Real Analysis II

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Assignment I.

1. **Problem 1.** Compute $1_{[0,1]} * 1_{[0,1]}(x)$, $x \in \mathbf{R}$.

2. **Problem 2.** Let f be a periodic (with period 1) integrable function on the real line, satisfying $\int\limits_0^1 f(t)dt = 0$. Prove that $\forall \alpha \in (0,1)$ there exists a set $A_\alpha \subset \mathbf{R}$ such that $\int\limits_{A_\alpha} f(t)dt = 0$, and $m(A_\alpha) = \alpha$.

   Hint. Where is Fubini? (periodicity is important!)

3. **Problem 3.** Prove that $(f * g) * h = f * (g * h)$, provided f, g, h are integrable on the real line.

4. **Problem 4.** Put $\phi(t) = 1 - \cos t$ if $0 \le t \le 2\pi$, $\phi(t) = 0$ for all other real t. For $-\infty < x < \infty$, define

   $$f(x) = 1, \qquad g(x) = \phi'(x), \qquad h(x) = \int\limits_{-\infty}^{x} \phi(t)dt.$$

   Verify the following statements about convolutions of these functions:

   (i) $f * g(x) = 0$ for all x.

   (ii) $g * h(x) = \phi * \phi(x) > 0$ on $(0, 4\pi)$.

   (iii) Therefore $(f*g)*h = 0$, whereas $f*(g*h)$ is a positive constant. But convolution is supposedly associative, by Fubini Theorem (see Problem 2). What went wrong?

5. **Problem 5.**

   Use Fubini Theorem and the relation

   $$\frac{1}{x} = \int\limits_0^\infty e^{-xt}dt, \qquad x > 0,$$

   to prove that

   $$\lim_{A\to\infty} \int\limits_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

6. **Problem 6.** Suppose A is a measurable subset of the real line, having finite positive measure. Show that the convolution $1_A * 1_A$ is continuous and not identically 0. Use this to prove that $A + A$ contains a segment.