Real Analysis II.

Instructor: Dmitry Ryabogin

Assignment X.

1. **Problem 1.**

   In class we considered the determinant D_m of the matrix, $(d_{ij})_{1\le i,j\le m}$, $d_{ij}=(a_i+b_i)^{-1}$, and showed that
   $$D_m = P_m/C_m, \qquad C_m=\Pi_{i,j=1}^m (a_i+b_j),$$
   where P_m is a polynomial of a_i and b_j.

   a) Prove that $deg(P_m)\le m^2-m$.

   b) Prove that $deg(A_m\,B_m)=m^2-m$,
   $$A_m\,B_m=\Pi_{1\le i<j\le m}(a_i-a_j)\,\Pi_{1\le i<j\le m}(b_i-b_j).$$

   c) Prove that
   $$\lim_{a_m\to\infty}\lim_{b_m\to\infty}\frac{a_m\,\Pi_{1\le i<j\le m}(a_i-a_j)(b_i-b_j)}{\Pi_{i,j=1}^m(a_i+b_j)} =$$
   $$\frac{\Pi_{1\le i<j\le m-1}(a_i-a_j)(b_i-b_j)}{\Pi_{i,j=1}^{m-1}(a_i+b_j)}.$$

2. **Problem 2.** A point $x_0\in C\subseteq H$ is called an **internal** point of C if for every $x\ne x_0$ there exists $\epsilon>0$ such that for every $0\le\lambda<\epsilon$ we have $(1-\lambda)x_0+\lambda x\in C$.

   a) Prove that every interior point is also an internal point.

   b) Let x be an internal point of convex C, and let $y\in C$. Then for all $0\le\lambda\le 1$, $\lambda x+(1-\lambda)y$ is an internal point of C.

   c) Find a set $C\subset\mathbf{R^2}$ such that the origin is an internal point of C but is not its interior point.

   d) Let $\{y_\alpha\}_{\alpha\in A}$ be an algebraic basis of H, and let $\|y_\alpha\|=1$ $\forall\alpha$. Pick a sequence $(x_n)_{n=1}^\infty$ of (countably many) vectors from the basis, and define C to be a set of vectors in H such that in the decomposition $y=\sum\limits_{k=1}^m c_{\alpha_k}y_{\alpha_k}$, $m=m(y)$, the coefficient near x_n is less than $1/n$ in absolute value.

   Prove that C is convex, that the origin is an internal point of C, but is not its interior point.

3. **Problem 3.**

   Let $g_1,g_2,...,g_n$ be vectors in $\mathbf{R^n}$, $g_k=(g_k(1),...,g_k(n)$. It is known that
   $$D(g_1,...,g_n)=|det\{(d_{ij})_{1\le i,j\le n}\}|,\qquad d_{ij}=g_j(i),$$

is a volume of a parallelepiped $\Pi_n(g_1, ..., g_n)$ generated by vectors $g_1, ..., g_n$.

a) Prove that $G(g_1, ..., g_n) = (D(g_1, ..., g_n))^2$, and conclude that $G(g_1, ..., g_n) = 0$ iff $g_1, ..., g_n$ are linearly dependent.

b) Consider $\mathbf{R^n}$ as a sub-space of $\mathbf{R^{n+1}}$, i.e.,

$$\mathbf{R^n} = \{x \in \mathbf{R^{n+1}} : x = (x(1), x(2), ..., x(n), 0)\}.$$

Let $g_1, ..., g_n \in \mathbf{R^n}$, $x \in \mathbf{R^{n+1}}$, and let $M = span\{g_1, ..., g_n\}$, $d := d(x, M) = \inf_{y \in M} \|x - y\|$. Prove that

$$d^2\, G(g_1, ..., g_n) = G(x, g_1, ..., g_n).$$

Hint:

$$vol_{n+1}(\Pi_{n+1}(x, g_1, ..., g_n)) = vol_n(\Pi_n(g_1, ..., g_n))\, height.$$

4. **Problem 4.** Let $\alpha > 0$, $f_\alpha(x) = e^{\alpha x}$, and let $H = L^2(-\infty, 0)$. Compute

$$dist(f_{\alpha_0}, span\{f_{\alpha_1}, ..., f_{\alpha_m}\}),$$

provided α_j, $j = 0, ..., m$ are pairwise different.