and $\mathcal{Q}^T_{n+2}$ to estimating the quadrature error in the Gauss quadrature rule $\mathcal{G}_n$. We will approximate the integral

$$I(f) = \int_{-1}^{1} f(x)\,\mathrm{d}\widehat{\lambda}(x)$$

for a few integrands and tabulate the error estimates

$$\begin{gathered} E_L = |\mathcal{Q}^L_{2n+1}(f) - \mathcal{G}_n(f)|, \qquad E_S = |\mathcal{Q}^S_{2n+1}(f) - \mathcal{G}_n(f)|, \\ E_{1/4} = |\mathcal{Q}^{1/4}_{2n+1}(f) - \mathcal{G}_n(f)|, \qquad E_W = |\mathcal{Q}^W_{2n+1}(f) - \mathcal{G}_n(f)|, \\ E_T = |\mathcal{Q}^T_{n+2}(f) - \mathcal{G}_n(f)|, \end{gathered} \tag{28}$$

for some values of γ, δ, and n. The actual errors in the tables are determined by letting the rule $\mathcal{A}_\ell$ in (5) be a Gauss quadrature rule $\mathcal{G}_\ell$ with ℓ large. The exact value $I = I(f)$ of the integral is approximated by $\mathcal{G}_\ell$.

Example 6.1. *Let*

$$f(x) = e^{-x^2} \quad \textit{and} \quad d\widehat{\lambda}(x) = \frac{x + 1.05}{x + 1.01} \cdot \frac{dx}{\sqrt{1 - x^2}}.$$

Table 8 depicts the maximum admissible β and the error estimates (28). The true value of the integral is $I = 2.4074780100748213295881078012\ldots$. All error estimates can be seen to be very accurate. Note that although the formula $\mathcal{Q}^L_{2n+1}$ is not internal, the integrand is well defined at all nodes.

n	$\beta - \frac{1}{4}$	E_L	E_S	E_W	$E_{1/4}$	E_T	*Actual Error*
5	-1.2676×10^{-2}	3.72×10^{-5}	3.72×10^{-5}	3.73×10^{-5}	3.72×10^{-5}	3.72×10^{-5}	3.72×10^{-5}
10	-5.5308×10^{-4}	1.19×10^{-12}	1.19×10^{-12}	1.19×10^{-12}	1.19×10^{-12}	1.19×10^{-12}	1.19×10^{-12}
15	-2.3736×10^{-5}	3.23×10^{-21}	3.23×10^{-21}	3.23×10^{-21}	3.23×10^{-21}	3.23×10^{-21}	3.23×10^{-21}
20	-1.0180×10^{-6}	1.69×10^{-30}	1.69×10^{-30}	1.69×10^{-30}	1.69×10^{-30}	1.69×10^{-30}	1.69×10^{-30}
30	-1.8721×10^{-9}	1.48×10^{-50}	1.48×10^{-50}	1.48×10^{-50}	1.48×10^{-50}	1.48×10^{-50}	1.48×10^{-50}

Table 8: Error estimates and the magnitude of true error for Example 6.1.

Example 6.2. *Let*

$$f(x) = 999.1^{\log_{10}(1+x)} \quad \textit{and} \quad d\widehat{\lambda}(x) = \frac{x + 1.01}{x + 1.0001} \cdot \frac{dx}{\sqrt{1 - x^2}}.$$

The true value of the integral is $I = 7.8989637058043261255845701792 7\ldots$. In this example, the rules $\mathcal{Q}^L_{2n+1}$, $\mathcal{Q}^S_{2n+1}$, and $\mathcal{Q}^{1/4}_{2n+1}$ have a node smaller than -1, which makes them unusable. Table 9 shows the error estimates (28) determined with the rules $\mathcal{Q}^W_{2n+1}$ and $\mathcal{Q}^T_{n+2}$.

n	$\beta - \frac{1}{4}$	E_W	E_T	*Actual Error*
5	-5.8655×10^{-2}	4.47×10^{-8}	4.45×10^{-8}	4.93×10^{-8}
10	-1.4642×10^{-2}	3.76×10^{-10}	2.90×10^{-10}	4.13×10^{-10}
15	-3.6145×10^{-3}	2.69×10^{-11}	1.58×10^{-11}	2.86×10^{-11}
20	-8.8325×10^{-4}	4.39×10^{-12}	2.06×10^{-12}	4.57×10^{-12}
30	-5.2392×10^{-5}	3.62×10^{-13}	1.20×10^{-13}	3.69×10^{-13}

Table 9: Error estimates and the magnitude of the true error for Example 6.2.

Example 6.3. *Consider*

$$f(x) = e^{-\frac{1}{x^2}} \quad \text{and} \quad d\widehat{\lambda}(x) = \frac{x+1.01}{x+1.1}\sqrt{1-x^2}\,dx.$$

All error estimates (28) can be computed and are listed in Table 10. The true value of the integral is $I = 0.071794048439686007272425252815929911 6\ldots$*. The quadrature rules* $\mathcal{Q}^L_{2n+1}$*,* $\mathcal{Q}^S_{2n+1}$*, and* $\mathcal{Q}^{1/4}_{2n+1}$ *can be seen to determine the most accurate error estimates.*

n	β	E_L	E_S	E_W	$E_{1/4}$	E_T	*Actual Error*
5	0.34543	4.70×10^{-3}	4.70×10^{-3}	4.17×10^{-3}	4.69×10^{-3}	5.12×10^{-3}	4.77×10^{-3}
10	0.29904	9.81×10^{-5}	9.81×10^{-5}	1.00×10^{-4}	9.81×10^{-5}	1.94×10^{-4}	9.97×10^{-5}
15	0.28304	4.80×10^{-6}	4.80×10^{-6}	4.04×10^{-6}	4.80×10^{-6}	6.45×10^{-6}	4.78×10^{-6}
20	0.27488	1.13×10^{-6}	1.13×10^{-6}	1.05×10^{-6}	1.13×10^{-6}	4.15×10^{-7}	1.13×10^{-6}
30	0.26662	4.48×10^{-9}	4.48×10^{-9}	5.13×10^{-9}	4.48×10^{-9}	4.40×10^{-8}	4.48×10^{-9}

TABLE 10: *Error estimates and the magnitude of the true error for Example 6.3.*

Example 6.4. *Let*

$$f(x) = (1-x)^3\ln(1-x) \quad \text{and} \quad d\widehat{\lambda}(x) = \frac{x-5}{x-1.25}\sqrt{\frac{1+x}{1-x}}\,dx.$$

The true value of the integral is $I = 1.4540374438682709252520276 9\ldots$*. The rules* $\mathcal{Q}^L_{2n+1}$*,* $\mathcal{Q}^S_{2n+1}$*, and* $\mathcal{Q}^{1/4}_{2n+1}$ *cannot be evaluated, so Table 11 shows only the error estimates induced by* $\mathcal{Q}^W_{2n+1}$ *and* $\mathcal{Q}^T_{n+2}$*, of which the former yields a more accurate error estimate than the latter.*

n	$\beta - \frac{1}{4}$	E_W	E_T	*Actual Error*
5	-1.2544×10^{-9}	6.14×10^{-4}	5.41×10^{-4}	6.11×10^{-4}
10	-1.3884×10^{-19}	7.02×10^{-6}	4.85×10^{-6}	6.98×10^{-6}
15	-1.5368×10^{-29}	4.95×10^{-7}	2.77×10^{-7}	4.92×10^{-7}
20	-1.7010×10^{-39}	7.36×10^{-8}	3.44×10^{-8}	7.31×10^{-8}
30	-2.0840×10^{-59}	4.86×10^{-9}	1.70×10^{-9}	4.82×10^{-9}

TABLE 11: *Error estimates and the magnitude of the true error for Example 6.4.*

7. Conclusion

This paper discusses the estimation of the quadrature error in n-node Gauss rules $\mathcal{G}_n(f)$ associated with modifications of Chebyshev measures of the first, second, and third kinds for some integrand f. Our default approach to estimate the quadrature error is to evaluate averaged rules $\mathcal{Q}^L_{2n+1}(f)$ or optimal averaged rules $\mathcal{Q}^S_{2n+1}(f)$ with $2n+1$ nodes. However, the rules $\mathcal{Q}^L_{2n+1}(f)$ and $\mathcal{Q}^S_{2n+1}(f)$ might not be internal, i.e., these rules might have nodes outside of the interval $[-1, 1]$, and this makes their application impossible when the integrand only is defined in the open interval $(-1, 1)$ or its closure. We therefore investigate whether the truncated rules $\mathcal{Q}_{n+1}^{T}(f)$ and the weighted averaged rules $\mathcal{Q}_{2n+1}^{\beta}(f)$ can be applied, where the parameter β determines the weighting. Computed examples show the latter rules to give more accurate estimates than the former. Moreover, we show inequalities for the parameter β, such that if β satisfies these inequalities, then the weighted averaged quadrature rules are guaranteed to be internal. These inequalities make it easy to determine a suitable weighting. The properties and performance of the quadrature rules considered are illustrated by numerous numerical examples.

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