POLYNOMIAL ZEROFINDERS BASED ON SZEGŐ POLYNOMIALS

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Abstract. The computation of zeros of polynomials is a classical computational problem. This paper presents two new zerofinders that are based on the observation that, after a suitable change of variable, any polynomial can be considered a member of a family of Szegő polynomials. Numerical experiments indicate that these methods generally give higher accuracy than computing the eigenvalues of the companion matrix associated with the polynomial.

Key words. Szegő-Hessenberg matrix, companion matrix, eigenvalue problem, continuation method, parallel computation.

1. Introduction.

The computation of the zeros of a polynomial

$$\psi_n(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_1 z + \alpha_0, \qquad \alpha_j \in \mathbb{C}, \tag{1}$$

is a fundamental problem in scientific computation that arises in many diverse applications. The conditioning of this problem has been investigated by Gautschi [7, 8]. Several classical methods for determining zeros of polynomials are described by Henrici [16, Chapter 6] and Stoer and Bulirsch [25, Chapter 5]. A recent extensive bibliography of zerofinders is provided by McNamee [20]. Among the most popular numerical methods for computing zeros of polynomials are the Jenkins-Traub algorithm [17], and the computation of the zeros as eigenvalues of the companion matrix

$$C_n = \begin{bmatrix} 0 & & & \cdots & 0 & -\alpha_0 \\ 1 & 0 & & \cdots & 0 & -\alpha_1 \\ & 1 & 0 & \cdots & 0 & -\alpha_2 \\ & & & & \vdots & \vdots \\ & & \ddots & \ddots & & \\ & & & 1 & 0 & -\alpha_{n-2} \\ 0 & & & & 1 & -\alpha_{n-1} \end{bmatrix} \in \mathbb{C}^{n\times n} \tag{2}$$

associated with the polynomial (1) by the QR algorithm after balancing; see Edelman and Murakami [6] and Moler [21]. Recently, Goedecker [9] compared these methods and found the latter approach to be competitive with several available implementations of the Jenkins-Traub algorithm with regard to both accuracy and execution time for polynomials of small to moderate degree.

This paper describes two new methods for computing zeros of polynomials. The methods are based on the observation that, after a change of variable, any polynomial can be considered a member of a family of Szegő polynomials. The new zerofinders use the recursion relation for the Szegő polynomials, which are defined as follows. Let

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ω be a nondecreasing distribution function with infinitely many points of increase on the unit circle in the complex plane and define the inner product

$$(f,g) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)\overline{g(z)}d\omega(t), \qquad z := \exp(it), \qquad i := \sqrt{-1}, \tag{3}$$

for polynomials f and g, where the bar denotes complex conjugation. We assume for notational convenience that $d\omega(t)$ is scaled so that $(1,1) = 1$. Introduce orthonormal polynomials with respect to this inner product, $\phi_0, \phi_1, \phi_2, \ldots$, where ϕ_j is of degree j with positive leading coefficient. These polynomials are known as Szegő polynomials and many of their properties are discussed by Grenander and Szegő [15]. In particular, they satisfy the recursion relation

$$\begin{array}{rcll} \phi_0(z) & = & \phi_0^*(z) = 1, & \\ \sigma_{j+1}\phi_{j+1}(z) & = & z\phi_j(z) + \gamma_{j+1}\phi_j^*(z), & \qquad j = 0, 1, 2 \ldots, n-1, \\ \sigma_{j+1}\phi_{j+1}^*(z) & = & \bar{\gamma}_{j+1} z\phi_j(z) + \phi_j^*(z), & \end{array} \tag{4}$$

where the recursion coefficients γ_{j+1} and the auxiliary coefficients σ_{j+1} are defined by

$$\begin{array}{l} \gamma_{j+1} = -\dfrac{(z\phi_j, 1)}{\delta_j}, \\ \sigma_{j+1} = \sigma_j(1 - |\gamma_{j+1}|^2), \qquad j = 0, 1, 2, \ldots , \\ \delta_{j+1} = \delta_j\sigma_{j+1}, \qquad \delta_0 = \sigma_0 = 1. \end{array} \tag{5}$$

It follows from (4) that the auxiliary polynomials ϕ_j^* satisfy

$$\phi_j^*(z) := z^j\bar{\phi}_j(1/z). \tag{6}$$

The zeros of the Szegő polynomials are strictly inside the unit circle and all recursion coefficients γ_j are of magnitude smaller than one; see, e.g., [1, 15]. The leading coefficient of ϕ_j is $1/\delta_j$.

The first step in the new zerofinders of this paper is to determine recursion coefficients $\{\gamma_j\}_{j=1}^n$, such that the Szegő polynomial ϕ_n satisfies

$$\delta_n\phi_n(\zeta) = \eta_1^n\psi_n(z), \tag{7}$$

where

$$\zeta = \eta_1 z + \eta_2, \tag{8}$$

and the constants η_1 and η_2 are chosen so that the zeros z_j of ψ_n are mapped to zeros ζ_j of ϕ_n inside the unit circle. We refer to this change of variable as a *rescaling* of the monic polynomial $\psi_n(z)$. Its construction is discussed in Section 2. Thus, the problem of determining the zeros of ψ_n is reduced to the problem of computing the zeros of a Szegő polynomial of degree n. Section 3 considers two methods for this purpose, based on a matrix formulation of the recursion relation (4). This gives an $n \times n$ upper Hessenberg matrix whose eigenvalues are the zeros of ϕ_n. We refer to this matrix, which is described in [10], as the *Szegő-Hessenberg matrix* associated with ϕ_n. Having computed the eigenvalues ζ_j of this matrix, we use the relation (8) to compute the zeros z_j of ψ_n.

A third method for computing the zeros of $\psi_n(z)$ is to use the power-basis coefficients of the monic Szegő polynomial $\Phi_n(\zeta) := \delta_n \phi_n(\zeta)$ of (7) to form the companion matrix associated with Φ_n, compute its eigenvalues, and transform these back to the z-variable using (8). In other words, to use the companion matrix of the rescaled monic polynomial Φ_n instead of that of ψ_n. This method is included in the numerical results we report in Section 4.

Section 4 compares the use of the QR algorithm with balancing for computing the eigenvalues of the Szegő-Hessenberg, the companion matrix (2) of ψ_n, and the companion matrix of the rescaled polynomial Φ_n. We note in passing that these are all upper Hessenberg matrices. Balancing is commonly used for improving the accuracy of the computed eigenvalues; see [6] for a discussion on balancing of the companion matrix. In our experiments we found that when the parameters η_1 and η_2 for the rescaling are chosen so that all zeros of ϕ_n are inside the unit circle and one zero is close to the unit circle, the computed eigenvalues of the Szegő-Hessenberg matrix and of the companion matrix of the rescaled polynomial (7) generally provide more accurate zeros of ψ_n than those of the companion matrix of ψ_n. This rescaling is achieved by application of the Schur-Cohn test as described in Section 3. Numerous computed examples, some of which are reported in Section 4, indicate that computing eigenvalues of the Szegő-Hessenberg matrix after balancing often gives the zeros of ψ_n with higher accuracy than computing eigenvalues of the companion matrix of the scaled polynomial (7) after balancing. Both methods, in general, give higher accuracy in the computed zeros than computing the zeros of ψ_n as eigenvalues of the balanced companion matrix.

The other zerofinder for Szegő polynomials discussed in Section 3 is the continuation method previously introduced in [2]. For many polynomials ψ_n, this method yields higher accuracy than the computation of the eigenvalues of the associated companion or Szegő-Hessenberg matrices. Section 4 presents numerical examples and Section 5 contains concluding remarks.

2. Computation of Szegő polynomials. Given a polynomial $\psi_n(z)$ in power-basis form (1), we compute the recursion coefficients $\{\gamma_j\}_{j=1}^n$ of the family of Szegő polynomials $\{\phi_j\}_{j=0}^n$, chosen so that ϕ_n satisfies (7), by first transforming the polynomial ψ_n so that the average of its zeros vanishes. Then we determine a disk centered at the origin that contains all zeros of the transformed polynomial. The complex plane is then scaled so that this disk becomes the unit disk. In this fashion, the problem of determining the zeros of the polynomial ψ_n has been transformed into an equivalent problem of determining the zeros of a polynomial with all zeros in the unit disk. We may assume that the latter polynomial has leading coefficient one, and identify it with the monic Szegő polynomial $\Phi_n = \delta_n \phi_n$. Given the power-basis coefficients of Φ_n, the recursion coefficients of the family of Szegő polynomials $\{\phi_j\}_{j=0}^n$ can be computed by the Schur-Cohn algorithm. The remainder of this section describes details of the computations outlined.

Let $\{z_j\}_{j=1}^n$ denote the zeros of ψ_n and introduce the average of the zeros

$$\rho := \frac{1}{n} \sum_{j=1}^{n} z_j. \tag{9}$$

We evaluate this quantity as $\rho = -\frac{\alpha_{n-1}}{n}$, and define the new variable $\hat{z} = z - \rho$. The polynomial $\hat{\psi}_n(\hat{z}) := \psi_n(z)$ can be written as

$$\hat{\psi}_n(\hat{z}) = \hat{z}^n + \hat{\alpha}_{n-2}\hat{z}^{n-2} + \cdots + \hat{\alpha}_1 \hat{z} + \hat{\alpha}_0. \tag{10}$$

The coefficients $\{\hat{\alpha}_j\}_{j=0}^{n-2}$ can be computed from the coefficients $\{\alpha_j\}_{j=0}^{n-1}$ in $\mathcal{O}(n^2)$ arithmetic operations.

We now scale the $\hat{z}$-plane in two steps in order to move the zeros of $\hat{\psi}_n$ inside the unit circle. Our choice of scaling is motivated by the following result mentioned by Ostrowski [22].

PROPOSITION 2.1. *Let χ_n be a polynomial of degree n of the form*

$$\chi_n(z) = z^n + \beta_{n-2}z^{n-2} + \cdots + \beta_1 z + \beta_0, \tag{11}$$

and assume that

$$\max_{0\leq j\leq n-2} |\beta_j| = 1.$$

Then all zeros of χ_n are contained in the open disk $\{z : |z| < \frac{1}{2}(1+\sqrt{5})\}$ in the complex plane.

Proof. Let z be a zero of χ_n and assume that $|z| > 1$. Then

$$z^n = -\beta_{n-2}z^{n-2} - \cdots - \beta_1 z - \beta_0,$$

and it follows that

$$|z|^n \leq \sum_{j=0}^{n-2} |z|^j = \frac{|z|^{n-1}-1}{|z|-1}.$$

This inequality can be written as

$$|z|^{n-1}(|z|^2 - |z| - 1) \leq -1. \tag{12}$$

Since $|z|^2 - |z| - 1 = (|z| - \frac{1}{2}(1-\sqrt{5}))(|z| - \frac{1}{2}(1+\sqrt{5}))$, inequality (12) can only hold for $|z| < \frac{1}{2}(1+\sqrt{5})$. $\square$

After the change of variable $\tilde{z} := \sigma\hat{z}$, where $\sigma > 0$ is chosen so that

$$\max_{2\leq j\leq n} \sigma^j |\hat{\alpha}_{n-j}| = 1,$$

the polynomial $\tilde{\psi}_n(\tilde{z}) := \sigma^n \hat{\psi}_n(\hat{z})$ satisfies the conditions of the proposition.

Define the scaling factor

$$\tau := \frac{2}{1+\sqrt{5}}. \tag{13}$$

By Proposition 2.1 the change of variables

$$\zeta := \tau\tilde{z} \tag{14}$$

yields a monic polynomial

$$\Phi_n^{(\tau)}(\zeta) := \tau^n \tilde{\psi}_n(\tilde{z}) \tag{15}$$

with all zeros inside the unit circle.

We identify $\Phi_n^{(\tau)}$ with the monic Szegő polynomial $\delta_n\phi_n$, and wish to compute the recursion coefficients $\{\gamma_j\}_{j=1}^{n}$ that determine polynomials of lower degree $\{\phi_j\}_{j=0}^{n-1}$ in the same family of Szegő polynomials; see (4). This can be done by using the relationship between the coefficients of ϕ_j in power form and the coefficients of the associated auxiliary polynomial. Specifically, it follows from (6) that if

$$\phi_j(z) = \sum_{k=0}^{j} \beta_{j,k} z^k, \tag{16}$$

then

$$\phi_j^*(z) = \sum_{k=0}^{j} \bar{\beta}_{j,k-j} z^k.$$

Thus, given the Szegő polynomial ϕ_n in power form, we can determine the coefficients of the associated auxiliary polynomial ϕ_n^* in power form and apply the recursion formula (4) "backwards" in order to determine the recursion coefficient γ_n and the coefficients of the polynomials ϕ_{n-1} and ϕ_{n-1}^* in power form. In this manner we can determine the recursion coefficients γ_j for decreasing values of the index j.

The Schur-Cohn algorithm, see, e.g., Henrici [16, Chapter 6], is an implementation of these computations. The algorithm requires $\mathcal{O}(n^2)$ arithmetic operations to determine the recursion coefficients $\{\gamma_j\}_{j=1}^n$ from the representation of ϕ_n in power form (16).

We remark that the Schur-Cohn algorithm is known for its use in determining whether a given polynomial, in power form, has all zeros inside the unit circle. In this context it is known as the Schur-Cohn test; see [16, Chapter 6]. All zeros being strictly inside the unit circle is equivalent with all recursion coefficients $\{\gamma_j\}_{j=1}^n$ being of magnitude strictly smaller than one. We will return to this property of the recursion coefficients in Section 3.

Perhaps the first application of the Schur-Cohn algorithm to the computation of zeros of polynomials was described by Lehmer [18], who covered the complex plane by disks and used the Schur-Cohn test to determine which disks contain zeros of the polynomial. Lehmer's method can be viewed as a generalization of the bisection method to the complex plane. It is discussed in [16, Chapter 6].

3. The zerofinders.

We present two zero finders for ϕ_n and assume that the recursion coefficients $\{\gamma_j\}_{j=1}^n$ as well as the auxiliary coefficients $\{\sigma_j\}_{j=1}^n$ are available.

3.1. An eigenvalue method.

Eliminating the auxiliary polynomials ϕ_j^* in the recursion formula (4) yields an expression for ϕ_{j+1} in terms of Szegő polynomials of lower degree. Writing the expressions for the first $n+1$ Szegő polynomials in matrix form yields

$$\begin{aligned}[\phi_0(z), \phi_1(z), \ldots, \phi_{n-1}(z)]H_n &= z[\phi_0(z), \phi_1(z), \ldots, \phi_{n-1}(z)] \\ &- [0, \ldots, 0, \phi_n(z)],\end{aligned} \tag{17}$$

where

$$H_n = \begin{bmatrix} -\gamma_1 & -\sigma_1\gamma_2 & -\sigma_1\sigma_2\gamma_3 & \cdots & -\sigma_1\cdots\sigma_{n-1}\gamma_n \\ \sigma_1 & -\overline{\gamma}_1\gamma_2 & -\overline{\gamma}_1\sigma_2\gamma_3 & \cdots & -\overline{\gamma}_1\sigma_2\cdots\sigma_{n-1}\gamma_n \\ & \sigma_2 & -\overline{\gamma}_2\gamma_3 & \cdots & -\overline{\gamma}_2\sigma_3\cdots\sigma_{n-1}\gamma_n \\ & & \ddots & & \vdots \\ & & \sigma_{n-2} & -\overline{\gamma}_{n-2}\gamma_{n-1} & -\overline{\gamma}_{n-2}\sigma_{n-1}\gamma_n \\ 0 & & & \sigma_{n-1} & -\overline{\gamma}_{n-1}\gamma_n \end{bmatrix} \in \mathbb{C}^{n\times n} \tag{18}$$

is the *Szegő-Hessenberg matrix* associated with the Szegő polynomials $\{\phi_j\}_{j=0}^n$; see [10]. Equation (17) shows that the eigenvalues of the upper Hessenberg matrix H_n are the zeros of ϕ_n. Thus, we can compute the zeros of ϕ_n by determining the eigenvalues of H_n.

Let ζ_j, $1 \leq j \leq n$, denote the zeros of ϕ_n. The scaling parameters η_1 and η_2 in (8) are chosen so that all zeros of ϕ_n are inside the unit circle. However, for some polynomials ψ_n, the scaling may be such that

$$\kappa_n := \max_{1\leq j\leq n} |\zeta_j| \ll 1.$$

We have noticed that we can determine the zeros of ψ_n with higher accuracy when the disk is rescaled to make κ_n close to one. Such a rescaling is easy to achieve by repeated application of the Schur-Cohn test as follows. Instead of scaling $\tilde{z}$ by the factor (13) in (14), we scale $\tilde{z}$ by $\tau := \sqrt{2}/(1+\sqrt{5})$ and apply the Schur-Cohn test to determine whether all zeros of the scaled polynomial (15) so obtained are inside the unit circle. If they are not, then we increase the scaling factor τ in (14) by a factor $\Delta\tau := (2/(1+\sqrt{5}))^{1/10}$ and check whether the (re)scaled polynomial (15) obtained has all zeros inside the unit circle. The scaling factor τ is increased repeatedly by the factor $\Delta\tau$ until the polynomial (15) has all its zeros inside the unit circle. On the other hand, if the polynomial (15) associated with the scaling factor $\tau = \sqrt{2}/(1+\sqrt{5})$ has all zeros inside the unit circle, we repeatedly decrease τ by a factor $(\Delta\tau)^{-1}$ until a scaling factor τ has been determined, such that all zeros of the polynomial $\Phi_n^{(\tau)}$ are inside the unit disk, but at least one zero of $\Phi_n^{(\tau/\Delta\tau)}$ is not. Our choice of scaling factor τ in (14) assures that the monic polynomial (15) has all its zeros inside the unit circle and (at least) one zero close to the unit circle.

The scaling factors τ in (14) for the computed examples reported in Section 4 have been determined as described above. In our experience, the time spent rescaling the disk is negligible compared to the time required to compute the eigenvalues of H_n, because each rescaling only requires $\mathcal{O}(n^2)$ arithmetic operations.

After determining the scaling factor τ as described above and computing the recursion coefficients $\{\gamma_j\}_{j=1}^n$ via the Schur-Cohn test, we form the Szegő-Hessenberg matrix (18), balance it, and compute its eigenvalues using the QR algorithm.

3.2. A continuation method. Similarly as in the method described in Subsection 3.1, we first determine the recursion coefficients of the Szegő polynomials $\{\phi_j\}_{j=0}^n$ such that equation (7) holds, as described above. We then apply the continuation method for computing zeros of Szegő polynomials developed in [2]. In this method the Szegő-Hessenberg matrix (18) is considered a function of the last recursion parameter γ_n. Denote this parameter by $t \in \mathbb{C}$ and the associated Szegő-Hessenberg matrix by $H_n(t)$. Thus, we write the matrix (18) as $H_n(\gamma_n)$. When $|t| = 1$, the Szegő-Hessenberg matrix $H_n(t)$ is unitary. Assume that $\gamma_n \neq 0$. Then $H_n(\gamma_n/|\gamma_n|)$ is the closest unitary matrix to $H_n(\gamma_n)$; see [2] for details. The continuation method for computing zeros of Szegő polynomials consists of the following steps:

i) Compute the eigenvalues of the unitary upper Hessenberg matrix $H_n(\gamma_n/|\gamma_n|)$.

ii) Apply a continuation method for tracking the path of each eigenvalue of the matrix $H_n(t)$ as t is moved from $\gamma_n/|\gamma_n|$ to γ_n.

Several algorithms that require only $\mathcal{O}(n^2)$ arithmetic operations for the computations of step i) are available; see, e.g., [4, 5, 11, 12, 13, 14]. If the coefficients α_j in (1) are real, then the method discussed in [3] can also be applied. These methods compute the eigenvalues of $H_n(\gamma_n/|\gamma_n|)$ without explicitly forming the matrix elements. In the numerical experiments reported in Section 4, we used the implementation [4] of the divide-and-conquer method described in [13, 14]. The computations required for this method can readily be implemented on a parallel computer. This may be of importance in the application of the zerofinder in real-time filter design; see, e.g., Parks and Burrus [23] and references therein for more on this application of polynomial zerofinders.

We have found that for many polynomials ψ_n, the continuation method determines the zeros with higher accuracy than the method discussed in Subsection 3.1. The continuation method determines the zeros of the Szegő polynomial ϕ_n close to the unit circle particularly rapidly. However, our present implementation of the continuation method may fail to determine all zeros for some polynomials ψ_n when the pathfollowing is complicated by (numerous) bifurcation points. These cases are easy to identify; see [2] for a discussion and remedies.

We remark that other continuation methods also are available, such as the method proposed by Li and Zeng [19] for computing the eigenvalues of a general Hessenberg matrix. This method does not use the structure of the Hessenberg matrices (18), i.e., the fact that the last recursion coefficient γ_n is a natural continuation parameter. However, it may be possible to apply some techniques developed in [19] to improve the performance of the continuation method of this paper; see [2] for a discussion and references to other continuation methods.

4. Computed examples. We present the results of several computed examples which illustrate the performance of the zerofinders discussed in Section 3. The computer programs used were all written in FORTRAN 77, and the numerical experiments were carried out on a SUN SparcStation 5 in single-precision arithmetic, i.e., with approximately 7 significant decimal digits of accuracy, except where explicitly stated otherwise. The eigenvalues of the companion and Szegő-Hessenberg matrices were computed by single-precision subroutines from EISPACK [24].

In our experiments, we input a set of n real or complex conjugate zeros of the polynomial ψ_n, see (1), and compute the coefficients α_j of the power-basis representation by a recursion formula. These computations are carried out in double-precision arithmetic, i.e., with about 15 significant digits, in order to avoid loss of accuracy. After their computation, the α_j are stored as single-precision real numbers. We now seek to determine the zeros of ψ_n, given the coefficients α_j, with one of several methods:

CB: The QR algorithm applied to the companion matrix (2) of ψ_n after balancing, using the EISPACK routines `balanc` and `hqr`.

CBS: The QR algorithm applied to the companion matrix of the monic Szegő polynomial Φ_n, after balancing, using the EISPACK routines `balanc` and `hqr`.

SHB: The QR algorithm applied to the Szegő-Hessenberg matrix after balancing, using the EISPACK routines `balanc` and `hqr`.

CM: The continuation method for real Szegő-Hessenberg matrices, described in [2].

We compare the following computed quantities:

Residuals: The maximum modulus of the values of the initial monic polynomial ψ_n in power form (1) at the computed roots.

Differences: The computed zeros are put into correspondence with the initial zeros, which were used to generate ψ_n as described above, and the maximum difference after this pairing is computed. Note that this is not exactly the error in the computed zeros; the error is the maximum difference of the computed roots and the exact roots of the monic polynomial ψ_n. However, since the coefficients of ψ_n were computed from the given zeros in floating-point arithmetic, the exact zeros of the ψ_n need not be close to the input zeros. Nevertheless, the computed differences provide a way to compare the various methods.

In the tables we also display in the column labeled ψ_n the residuals computed at the input zeros; i.e. at the zeros that were used to compute the power-basis coefficients of ψ_n. This provides some indication of how ill-conditioned the roots of ψ_n and the computation of its power-basis coefficients are, as well as an indication of the significance of the differences and the other computed residuals that are displayed.

The polynomials ψ_n in all computed examples except those for Tables 7-8 have real or complex conjugate zeros uniformly distributed in a disk

$$D_R := \{z : |z| \leq R\} \subset \mathbb{C}. \tag{19}$$

In particular, the coefficients α_j in the representation (1) are real. We generate zeros of ψ_n in D_R as follows. Two random numbers are determined according to a uniform distribution on the interval $[-R, R]$ and used as the real and imaginary parts of a candidate zero z. If $z \in D_R$ and $\mathrm{Im}(z) > 1 \cdot 10^{-6}$, then both z and $\overline{z}$ are accepted as zeros of ψ_n. If $z \in D_R$ and $\mathrm{Im}(z) \leq 1 \cdot 10^{-6}$ then $\mathrm{Re}(z)$ is accepted as a real zero of ψ_n. The purpose of the condition on the imaginary part of z is to avoid that ψ_n has very close zeros. We generate candidate points until n zeros of ψ_n have been determined. When n is odd, then at least one of the zeros of ψ_n is in the real interval $[-R, R]$.

Table 1 shows results for 10 polynomials ψ_{15} generated in this manner with zeros in the disk D_1. We display the maximum modulus of the residuals and the maximum difference of the computed zeros with the input zeros for the methods CB, SHB, CM, and CBS. The results for CM for one of these 10 polynomials are marked with a "-" to indicate that the continuation method did not yield all n zeros. The averages for CM ignore the entries marked by -. In Table 1 the standard companion matrix approach (CB) consistently yields the least accuracy as measured both by the residuals and by the differences with the input zeros.

The integer arrays at the bottom of Table 1 display the relative performance of the algorithms. The (j, k) entry for $j > k$ is the number of times the jth algorithm gave smaller maximal differences or residuals than the kth algorithm, and the (j, j) entry indicates the number of times the jth algorithm gave the smallest maximal differences or residuals among the four methods compared. For example, the arrays for Table 1 show that CM produces the smallest residuals for 7 of the 10 polynomials generated. This count includes the polynomial for which CM failed to determine all zeros. The maximum residual for CM was smaller than for CB, SHB, and CBS for 9, 8, and 8 polynomials, respectively. CB produced larger residuals than any of the other three methods for all polynomials, except for the polynomial for which CM failed to determine all zeros.

Table 2 gives the results for 10 polynomials of degree 15 with uniformly distributed real and complex conjugate zeros in the disk D_2. In this experiment, CM successfully determined all zeros of all polynomials.

Tables 3-4 show summary data for 100 polynomials of each of several degrees n with uniformly distributed real and complex conjugate zeros in the disk D_1. We display in Tables 3 the average of the maximum differences and the average of the maximum residuals for the methods CB, SHB and CBS over all polynomials. For CM we compute these averages only over those polynomials for which the method successfully determined all zeros. The number of those polynomials of each degree n, out of 100, is denoted by N and is displayed in the last column of Table 3.

Table 1

Ten polynomials of degree $n = 15$ with zeros in D_1

Differences:	CB	SHB	CM	CBS
	6.67E-05	4.89E-06	4.57E-06	6.82E-06
	1.66E-03	7.57E-05	5.49E-05	2.11E-04
	1.20E-01	3.06E-03	-	1.83E-02
	8.41E-04	2.45E-05	3.91E-05	6.22E-04
	9.66E-04	5.88E-05	5.82E-05	1.51E-04
	2.75E-05	5.20E-07	1.79E-07	2.40E-06
	3.34E-05	5.75E-06	2.71E-07	2.05E-05
	1.67E-05	2.85E-06	2.25E-06	5.52E-05
	2.72E-04	6.60E-06	7.48E-07	3.77E-05
	7.60E-05	1.16E-06	7.40E-07	3.30E-06
Averages:	1.24E-02	3.24E-04	1.79E-05	1.94E-03

Residuals:	CB	SHB	CM	CBS	ψ_n
	3.85E-06	9.06E-07	4.89E-07	1.10E-06	6.94E-07
	3.31E-07	9.68E-08	2.05E-08	1.15E-07	1.47E-08
	3.16E-05	1.30E-05	-	2.41E-05	5.80E-07
	2.48E-06	9.15E-07	3.16E-07	1.47E-06	6.62E-08
	5.24E-06	6.74E-07	1.18E-06	1.50E-06	3.58E-07
	8.64E-08	2.13E-08	1.47E-08	4.12E-08	2.18E-09
	1.87E-06	6.88E-07	5.66E-07	8.80E-07	2.92E-08
	2.93E-06	2.48E-06	2.76E-07	2.71E-06	4.34E-08
	2.14E-07	7.87E-08	6.35E-08	3.23E-08	6.32E-09
	1.07E-06	4.44E-07	9.72E-08	9.11E-07	2.11E-08
Averages:	4.97E-06	1.93E-06	3.36E-07	3.28E-06	1.82E-07

	Differences				Residuals			
CB	0				0			
SHB	10	2			10	2		
CM	9	8	8		9	8	7	
CBS	9	0	1	0	10	1	2	1

In the experiments for Tables 5-6, we generated 100 polynomials of degree 20 with uniformly distributed real or complex conjugate zeros in disks (19) of radius R for several different values of R. The entries in the columns "Average Differences" and "Average Residuals" of Table 5 are computed as for Table 3. We display results obtained for disks with radii between 0.2 and 3.

Finally, Tables 7-8 illustrate the performance of the zerofinders for polynomials ψ_{20} with real zeros only. The zeros are uniformly distributed in the interval $[-1, 1]$. Tables 7-8 are analogous to Tables 3-4. We see that CBS often gives significantly higher accuracy than CB, and SHB usually yields slightly higher accuracy than CBS. Our present implementation of CM is able to accurately determine all or most zeros for the polynomials in this experiment of fairly low degree, $n \leq 10$, only, due to numerous bifurcation points encountered during pathfollowing. The performance of CM might be improved by using a more sophisticated pathfollowing method; see [2] for a discussion.

Table 2
Ten polynomials of degree $n = 15$ with zeros in D_2

Differences:	CB	SHB	CM	CBS
	3.06E-04	4.98E-05	4.25E-05	9.49E-05
	1.47E-04	4.30E-05	4.22E-05	8.30E-05
	9.99E-06	2.40E-06	2.67E-07	8.38E-06
	5.97E-06	3.04E-05	2.09E-06	1.59E-05
	2.72E-04	3.44E-05	3.05E-05	3.37E-05
	1.10E-06	1.77E-06	5.06E-07	1.53E-06
	4.77E-04	1.56E-05	1.78E-05	5.08E-05
	7.30E-04	1.02E-03	8.53E-04	8.76E-04
	7.92E-06	2.82E-06	1.53E-06	6.90E-06
	4.88E-04	8.80E-05	1.33E-05	1.55E-04
Averages:	2.44E-04	1.29E-04	1.00E-04	1.33E-04

Residuals:	CB	SHB	CM	CBS	ψ_n
	5.85E-02	6.82E-03	1.11E-02	6.22E-03	1.06E-03
	1.50E-01	3.04E-02	1.96E-02	4.09E-02	1.95E-02
	8.29E-02	1.90E-02	4.67E-03	1.26E-02	2.27E-03
	4.56E-01	4.67E-01	2.94E-02	2.13E-01	7.14E-03
	1.98E-03	2.93E-03	8.92E-04	8.11E-04	1.00E-03
	1.77E-02	1.92E-02	7.89E-03	7.24E-03	1.30E-03
	7.42E-01	3.88E-01	4.22E-01	5.35E-01	1.84E-02
	9.64E-03	7.14E-03	3.95E-03	1.23E-02	4.08E-03
	7.70E-02	2.89E-02	2.19E-02	1.21E-01	4.53E-03
	3.02E-02	3.05E-03	6.00E-04	4.11E-04	2.43E-03
Averages:	1.62E-01	9.73E-02	5.22E-02	9.50E-02	6.17E-03

	Differences				Residuals			
CB	1				0			
SHB	7	1			7	1		
CM	9	9	8		10	8	5	
CBS	7	4	0	0	8	6	4	4

In addition to the examples reported above, we carried out numerous numerical experiments with the zerofinders applied to polynomials whose zeros were uniformly distributed in squares and wedges in the complex plane. The performance of the zerofinders for these problems is similar to the performance reported in the Tables 1-6, and we therefore omit the details. We noted that for some classes of problems CBS performed comparatively better than in the Tables 1-6, and gave about the same accuracy as SHB. In all examples considered, CB gave the poorest overall accuracy.

5. Conclusions. Numerous numerical experiments, some of which have been presented in Section 4, indicate that the polynomial zerofinders CBS, CM and SHB presented in this paper, in general, yield higher accuracy than computing eigenvalues of the associated balanced companion matrix, the CB method. When CM finds all zeros, this method typically yields the highest accuracy. Presently, we are using a fairly simple pathfollowing scheme described in [2], and this implementation of CM may oc-

TABLE 3
Comparison of methods for 100 polynomials of each degree n with zeros in D_1

Average Differences						
n	CB	SHB	CM	CBS		N
10	1.20E-03	1.78E-05	1.75E-05	2.08E-05		99
15	3.12E-03	1.34E-04	1.14E-02	3.22E-04		94
20	3.48E-02	6.59E-03	7.27E-03	9.89E-03		86
30	1.75E-01	5.28E-02	1.67E-03	1.04E-01		47
40	3.95E-01	1.60E-01	1.07E-03	3.20E-01		12
Average Residuals						
n	CB	SHB	CM	CBS	ψ_n	N
10	1.70E-06	1.09E-06	3.58E-07	9.85E-07	1.20E-07	99
15	6.99E-06	2.89E-06	7.89E-07	3.39E-06	2.95E-07	94
20	4.06E-03	9.95E-06	1.90E-06	2.35E-05	8.01E-07	86
30	3.08E+01	7.52E-03	1.36E-05	1.03E-03	4.83E-06	47
40	1.05E+04	3.92E-02	6.31E-06	4.30E-02	4.64E-05	12

casionally fail to find all zeros. Our numerical experiments suggest that CM with an improved pathfollowing scheme would be an attractive zerofinder. Alternatively, one can use CM as presently implemented and switch to a different zerofinding method when CM fails to determine all zeros. This approach has the advantage of allowing us to keep the pathfollowing scheme simple. The numerical examples of Section 4, as well as other examples not reported, indicate that the SHB method may be a good method to switch to. It is simple to implement and often gives higher accuracy than the CB and CBS methods.

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Table 4
Comparative counts for 100 polynomials for each degree n with zeros in D_1

$n = 10$	Differences				Residuals			
CB	0				2			
SHB	97	12			79	13		
CM	99	83	71		95	81	74	
CBS	90	28	20	17	78	39	15	11
$n = 15$	Differences				Residuals			
CB	0				2			
SHB	100	17			77	8		
CM	97	80	78		95	85	81	
CBS	88	16	9	5	71	48	12	9
$n = 20$	Differences				Residuals			
CB	0				4			
SHB	97	15			79	16		
CM	88	78	77		88	79	73	
CBS	85	19	17	8	70	32	18	7
$n = 30$	Differences				Residuals			
CB	1				8			
SHB	97	42			84	29		
CM	61	55	53		58	50	46	
CBS	73	6	40	4	73	29	48	17
$n = 40$	Differences				Residuals			
CB	4				8			
SHB	94	74			88	55		
CM	26	17	16		19	13	12	
CBS	61	8	78	6	76	29	83	25

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Table 5
Comparison of methods for 100 polynomials of degree $n = 20$ for each radius R

Average Differences						
R	CB	SHB	CM	CBS		N
0.2	2.96E-01	1.54E-03	1.25E-03	2.01E-03		86
0.7	9.76E-02	4.45E-03	5.59E-04	6.54E-03		84
1.0	3.48E-02	6.59E-03	7.27E-03	9.89E-03		86
1.5	2.20E-02	1.16E-02	8.55E-04	1.84E-02		83
3.0	6.81E-02	2.32E-02	1.71E-03	3.38E-02		83
Average Residuals						
R	CB	SHB	CM	CBS	ψ_n	N
0.2	6.79E-10	1.32E-19	2.27E-20	1.15E-19	7.20E-21	86
0.7	4.69E-07	7.86E-09	1.73E-09	7.43E-09	5.79E-10	84
1.0	4.06E-03	9.95E-06	1.90E-06	2.35E-05	8.01E-07	86
1.5	6.91E+01	4.03E-02	5.13E-03	7.10E-02	3.15E-03	83
3.0	1.17E+08	4.06E+04	5.50E+03	6.79E+04	3.30E+03	83

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Table 6

Comparative counts for 100 polynomials of degree $n = 20$ for each radius R

$R = 0.2$	Differences				Residuals			
CB	0				0			
SHB	100	22			100	12		
CM	100	73	72		100	79	73	
CBS	100	23	15	6	100	48	20	15
$R = 0.7$	Differences				Residuals			
CB	0				0			
SHB	100	19			100	10		
CM	91	81	76		90	80	75	
CBS	100	15	17	5	100	35	20	15
$R = 1.0$	Differences				Residuals			
CB	0				4			
SHB	97	15			79	16		
CM	88	78	77		88	79	73	
CBS	85	19	17	8	70	32	18	7
$R = 1.5$	Differences				Residuals			
CB	7				4			
SHB	58	18			77	21		
CM	87	76	72		85	72	67	
CBS	36	16	17	3	72	37	21	8
$R = 3.0$	Differences				Residuals			
CB	4				0			
SHB	66	16			95	22		
CM	89	78	76		89	69	66	
CBS	44	17	17	4	90	37	22	12

Table 7

Comparison of methods for 100 polynomials of each degree n zeros in $[-1, 1]$

Average Differences				
n	CB	SHB	CBS	
10	8.73E-03	1.53E-03	3.16E-03	
15	5.83E-02	1.43E-02	3.47E-02	
20	2.07E-01	8.64E-02	1.67E-01	
30	4.97E-01	2.93E-01	5.62E-01	
40	7.18E-01	5.62E-01	7.94E-01	
Average Residuals				
n	CB	SHB	CBS	ψ_n
10	7.90E-07	4.64E-07	4.23E-07	6.92E-08
15	1.59E-06	8.51E-07	1.48E-06	9.62E-08
20	1.03E-05	4.05E-06	9.74E-06	2.69E-07
30	3.07E-04	5.24E-05	8.11E-05	7.90E-07
40	3.70E+01	5.01E-02	6.71E-02	3.34E-06

TABLE 8

Comparative counts for 100 polynomials of each degree n with zeros in $[-1, 1]$

$n = 10$	Differences			Residuals		
CB	2			6		
SHB	96	31		71	23	
CBS	74	9	7	74	59	28
$n = 15$	Differences			Residuals		
CB	2			17		
SHB	98	63		75	50	
CBS	77	5	4	60	31	26
$n = 20$	Differences			Residuals		
CB	2			17		
SHB	98	95		77	68	
CBS	71	4	3	59	17	15
$n = 30$	Differences			Residuals		
CB	10			7		
SHB	89	86		93	85	
CBS	39	6	4	64	11	8
$n = 40$	Differences			Residuals		
CB	19			10		
SHB	78	67		88	80	
CBS	37	20	13	53	11	10